

Math 225, Fall 2025

Exam #1

Name: _____

- Please make an effort to write neatly, and *insert a few words where necessary* to get your ideas across. It's difficult to understand (and evaluate) mathematics in the absence of guiding words.

I will award up to 2 bonus points if I find your work well-documented and easy to read and understand.

- *No books, notes, calculators or other electronic devices on this exam.*
- Use the reverse sides of the pages or the extra blank sheets at the end if you need them.
- There are four problems, each worth 10 points.

Cheers!

1. A tank initially contains 10 gallons of fresh water. A salt solution containing 1 pound of salt per gallon of water is poured into the tank at the rate of 2 gal/min, and the mixture is drained from the tank at the rate of 1 gal/min. What is the amount of salt in the tank at any time t ? Assume that the tank does not overflow.

Solution: [Like exercise 3 of Section 2.4]

Let $V(t)$ be the volume of the brine in the tank and $Q(t)$ be the mass of salt in the tank, at any time t . The concentration of the salt in the tank is $c(t) = Q(t)/V(t)$. The tank initially contains fresh water, that is $Q(0) = 0$.

The conservation of salt's mass is expressed through

$$\frac{dQ}{dt} = F_{\text{in}}c_{\text{in}} - F_{\text{out}}c_{\text{out}},$$

where F_{in} and F_{out} are the (volumetric) rates of inflow and outflow, and c_{in} and c_{out} are the salt concentrations at the input and output. The salt concentration at output is the same as the concentration in the tank, that is, $c_{\text{out}} = c(t)$.

We are given $F_{\text{in}} = 2$ gal/min, $F_{\text{out}} = 1$ gal/min. Therefore the volume of the brine in the tank increases at the rate of 1 gal/min. Considering that the initial volume is 10 gal, the volume at any time is $V(t) = 10 + t$, where t is measured in minutes. We conclude that the equation of conservation of mass takes the form

$$\frac{dQ(t)}{dt} = 2 \times 1 - 1 \times \frac{Q(t)}{10 + t},$$

which we rearrange into the standard form and arrive at the initial value problem

$$\frac{dQ}{dt} + \frac{1}{10 + t}Q = 2, \quad Q(0) = 0,$$

and proceed to solve it through the method of integrating factors. We have

$$\int \frac{1}{10 + t} dt = \ln(10 + t)$$

and therefore the integrating factor is

$$\mu(t) = e^{\ln(10+t)} = 10 + t.$$

We multiply the differential equation through by the integrating factor

$$(10 + t) \frac{dQ}{dt} + Q = 2(10 + t),$$

and group the terms into

$$\left((10 + t)Q(t) \right)' = 2(10 + t) = 20 + 2t,$$

and integrate to get

$$(10 + t)Q(t) = 20t + t^2 + c.$$

The integration constant c is determined by applying the initial condition $Q(0) = 0$, which yields $c = 0$. We conclude that

$$Q(t) = \frac{20t + t^2}{10 + t}.$$

2. A cup of hot cocoa at the temperature of 220°F is served at the ski slope where the air temperature is 20°F . (That's cold!)

- (a) (2 pts) Write down the differential equation that expresses the rate of change of the temperature $T(t)$ of the cocoa according to *Newton's Law of Cooling*.
- (b) (4 pts) Solve the differential equation to determine $T(t)$ at any time t .
- (c) (4 pts) The drink's temperature drops to 120°F within two minutes. If not drunk, when will the cocoa reach the freezing temperature of 32°F ?

Solution: [Like exercises of Section 2.5]

(a) Newton's Law of Cooling says

$$\frac{d}{dt}T(t) = -k(T(t) - M)$$

where M is the ambient temperature. In our case we have $M(t) = 20$, and therefore

$$\frac{d}{dt}T(t) = -k(T(t) - 20).$$

(b) We put the differential equation into the standard form

$$\frac{d}{dt}T(t) + kT(t) = 20k.$$

The integrating factor is $\mu(t) = e^{\int k dt} = e^{kt}$. We multiply the equation by the integrating factor

$$e^{kt} \frac{d}{dt}T(t) + ke^{kt}T(t) = 20ke^{kt},$$

simplify

$$\frac{d}{dt}(e^{kt}T(t)) = 20ke^{kt},$$

and integrate:

