Chapter 11

Solution to Exercise 11.1. We have seen that after the change of variable $u = v + \eta$ where η is as in (11.11), the IBVP (11.10) changes to (11.12) which is quite similar to the problem solved in 10.5. The heat source term was $\sigma \sin \omega t$ in that exercise, while in the current case it is $-\omega\sigma\left(1 - \frac{x}{\ell}\right)\cos \omega t$. The eigenvalues and eigenfunctions remain as before:

$$\gamma_n = \frac{n\pi}{\ell}, \quad \lambda_n = \gamma_n^2 = \left(\frac{n\pi}{\ell}\right)^2, \quad X_n(x) = \sin \gamma_n x = \sin \frac{n\pi}{\ell}, \quad n = 1, 2 \ldots.$$

We expand the solution u, the heat source term, and the initial condition f into series of eigenfunction as in (10.19). Since the initial condition is zero in this case, the coefficients F_n are all zero. We calculate the coefficients $Q_n(t)$:

$$\begin{aligned} Q_n(t) &= \frac{2}{\ell} \int_0^\ell -\omega\sigma\left(1 - \frac{x}{\ell}\right)\cos \omega t \sin \gamma_n x \, dx \\ &= -\frac{2\sigma\omega \cos \omega t}{\ell} \int_0^\ell \left(1 - \frac{x}{\ell}\right)\sin \gamma_n x \, dx. \end{aligned}$$

We perform the integration according to Kronecker's algorithm

$$\int \left(1 - \frac{x}{\ell}\right)\sin \gamma_n x \, dx = \left(1 - \frac{x}{\ell}\right)\left[-\frac{1}{\gamma_n}\cos \gamma_n x\right] - \left(-\frac{1}{\ell}\right)\left[-\frac{1}{\gamma_n^2}\sin \gamma_n x\right],$$

and evaluate the result at ℓ and 0 and subtract, to arrive at

$$\int_0^\ell \left(1 - \frac{x}{\ell}\right)\sin \gamma_n x \, dx = \frac{1}{\gamma_n} = \frac{\ell}{n\pi},$$

and conclude that

$$Q_n(t) = -\frac{2\sigma\omega \cos \omega t}{n\pi} = -J_n \cos \omega t,$$

where we have let $J_n = \dfrac{2\sigma\omega}{n\pi}$ for convenience. Then, the initial value problem (10.21) takes the form

$$U_n'(t) + k\lambda_n U_n(t) = -J_n \cos \omega t, \quad U_n(0) = 0, \quad n = 1, 2, \ldots,$$

which may be solved in a number of ways. Here we do it through Laplace transform[18], whence

[18] Alternatively, duplicate the method illustrated in the solution of Example 10.5 on page 143.

$$\begin{aligned} \mathscr{L}\{U_n(t)\} &= \frac{-J_n s}{(s + k\lambda_n)(s^2 + \omega^2)} \\ &= \frac{J_n}{\omega^2 + k^2\lambda_n^2}\left[\frac{k\lambda_n}{s + k\lambda_n} - \frac{k\lambda_n s}{s^2 + \omega^2} - \frac{\omega^2}{s^2 + \omega^2}\right]. \quad \text{(partial fractions)} \end{aligned}$$

We conclude that

$$U_n(t) = \frac{J_n}{\omega^2 + k^2\lambda_n^2}\left[k\lambda_n e^{-k\lambda_n t} - k\lambda_n \cos \omega t - \omega \sin \omega t\right],$$

and therefore

$$v(x,t) = \sum_{n=1}^{\infty} \frac{J_n}{\omega^2 + k^2\lambda_n^2} \left[-k\lambda_n e^{-k\lambda_n t} + k\lambda_n \cos\omega t + \omega \sin\omega t\right] \sin\gamma_n x$$

and

$$u(x,t) = \sigma\left(1 - \frac{x}{\ell}\right)\sin\omega t + \sum_{n=1}^{\infty} \frac{J_n}{\omega^2 + k^2\lambda_n^2} \left[k\lambda_n e^{-k\lambda_n t} - k\lambda_n \cos\omega t - \omega \sin\omega t\right] \sin\gamma_n x,$$

which is equivalent to (11.13) since $J_n = \frac{2\sigma\omega}{n\pi}$.

