Solution to Exercise 13.2. The boundary value problem is

$$\begin{aligned} &u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 && \text{in } \Omega, \\ &u(r,0) = 0 && 0 < r < a, \\ &u(r,\pi) = 0 && 0 < r < a, \\ &u(a,\theta) = f(\theta) && 0 < \theta < \pi. \end{aligned}$$

We look for solutions of the form $u(r,\theta) = R(r)\Psi(\theta)$ and arrive at the familiar eigenvalue problem

$$\Psi''(\theta) + \lambda\Psi(\theta) = 0, \quad \Psi(0) = 0, \quad \Psi(\pi) = 0$$

whose solution is given by

$$\gamma_n = n, \quad \lambda_n = \gamma_n^2 = n^2, \quad \Psi_n(\theta) = \sin\gamma_n\theta = \sin n\theta.$$

Then for $R(r)$ we have the Euler equation

$$r^2R_n''(r) + rR_n'(r) - n^2R_n(r) = 0,$$

whose general solution is

$$R_n(r) = A_n r^n + \frac{B_n}{r^n}.$$

We set $B_n = 0$ to avoid blowup at the origin. Thus we have constructed a series of functions $u_n(r,\theta) = A_n r^n \sin n\theta$ that satisfy the PDE and the boundary conditions on the domain's straight edges. To meet the boundary condition on the curved edge, we form the solution candidate

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

Enforcing the boundary condition leads to

$$f(\theta) = \sum_{n=1}^{\infty} A_n a^n \sin n\theta,$$

which is the usual Fourier sine series for f, and therefore

$$a^n A_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\sin n\theta\,d\theta,$$

whence we obtain

$$A_n = \frac{2}{a^n\pi}\int_0^{\pi} f(\theta)\sin n\theta\,d\theta,$$

and arrive at the solution

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta, \quad \text{where} \quad A_n = \frac{2}{a^n\pi}\int_0^{\pi} f(\theta)\sin n\theta\,d\theta.$$

Solution to Exercise 13.3. The boundary value problem is

$$
\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 && \text{in } \Omega,\\
&u(r,0) = 0 && a < r < b,\\
&u(r,\pi) = 0 && a < r < a,\\
&u(a,\theta) = 0 && 0 < \theta < \pi\\
&u(b,\theta) = \sin\theta && 0 < \theta < \pi.
\end{aligned}
$$

We look for solutions of the form $u(r,\theta) = R(r)\Psi(\theta)$ and arrive at the familiar eigenvalue problem

$$\Psi''(\theta) + \lambda\Psi(\theta) = 0, \quad \Psi(0) = 0, \quad \Psi(\pi) = 0$$

whose solution is given by

$$\gamma_n = n, \quad \lambda_n = \gamma_n^2 = n^2, \quad \Psi_n(\theta) = \sin\gamma_n\theta = \sin n\theta.$$

Then for $R(r)$ we have the Euler equation

$$r^2 R_n''(r) + rR_n'(r) - n^2 R_n(r) = 0,$$

whose general solution is

$$R_n(r) = A_n r^n + \frac{B_n}{r^n}.$$

Thus, we form the solution candidate

$$u(r,\theta) = \sum_{n=1}^{\infty}\Big(A_n r^n + \frac{B_n}{r^n}\Big)\sin n\theta.$$

Applying the boundary conditions specified on the curved edges, we get

$$
\begin{aligned}
0 &= \sum_{n=1}^{\infty}\Big(A_n a^n + \frac{B_n}{a^n}\Big)\sin n\theta,\\
\sin\theta &= \sum_{n=1}^{\infty}\Big(A_n b^n + \frac{B_n}{b^n}\Big)\sin n\theta.
\end{aligned}
$$

These are the Fourier sine series expansions of the functions 0 and $\sin\theta$. We conclude that

$$A_n a^n + \frac{B_n}{a^n} = 0 \quad \text{for all } n, \quad \text{and} \quad A_n b^n + \frac{B_n}{b^n} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Specifically, when $n = 1$ we have

$$A_1 a + \frac{B_1}{a} = 0, \quad A_1 b + \frac{B_1}{b} = 1.$$

We solve this linear system of two equations for A_1 and B_1 and obtain

$$A_1 = \frac{b}{b^2 - a^2}, \quad B_1 = -\frac{a^2 b}{b^2 - a^2}.$$

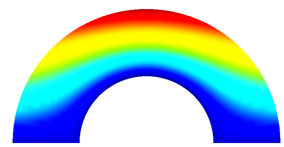
When $n > 1$ we have

$$A_n a + \frac{B_n}{a} = 0, \quad A_n b + \frac{B_n}{b} = 0,$$

which implies $A_n = B_n = 0$. We conclude that the infinite series of the solution actually consists of a single term:

$$u(r, \theta) = \frac{br}{b^2 - a^2} - \frac{a^2 b}{r(b^2 - a^2)} \sin\theta,$$

which may be rearranged into the more presentable form



$$u(r, \theta) = \frac{ab}{b^2 - a^2}\left(\frac{r}{a} - \frac{a}{r}\right) \sin\theta.$$

Solution to Exercise 13.4. We need to solve the BVP

$$\begin{aligned}
&u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + 1 = 0 && \text{in } \Omega,\\
&u(r,0) = 0 && 0 \le r \le a,\\
&u(r,\pi) = 0 && 0 \le r \le a,\\
&u(a,\theta) = 0 && 0 \le \theta \le \pi.
\end{aligned}$$