$$e^{kt}T(t) = 20e^{kt} + C.$$

Evaluating this at $t = 0$ and applying the initial condition $T(0) = 220$ we obtain $220 = 20 + C$, whence $C = 200$. We thus arrive at $e^{kt}T(t) = 20e^{kt} + 200$, whence

$$T(t) = 20 + 200e^{-kt}.$$

(c) We are given that

$$120 = 20 + 200e^{-2k}.$$

Therefore $200e^{-2k} = 100$, that is, $e^{2k} = 2$. We conclude that $2k = \ln 2$, and so

$$k = \frac{1}{2} \ln 2.$$

The time to reach the temperature of 32°F is obtained by solving the equation $32 = 20 + 200e^{-kt}$, that is, $12 = 200e^{-kt}$, and therefore $e^{kt} = 200/12$, and thus $kt = \ln(200/12)$. Substituting for the value of k calculated earlier, this become

$$\left(\frac{1}{2} \ln 2\right) t = \ln \frac{200}{12}.$$

We conclude that

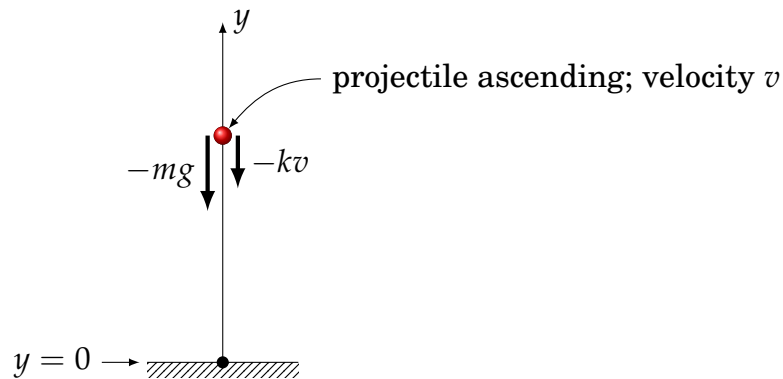
$$t = \frac{2 \ln \frac{200}{12}}{\ln 2} = \frac{2 \ln \frac{50}{3}}{\ln 2}.$$

That's approximately 8 minutes.

3. A projectile is launched straight up (vertically) with the initial velocity v_0 . The force of air resistance is $-kv$ where v is the velocity as a function of time t during the flight.
- (3 pts) Set up a vertical coordinate axis y , pointing up, along which the projectile moves. Set the origin at the launch point. Draw a diagram that shows the forces that act on the projectile during its upward flight. Apply that information to write down the differential equation that expresses Newton's law of motion, and its initial condition.
 - (4 pts) Solve the initial value problem and thus obtain the velocity $v(t)$ of the projectile as a function of time.
 - (3 pts) How long does it take for the projectile to reach the maximum height before it reverses its motion and begins to fall down?

Solution: [This is the first half of Exercise 16 of Section 2.6]

The forces acting on the projectile are the force of gravity $-mg$, and the air resistance $-kv$. As the projectile ascends, both of those forces point down, as shown in the following diagram.



According to Newton's Law of Motion, the resultant of the applied forces equals mass times acceleration, that is,

$$m \frac{d}{dt} v(t) = -mg - kv(t), \quad v(0) = v_0,$$

where $v(t)$ is the velocity at any time t .

- (b) We normalize the equation motion

$$\frac{d}{dt} v(t) + \frac{k}{m} v(t) = -g,$$

and calculate the integrating factor

$$\mu(t) = e^{\int k/m dt} = e^{kt/m}.$$

We multiply the equation by $\mu(t)$

$$e^{kt/m} \frac{d}{dt} v(t) + e^{kt/m} \frac{k}{m} v(t) = -ge^{kt/m}$$

and combine the terms on the left and get

$$\frac{d}{dt} \left(e^{kt/m} v(t) \right) = -g e^{kt/m}.$$

Integrating this we obtain

$$e^{kt/m} v(t) = -\frac{mg}{k} e^{kt/m} + c,$$

where c is the integration constant. We determine c by applying the initial condition $v(0) = v_0$:

$$v_0 = -\frac{mg}{k} + c,$$

and therefore

$$c = v_0 + \frac{mg}{k}.$$

We thus arrive at

$$e^{kt/m} v(t) = -\frac{mg}{k} e^{kt/m} + v_0 + \frac{mg}{k},$$

and therefore

$$v(t) = \left(v_0 + \frac{mg}{k} \right) e^{-kt/m} - \frac{mg}{k}.$$

- (c) The projectile's velocity is zero at the maximum height. That enables us to calculate the time of that event:

$$0 = \left(v_0 + \frac{mg}{k} \right) e^{-kt/m} - \frac{mg}{k}.$$

We rearrange that into

$$\left(v_0 + \frac{mg}{k} \right) e^{-kt/m} = \frac{mg}{k},$$

and isolate the exponential term

$$e^{kt/m} = \frac{v_0 + \frac{mg}{k}}{\frac{mg}{k}} = 1 + \frac{kv_0}{mg}.$$

Taking the logarithm of both sides we get

$$\frac{kt}{m} = \ln \left(1 + \frac{kv_0}{mg} \right),$$

and conclude that

$$t = \frac{m}{k} \ln \left(1 + \frac{kv_0}{mg} \right).$$

4. A bowling ball weighing $W = 16$ pounds is suspended from the ceiling through a spring. At equilibrium, it stretches the spring by $\Delta L = 6$ inches. The ball is then pulled down by an additional 1 foot and released with a downward velocity of 8 ft/sec.
- Write down the initial value problem that describes the mass's motion.
 - Solve the initial value problem and determine the amplitude of the oscillations.
 - How long does it take for the ball to arrive at the lowest point for the first time?

The acceleration due to gravity is $g = 32 \text{ ft/sec}^2$.

Solution: [Like exercise 8 of Section 3.9]

(a) Set up a vertical axis, pointing downward, along which the motion takes place. Take the origin to be the mass's equilibrium position, and let $u(t)$ be the mass's position along that axis at any time t .

We have seen that the differential equation of motion is $m\ddot{u} + ku = 0$, where m is the moving mass, and k is the spring constant. However, the spring constant k is defined through $W = k\Delta L$, and the mass m is related to the weight W through $W = mg$. Therefore the differential equation may be written as $\frac{W}{g}\ddot{u} + \frac{W}{\Delta L}u = 0$ which simplifies to $\ddot{u} + \frac{g}{\Delta L}u = 0$. We see that W drops out, so the specification of $W = 16$ pounds is immaterial.

As to the rest of the data, we have $g = 32 \text{ ft/sec}^2$ and $\Delta L = 6 \text{ in} = 1/2 \text{ ft}$, and therefore $\frac{g}{\Delta L} = 64$. Thus we arrive at the initial value problem

$$\ddot{u} + 64u = 0, \quad u(0) = 1, \quad \dot{u}(0) = 8.$$

(b) The characteristic equation of the DE is $r^2 + 64 = 0$ which has roots $r = \pm 8i$. Therefore the general solution of the DE is

$$u(t) = c_1 \cos 8t + c_2 \sin 8t,$$

and in particular

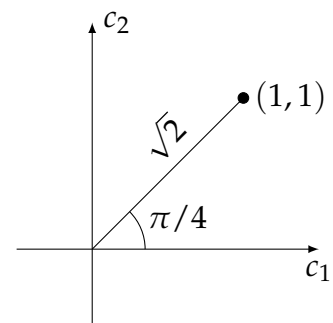
$$\dot{u}(t) = -8c_1 \sin 8t + 8c_2 \cos 8t.$$

Plugging in the initial data, we obtain $1 = c_1$ and $8 = 8c_2$, whence $c_1 = c_2 = 1$. We conclude that the solution of the initial value problem is

$$u(t) = \cos 8t + \sin 8t.$$

In the adjacent c_1 - c_2 diagram, we see that the amplitude of motion is $\sqrt{2}$ and the phase angle is $\phi = \pi/4$. Therefore the solution may be expressed as

$$u(t) = \sqrt{2} \cos\left(8t - \frac{\pi}{4}\right).$$



(c) The mass reaches the lowest position when $u(t)$ takes on the largest value. That occurs when cosine evaluates to 1 (note that downward is positive!) that is, $\cos\left(8t - \frac{\pi}{4}\right) = 1$, and therefore $8t - \frac{\pi}{4} = 0$, whence

$$t = \frac{\pi}{32}.$$