The exponential term in the square brackets is the transient; it dies out as t grows. The remaining terms correspond to steady-state oscillations at angular frequency ω.

Solution to Exercise 11.4. The first step toward obtaining the solution is to eliminate the nonhomogeneous boundary condition through a change of variable. The boundary conditions here correspond to the general formulation (11.15b) and (11.15c) with

$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad \alpha(t) = 0, \quad \beta_1 = 0, \quad \beta_2 = 1, \quad \beta(t) = \sigma \sin \omega t.$$

Substituting these into (11.17) we obtain:

$$c_0(t) = 0, \quad c_1(t) = \sigma \sin \omega t. \tag{14.46}$$

Appealing to a "black-box" formula such as (11.17), however, is overkill; that can be error-prone, and is not recommended. It is in fact quite straightforward to do the calculation from scratch for the specific problem at hand rather than to appeal to a general formula such as (11.17).

To illustrate that, introduce $v(x,t)$ through $u(x,t) = v(x,t) + \eta(x,t)$, where η is any function that satisfies the IBVP's boundary conditions. As u and η satisfy identical boundary conditions, v will satisfy homogeneous boundary conditions, and we may apply the previous chapter's techniques to calculate v.

There is quite a bit of flexibility in selecting η. Following the idea offered in Section 11.2, we take

$$\eta(x,t) = c_0(t) + c_1(t)x, \tag{14.47}$$

where c_0 and c_1 are to be determined by requiring η to satisfy the IBVP's boundary conditions, that is

$$\eta(0,t) = 0, \quad \eta_x(\ell,t) = \sigma \sin \omega t. \tag{14.48}$$

Substituting the form (14.47) into (14.48), we get

$$c_0(t) = 0, \quad c_1(t) = \sigma \sin \omega t.$$

This is the same as (14.46) but we have obtained it without an appeal to an obscure formula.

In any case, we conclude that $\eta(x,t) = \sigma x \sin \omega t$, and therefore

$$u(x,t) = v(x,t) + \sigma x \sin \omega t.$$

We substitute this expression into the original IBVP and we arrive at the following IBVP for v:

$$v_t = k v_{xx} - \sigma \omega x \cos \omega t \qquad 0 < x < \ell, \quad t > 0, \tag{14.49a}$$
$$v(0,t) = 0 \qquad t > 0, \tag{14.49b}$$
$$v_x(\ell,t) = 0 \qquad t > 0, \tag{14.49c}$$
$$v(x,0) = 0 \qquad 0 < x < \ell. \tag{14.49d}$$

Following the usual procedure, we look for separable solutions $v(x,t) = X(x)T(t)$ of the *corresponding homogeneous problem*

$$\begin{aligned} &v_t = kv_{xx} && 0 < x < \ell, \quad t > 0, \\ &v(0,t) = 0 && t > 0, \\ &v_x(\ell,t) = 0 && t > 0, \end{aligned}$$

which leads to the eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X'(\ell) = 0,$$

This is identical to the problem analyzed in Example (10.1) on page 135, where we saw that the eigenvalues λ_n and eigenfunctions X_n are given through

$$\gamma_n = \frac{(2n-1)\pi}{2\ell}, \quad \lambda_n = \gamma_n^2, \quad X_n(x) = \sin\gamma_n x, \quad \|X_n\|^2 = \frac{\ell}{2}, \quad n = 1, 2, \ldots.$$

Returning to the IBVP (14.49), we expand $v(x,t)$ and the heat source term $-\sigma\omega x\cos\omega t$ into series of eigenfunctions:

$$v(x,t) = \sum_{n=1}^{\infty} V_n(t)X_n(x), \tag{14.50a}$$

$$-\sigma\omega x\cos\omega t = \sum_{n=1}^{\infty} Q_n(t)X_n(x). \tag{14.50b}$$

From the general formula of the coefficients of the sine series applied to (14.50b) we get

$$Q_n(t) = \frac{2}{\ell}\int_0^{\ell} -\sigma x\cos\omega t\, X_n(x)\,dx = -\frac{2\sigma\omega\cos\omega t}{\ell}\int_0^{\ell} xX_n(x)\,dx.$$

