Solution to Exercise 13.7. This is very similar to the problem solved in Sections 13.3 and 13.4. The only difference is the boundary condition f, which was given in (13.32), is now changed to

$$f(\theta) = \begin{cases} 1 & \text{if } 0 \le \theta < \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

We begin with calculating f's Fourier coefficients according to (13.30):

$$\phi_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{\pi/2} 1\,d\theta = \frac{1}{4},$$
$$\phi_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta = \frac{1}{\pi}\int_0^{\pi/2} \cos n\theta\,d\theta = \frac{1}{n\pi}\sin\frac{n\pi}{2},$$
$$\psi_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta = \frac{1}{\pi}\int_0^{\pi/2} \sin n\theta\,d\theta = \frac{1}{n\pi}\Big[1-\cos\frac{n\pi}{2}\Big],$$

and therefore

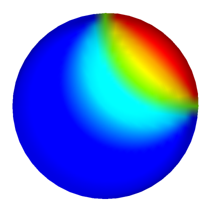
$$f(\theta) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin\frac{n\pi}{2}\cos n\theta + \sum_{n=1}^{\infty}\frac{1}{n\pi}\Big[1-\cos\frac{n\pi}{2}\Big]\sin n\theta.$$

The solution candidate $u(r,\theta)$ is the same as that in (13.34):

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} A_n r^n \cos n\theta + \sum_{n=1}^{\infty} B_n r^n \sin n\theta.$$

We evaluate this at $r = a$, equate the result to f's coefficients, and arrive at

$$A_0 = \frac{1}{4}, \quad A_n a^n = \frac{1}{n\pi}\sin\frac{n\pi}{2}, \quad B_n a^n = \frac{1}{n\pi}\Big[1-\cos\frac{n\pi}{2}\Big].$$



We conclude that

$$u(r,\theta) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{r}{a}\Big)^n \sin\frac{n\pi}{2}\cos n\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\Big(\frac{r}{a}\Big)^n\Big[1-\cos\frac{n\pi}{2}\Big]\sin n\theta.$$

Solution to Exercise 13.8. This is very similar to the problem solved in Sections 13.3 and 13.4. The solution in expressed in the series (13.29a), that is

$$u(r,\theta) = a_0(r) + \sum_{n=1}^{\infty} a_n(r)\cos n\theta + \sum_{n=1}^{\infty} b_n(r)\sin n\theta,$$

where the coefficients are the solutions of the ODEs (13.31). The outer boundary condition is the same as f in (13.32) whose Fourier coefficients are calculated in (13.33). We thus have

$$f(\theta) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}\sin n\theta. \tag{14.64}$$

As before, the solutions of the ODEs (13.31) are

$$a_0(r) = A_0 + \tilde{A}_0 \ln r, \quad a_n(r) = A_n r^n + \tilde{A}_n r^{-n}, \quad b_n(r) = B_n r^n + \tilde{B}_n r^{-n},$$

but unlike in the previous case, we don't discard $\tilde{A}$ and $\tilde{B}$ since the origin is not a part of the domain Ω. We thus arrive at the solution candidate

$$u(r,\theta) = A_0 + \tilde{A}_0 \ln r + \sum_{n=1}^{\infty} (A_n r^n + \tilde{A}_n r^{-n})\cos n\theta + \sum_{n=1}^{\infty} (B_n r^n + \tilde{B}_n r^{-n})\sin n\theta. \tag{14.65}$$

We evaluate this at $r = a$ and equate the result to zero:

$$A_0 + \tilde{A}_0 \ln a + \sum_{n=1}^{\infty} (A_n a^n + \tilde{A}_n a^{-n})\cos n\theta + \sum_{n=1}^{\infty} (B_n a^n + \tilde{B}_n a^{-n})\sin n\theta = 0,$$

and conclude that

$$A_0 + \tilde{A}_0 \ln a = 0, \tag{14.66a}$$
$$A_n a^n + \tilde{A}_n a^{-n} = 0, \tag{14.66b}$$
$$B_n a^n + \tilde{B}_n a^{-n}. \tag{14.66c}$$

We also evaluate (14.65) at $r = b$ and equate the result to (14.64):

$$A_0 + \tilde{A}_0 \ln b + \sum_{n=1}^{\infty} (A_n b^n + \tilde{A}_n b^{-n})\cos n\theta + \sum_{n=1}^{\infty} (B_n b^n + \tilde{B}_n b^{-n})\sin n\theta = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}\sin n\theta$$

and conclude that

$$A_0 + \tilde{A}_0 \ln b = \frac{1}{2}, \tag{14.66d}$$

$$A_n b^n + \tilde{A}_n b^{-n} = 0, \tag{14.66e}$$

$$B_n b^n + \tilde{B}_n b^{-n} = Q_n, \tag{14.66f}$$

where we have set $Q_n = \frac{1-(-1)^n}{n\pi}$. We solve the six equations (14.66) for the six unknowns A_0, $\tilde{A}_0$, A_n, $\tilde{A}_n$, B_n, $\tilde{B}_n$, and obtain

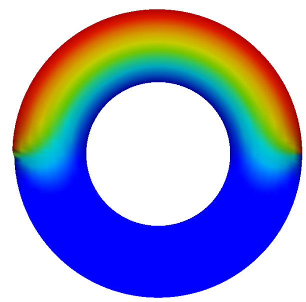
$$A_0 = -\frac{\ln a}{2\ln(b/a)}, \quad \tilde{A}_0 = \frac{1}{2\ln(b/a)}, \quad A_n = 0, \quad \tilde{A}_n = 0,$$
$$B_n = \frac{b^n}{b^{2n} - a^{2n}} Q_n, \quad \tilde{B}_n = -\frac{a^{2n} b^n}{b^{2n} - a^{2n}} Q_n,$$

We conclude that

$$a_0(r) = \frac{\ln(r/a)}{2\ln(b/a)}, \quad a_n(r) = 0,$$
$$b_n(r) = \left[\frac{(r/b)^n}{1-(a/b)^{2n}} - \frac{(b/r)^n}{(b/a)^{2n} - 1}\right] Q_n,$$

and therefore

$$u(r,\theta) = \frac{\ln(r/a)}{2\ln(b/a)} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n}\left[\frac{(r/b)^n}{1-(a/b)^{2n}} - \frac{(b/r)^n}{(b/a)^{2n}-1}\right] \sin n\theta.$$



The adjacent diagram shows the temperature distribution in the annulus with $a = 1$, $b = 2$. Red is hot, blue is cold.