We look for solutions of the form $u(r,\theta) = R(r)\Psi(\theta)$ *of the homogeneous PDE* and arrive at the familiar eigenvalue problem

$$\Psi''(\theta) + \lambda\Psi(\theta) = 0, \quad \Psi(0) = 0, \quad \Psi(\pi) = 0$$

whose solution is given by

$$\gamma_n = n, \quad \lambda_n = \gamma_n^2 = n^2, \quad \Psi_n(\theta) = \sin\gamma_n\theta = \sin n\theta.$$

Then we expand the unknown solution $u(r,\theta)$ and the give heat source $q(r,\theta) \equiv 1$ into series of the eigenfunction $\Psi_n(\theta)$.

$$\begin{aligned}
u(r,\theta) &= \sum_{n=1}^{\infty} R_n(r)\Psi_n(\theta),\\
1 &= \sum_{n=1}^{\infty} Q_n\Psi_n(\theta).
\end{aligned}$$

The coefficients Q_n are readily found through the usual formula

$$\begin{aligned}
Q_n = \frac{2}{\pi}\int_0^\pi \Psi_n(\theta)\,d\theta = \frac{2}{\pi}\int_0^\pi \sin n\theta\,d\theta = -\frac{2}{n\pi}\cos n\theta\Big|_0^\pi\\
= \frac{2}{n\pi}\Big[1 - \cos n\pi\Big] = \frac{2}{n\pi}\Big[1 - (-1)^n\Big].
\end{aligned}$$

We now substitute the solution candidate into the PDE:

$$\sum_{n=1}^{\infty}\left[R_n''(r)\Psi_n(\theta) + \frac{1}{r}R_n'(r)\Psi_n(\theta) + \frac{1}{r^2}R_n(r)\Psi_n''(\theta)\right] + \sum_{n=1}^{\infty} Q_n\Psi_n(\theta) = 0.$$

Considering that $\Psi_n''(\theta) = -\lambda_n\Psi_n(\theta) = -n^2\Psi_n(\theta)$, we simplify and rearrange the result into

$$\sum_{n=1}^{\infty}\left[R_n''(r) + \frac{1}{r}R_n'(r) - \frac{n^2}{r^2}R_n(r) + Q_n\right]\Psi_n(\theta) = 0,$$

and we conclude that

$$R_n''(r) + \frac{1}{r}R_n'(r) - \frac{n^2}{r^2}R_n(r) + Q_n = 0,$$

which is better expressed as

$$r^2R_n''(r) + rR_n'(r) - n^2R_n(r) = -r^2Q_n, \quad n = 1,2,\ldots.$$

These are nonhomogeneous versions of Euler's equation. The general solution of the corresponding *homogeneous equation* for each n is given in (13.11). To avoid the singularity at the origin, we take $B_n = 0$. All there remains is to find a particular solution of the nonhomogeneous equation and add to the solution of the homogeneous equation.

Considering the special structure of the equation, it makes sense to look for a particular solution of the form Cr^2. Plugging that guess into the ODE we obtain $4Cr^2 - n^2Cr^2 = -Q_n r^2$, whence $C = \frac{Q_n}{n^2-4}$. We conclude that

$$R_n(r) = A_n r^n + \frac{Q_n}{n^2-4} r^2.$$

That is good for all positive integers n except for $n = 2$. So we take a closer look at the $n = 2$ case, where we have

$$r^2 R_2''(r) + rR_2'(r) - 4R_n(r) = -Q_2 r^2.$$

Luckily, $Q_2 = 0$, so that equation reduces to

$$r^2 R_2''(r) + rR_2'(r) - 4R_n(r) = 0,$$

whose general solution is $R_2(r) = A_2 r^2 + B_2/r^2$. We set $B_2 = 0$ to avoid the singularity, and therefore

$$R_2(r) = A_2 r^2.$$

We thus arrive at the solution candidate

$$u(r,\theta) = \Big[A_1 r + \frac{Q_1}{1^2-4} r^2\Big] \sin\theta + A_2 r^2 \sin 2\theta + \sum_{n=3}^{\infty} \Big[A_n r^n + \frac{Q_n}{n^2-4} r^2\Big] \sin n\theta. \quad (14.63)$$

This satisfies the PDE and the boundary conditions on the lamina's straight edges. The boundary condition on the curved edge implies that

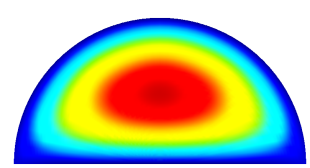
$$\Big[A_1 a + \frac{Q_1}{1^2-4} a^2\Big] \sin\theta + A_2 a^2 \sin 2\theta + \sum_{n=3}^{\infty} \Big[A_n a^n + \frac{Q_n}{n^2-4} a^2\Big] \sin n\theta = 0.$$

It follows that

$$A_1 = \frac{1}{3} Q_1 a \quad , A_2 = 0, \quad A_n = -\frac{Q_n a^2}{a^n(n^2-4)}, \quad n = 3, 4, \ldots.$$

Substituting this into (14.63) and simplifying, we arrive at

$$u(r,\theta) = \frac{4a^2}{3\pi} \frac{r}{a}\Big(1 - \frac{r}{a}\Big) \sin\theta + \frac{2a^2}{\pi} \sum_{n=3}^{\infty} \frac{1-(-1)^n}{n(n^2-4)} \Big[\Big(\frac{r}{a}\Big)^2 - \Big(\frac{r}{a}\Big)^n\Big] \sin n\theta.$$



The adjacent diagram shows the temperature distribution in the lamina. Red is hot, blue is cold.