We evaluate the integral of $xX_n(x)$ through Kronecker's algorithm:

$$\begin{aligned} \int xX_n(x)\,dx &= \int x\sin\gamma_n x\,dx \\ &= (x)\Big(-\frac{1}{\gamma_n}\cos\gamma_n x\Big) - (1)\Big(-\frac{1}{\gamma_n^2}\sin\gamma_n x\Big) \\ &= \frac{1}{\gamma_n^2}(\sin\gamma_n x - \gamma_n x\cos\gamma_n x), \end{aligned}$$

and therefore

$$\begin{aligned} \int_0^{\ell} xX_n(x)\,dx &= \frac{1}{\gamma_n^2}(\sin\gamma_n x - \gamma_n x\cos\gamma_n x)\Big|_0^{\ell} \\ &= \frac{1}{\gamma_n^2}(\sin\gamma_n \ell - \gamma_n\cos\gamma_n \ell). \end{aligned}$$

Since $\gamma_n = \frac{(2n-1)\pi}{2\ell}$, we have $\gamma_n\ell = (2n-1)\frac{\pi}{2}$, that is, $\gamma_n\ell$ is an odd multiple of $\pi/2$, we have

$$\sin\gamma_n\ell = -(-1)^n, \quad \cos\gamma_n\ell = 0.$$

We conclude that

$$\int_0^\ell xX_n(x)\,dx = -\frac{1}{\gamma_n^2}(-1)^n,$$

and therefore

$$Q_n(t) = \frac{2\sigma\omega\cos\omega t}{\ell\gamma_n^2}(-1)^n = \frac{2\sigma\omega}{\ell\gamma_n^2}(-1)^n\cos\omega t.$$

Letting

$$J_n = \frac{2\sigma\omega}{\ell\gamma_n^2}(-1)^n,$$

we write $Q_n(t)$ in the compact form

$$Q_n(t) = J_n\cos\omega t.$$

Having obtained $Q_n(t)$, we substitute the series (14.50) into the PDE (14.49a):

$$\sum_{n=1}^{\infty} V_n'(t)X_n(x) = k\sum_{n=1}^{\infty} V_n(t)X_n''(x) + \sum_{n=1}^{\infty} Q_n(t)X_n(x).$$

We eliminate $X_n''(x)$ in favor of $X_n(x)$ courtesy of $X''(x) + \lambda_n X(x) = 0$, and combine the two sums on the right-hand side, and arrive at

$$\sum_{n=1}^{\infty}\Big[V_n'(t) + k\lambda_n V_n(t)\Big]\,X_n(x) = \sum_{n=1}^{\infty} Q_n(t)X_n(x).$$

We conclude that $V_n'(t) + k\lambda_n V_n(t) = Q_n(t)$, that is

$$V_n'(t) + k\lambda_n V_n(t) = J_n\cos\omega t, \quad n = 1, 2, \ldots,$$

which is a linear first order ODE. The ODE's initial condition is obtained by applying the IBVP's initial condition (14.49d) to the series (14.50a) whereby we get

$$0 = \sum_{n=1}^{\infty} V_n(0)X_n(x),$$

which implies that $V_n(0) = 0$ for all n. Thus, $V_n(t)$ are the solutions of the initial value problem

$$V_n'(t) + k\lambda_n V_n(t) = J_n\cos\omega t, \quad V_n(0) = 0, \quad n = 1, 2, \ldots. \tag{14.51}$$

This may be solved through (i) Laplace transform, or (ii) splitting the solution into the sum of the homogeneous and particular solutions, or (iii) integrating factors. Here we provide solutions via methods (i) and (ii).

Solving (14.51) *through the Laplace transform.*

We apply the Laplace transform to the ODE in (14.51)

$$s\mathscr{L}\{V_n(t)\} - V_n(0) + k\lambda_n \mathscr{L}\{V_n(t)\} = \frac{J_n s}{s^2+\omega^2}.$$

Considering that $V_n(0) = 0$, this leads to[19]

$$\begin{aligned}\mathscr{L}\{V_n(t)\} &= \frac{J_n s}{(s+k\lambda_n)(s^2+\omega^2} \\ &= \frac{J_n}{\omega^2+k^2\lambda_n^2}\left[[-\frac{k\lambda_n}{s+k\lambda_n} + \frac{k\lambda_n s+\omega^2}{\omega^2+s^2}\right], \qquad \text{(partial fractions)}\end{aligned}$$

[19] In pursuing option (i) of the three options noted above, the bulk of the effort is directed toward partial fraction decomposition.

and therefore, by taking the inverse Laplace transform, we arrive at

$$V_n(t) = \frac{J_n}{\omega^2+k^2\lambda_n^2}\left[-k\lambda_n e^{-k\lambda_n t} + k\lambda_n \cos\omega t + \omega\sin\omega t\right].$$

Solving (14.51) *through basic ODEs.*

The homogeneous equation corresponding to (14.51) has the general solution

$$V_n^{(h)}(t) = C_n e^{-k\lambda_n t},$$

where C_n is an arbitrary constant. As to a particular solution, we look for an expression of the form[20]

$$V_n^{(p)}(t) = A_n\cos\omega t + B_n\sin\omega t,$$

[20] In pursuing option (ii) of the three options noted above, the bulk of the effort is directed toward finding a particular solution. If you were to pursue option (iii), the bulk of your effort would be directed toward integrating by parts. One way or another, you pay a price. There is no free lunch!

and determine A_n and B_n by plugging that expression into the ODE. We get

$$\big(-A_n\omega\sin\omega t + B_n\omega\cos\omega t\big) + k\lambda_n\big(A_n\cos\omega t + B_n\sin\omega t\big) = J_n\cos\omega t,$$

whence

$$\begin{aligned} k\lambda_n A_n + \omega B_n &= J_n, \\ -\omega A_n + k\lambda_n B_n &= 0.\end{aligned}$$

We solve for A_n and B_n and arrive at

$$A_n = \frac{k\lambda_n}{\omega^2+k^2\lambda_n^2}J_n, \quad B_n = \frac{\omega}{\omega^2+k^2\lambda_n^2}J_n.$$

We thus arrive at the general solution of the ODE

$$V_n(t) = C_n e^{-k\lambda_n t} + \frac{k\lambda_n}{\omega^2+k^2\lambda_n^2}J_n\cos\omega t + \frac{\omega}{\omega^2+k^2\lambda_n^2}J_n\sin\omega t.$$

We apply the initial condition $V_n(0) = 0$ and solve for C_n:

$$C_n = -\frac{k\lambda_n}{\omega^2+k^2\lambda_n^2}J_n.$$

Plugging this back into the expression for $V_n(t)$, we conclude that

$$V_n(t) = \frac{k\lambda_n J_n}{\omega^2 + k^2\lambda_n^2}\left[-e^{-k\lambda_n t} + \cos\omega t + \frac{\omega}{k\lambda_n}\sin\omega t\right],$$

which agrees with the solution obtained earlier through the Laplace transform.

Now we resume the solution of the original IBVP. In view of (14.50a), we obtain

$$v(x,t) = \sum_{n=1}^{\infty} \frac{k\lambda_n J_n}{\omega^2 + k^2\lambda_n^2}\left[-e^{-k\lambda_n t} + \cos\omega t + \frac{\omega}{k\lambda_n}\sin\omega t\right] X_n(x),$$

and finally, since $u(x,t) = v(x,t) + \eta(x,t)$, we arrive at

$$\begin{aligned} u(x,t) &= \sigma x \sin\omega t \\ &\quad + \sum_{n=1}^{\infty} \frac{k\lambda_n J_n}{\omega^2 + k^2\lambda_n^2}\left[-e^{-k\lambda_n t} + \cos\omega t + \frac{\omega}{k\lambda_n}\sin\omega t\right] X_n(x), \end{aligned}$$

where

$$J_n = \frac{2\sigma\omega}{\ell\gamma_n^2}(-1)^n, \quad X_n(x) = \sin\gamma_n x, \quad \lambda_n = \gamma_n^2, \quad \gamma_n = \frac{(2n-1)\pi}{2\ell}.$$

Chapter 12

Solution to Exercise 12.1. We have $\eta = \frac{2y}{b}x(a-x)$. We evaluate η along Ω's four edges:

$$\begin{aligned}
&\text{on the left edge:} && \eta(0,y) = 0,\\
&\text{on the bottom edge:} && \eta(x,0) = 0,\\
&\text{on the right edge:} && \eta(a,y) = 0,\\
&\text{on the top edge:} && \eta(a,b) = 2x(a-x),
\end{aligned}$$

and see that it satisfies the same boundary conditions as the membrane. Therefore $v(x,y) = u(x,y) - \eta(x,y)$ is zero all around Ω's boundary.

The membrane's displacement, $u(x,y)$ satisfies the equation (12.3a) with $q = 0$, that is, $u_{xx} + u_{yy} = 0$. Changing to the v variable we get

$$(v+\eta)_{xx} + (v+\eta)_{yy} = 0.$$

But $\eta_{xx} = -\frac{4}{b}y$ and $\eta_{yy} = 0$. Therefore

$$v_{xx} + v_{yy} - \frac{4}{b}y = 0.$$

We conclude that v is the solution of the BVP (12.3) with $q(x,y) = -\frac{4}{b}y$. The solution of that BVP for arbitrary forcing function q was obtained in (12.9), that is:

$$v(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn} X_m(x) Y_n(y),$$

and where U_{mn} and Q_{mn} are given in (12.11) and (12.8). For the current case's specific forcing function we have

$$\begin{aligned}
Q_{mn} &= \frac{4}{ab}\int_0^b \Big[\int_0^a \Big(-\frac{4}{b}y\Big) X_m(x)\,dx\Big] Y_n(y)\,dy\\
&= -\frac{16}{ab^2}\Big[\int_0^a X_m(x)\,dx\Big]\Big[\int_0^b yY_n(y)\,dy\Big]\\
&= -\frac{16}{ab^2}\Big[\int_0^a \sin\frac{m\pi x}{a}\,dx\Big]\Big[\int_0^b y\sin\frac{n\pi y}{b}\,dy\Big].
\end{aligned}$$

We calculate

$$\begin{aligned}
\int_0^a \sin\frac{m\pi x}{a}\,dx &= -\frac{a}{m\pi}\cos\frac{m\pi x}{a}\Big|_0^a\\
&= -\frac{a}{m\pi}\Big[1-\cos m\pi\Big] = \frac{a}{m\pi}\Big[1-(-1)^m\Big],
\end{aligned}$$

and[25]

[25] We apply Kronecker's method of integration by parts. See Appendix 8.10 on page 119

$$\int_0^b y \sin\frac{n\pi y}{b}\,dy = \left[(y)\left(-\frac{b}{n\pi}\cos\frac{n\pi y}{b}\right) - (1)\left(-\frac{b^2}{n^2\pi^2}\sin\frac{n\pi y}{b}\right)\right]_0^b$$
$$= -\frac{b^2}{n\pi}\cos n\pi + \frac{b^2}{n^2\pi^2}\sin n\pi = -\frac{b^2}{n\pi}(-1)^n,$$

and thus arrive at

$$Q_{mn} = \frac{16\left[1-(-1)^m\right](-1)^n}{mn\pi^2}.$$

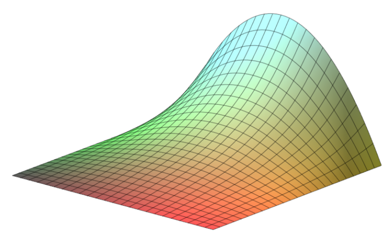
Then, from (12.11) we get

$$U_{mn} = \frac{16\left[1-(-1)^m\right](-1)^n}{mn\pi^2\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)} = \frac{16}{\pi^4}\cdot\frac{\left[1-(-1)^m\right](-1)^n}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}.$$

We conclude that

$$v(x,y) = \frac{16}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^m\right](-1)^n}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

and therefore the membrane's shape is



$$u(x,y) = \frac{2y}{b}x(a-x) + \frac{16}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^m\right](-1)^n}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

The adjacent figure shown the membrane with the choices of $a = b = 1$ and $m = n = 5$.