Solution to Exercise 10.7. We substitute the series (10.23a) into the PDE (10.22a) and obtain

$$\sum_{n=0}^{\infty} U_n''(t) X_n(x) = c^2 \sum_{n=0}^{\infty} U_n(t) X_n''(x) + q(x,t).$$

Recalling (10.18a), we replace $X_n''(x)$ by $-\lambda_n X_n(x)$ and then rearrange the equation into

$$\sum_{n=0}^{\infty} \big[U_n''(t) + c^2 \lambda_n U_n(t) \big] X_n(x) = q(x,t).$$

Comparing with (10.23b) we conclude that

$$U_n''(t) + c^2 \lambda_n U_n(t) = Q_n(t),$$

which agrees with the ODE in (10.24).

To enforce the initial conditions (10.22d) and (10.22e), we note that

$$u_t(x,t) = \sum_{n=0}^{\infty} U_n'(t) X_n(x),$$

and then we substitute $t = 0$ into this and also in (10.23a), and arrive at

$$\sum_{n=0}^{\infty} U_n(0) X_n(x) = f(x),$$
$$\sum_{n=0}^{\infty} U_n'(0) X_n(x) = g(x).$$

Comparing these with (10.23c) and (10.23d) we conclude that

$$U_n(0) = F_n, \quad U_n'(0) = G_n,$$

which confirm the initial conditions in (10.24).

Solution to Exercise 10.10. This is identical to Example 10.6 with the only difference that the coefficients F_n are zero and the coefficients G_n are

$$G_n = \frac{2}{\ell}\int_0^\ell g(x)\sin\gamma_n x\,dx.$$

The initial value problems (10.36) changes to

$$\begin{aligned}
&U_n''(t) + c^2\gamma_n^2 U_n(t) = 0,\\
&U_n(0) = 0,\\
&U_n'(0) = G_n,
\end{aligned}$$

$n = 1, 2, \ldots$, whose solution is

$$U_n(t) = \frac{1}{c\gamma_n} G_n \sin\gamma_n ct.$$

We conclude that

$$u(x,t) = \sum_{n=1}^{\infty} \frac{1}{c\gamma_n} G_n \sin\gamma_n x \sin\gamma_n ct,$$

where G_n are given above, and $\gamma_n = (2n-1)\frac{\pi}{2\ell}$.

Solution to Exercise 10.11. The equation of motion is

$$\begin{aligned}
&u_{tt} = c^2 u_{xx} + \sigma \sin \omega t && 0 < x < \ell, \quad t > 0,\\
&u(0,t) = 0 && t > 0,\\
&u(\ell,t) = 0 && t > 0,\\
&u(x,0) = 0 && 0 < x < \ell,\\
&u_t(x,0) = 0 && 0 < x < \ell.
\end{aligned}$$

This is a special case of the IBVP (10.24) with $f = 0$, $g = 0$, and $q(x,t) = \sigma \sin \omega t$. The eigenfunctions are obtained from the eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(\ell) = 0,$$

whence

$$\gamma_n = \frac{n\pi}{\ell}, \quad \lambda_n = \gamma_n^2 = \left(\frac{n\pi}{\ell}\right)^2, \quad X_n(x) = \sin \gamma_n x = \sin \frac{n\pi x}{\ell}.$$

In the expansions (10.23), the coefficients F_n and G_n are zero. We calculate the coefficients Q_n:

$$\begin{aligned}
Q_n(t) &= \frac{2}{\ell}\int_0^\ell \sigma \sin \omega t \sin \gamma_n x \, dx\\
&= \frac{2\sigma}{\gamma_n \ell}\Big(-\cos \gamma_n x\Big)\Big|_0^\ell \sin \omega t\\
&= \frac{2\sigma}{\gamma_n \ell}\Big(1 - \cos \gamma_n \ell\Big) \sin \omega t.
\end{aligned}$$

But $\gamma_n \ell = n\pi$ and therefore $\cos \gamma_n \ell = \cos n\pi = (-1)^n$. We conclude that

$$Q_n(t) = \frac{2\sigma\big(1 - (-1)^n\big)}{n\pi} \sin \omega t.$$

For convenience, we let

$$J_n = \frac{2\sigma\big(1 - (-1)^n\big)}{n\pi},$$

and express Q_n as

$$Q_n(t) = J_n \sin \omega t.$$

Then the initial value problem (10.24) takes the form

$$U_n''(t) + U^2 \gamma_n^2 c_n(t) = J_n \sin \omega t, \tag{14.45a}$$

$$U_n(0) = 0, \tag{14.45b}$$

$$U_n'(0) = 0, \tag{14.45c}$$

which may be solved in a number of ways; see suggestions in Example 10.6. Here we chose to solve through the Laplace transform since that is a little quicker. Applying the transform, and accounting for the null initial conditions, we get

$$\begin{aligned}\mathscr{L}\{U_n(t)\} &= \frac{\omega J_n}{(s^2+c^2\gamma_n^2)(s^2+\omega^2)} \\ &= \frac{J_n}{c^2\gamma_n^2-\omega^2}\left(\frac{\omega}{s^2+\omega^2}-\frac{\omega}{s^2+c^2\gamma_n^2}\right) \qquad \text{(partial fractions)},\end{aligned}$$

whence

$$U_n(t) = \frac{J_n}{c^2\gamma_n^2-\omega^2}\Big(\sin\omega t - \frac{\omega}{c\gamma_n}\sin c\gamma_n t\Big).$$

We conclude that

$$u(x,t) = \sum_{n=1}^{\infty}\frac{J_n}{c^2\gamma_n^2-\omega^2}\Big(\sin\omega t - \frac{\omega}{c\gamma_n}\sin c\gamma_n t\Big)\sin\gamma_n x.$$

Observe that the solution is *invalid* if $\omega = c\gamma_n$ for any integer n since that results in a division by zero. In those exceptional cases, the imposed frequency ω matches one of the string's natural frequencies and causes resonance. For an instructive extra exercise, solve the initial value problem (14.45) with $n = 1$ and $\omega = c\gamma_1$ to see what happens.

Solution to Exercise 10.12. Mutiplying the ODE (10.32a) by the integrating factor $e^{k\lambda_n t}$, we get

$$\left(e^{k\lambda_n t}U_n(t)\right)' = J_n e^{k\lambda_n t}\sin\omega t.$$

We integrate both sides[16] and arrive at

$$e^{k\lambda_n t}U_n(t) = \frac{J_n e^{k\lambda_n t}}{\omega^2 + k^2\lambda_n^2}\left(k\lambda_n \sin\omega t - \omega\cos\omega t\right) + C,$$

where C is the integration constant. We evaluate this at $t = 0$ and apply the initial condition $U_n(0) = 0$,

$$0 = -\frac{\omega J_n}{\omega^2 + k^2\lambda_n^2} + C,$$

and solve for C:

$$C = \frac{\omega J_n}{\omega^2 + k^2\lambda_n^2}.$$

Thus we arrive at

$$e^{k\lambda_n t}U_n(t) = \frac{J_n e^{k\lambda_n t}}{\omega^2 + k^2\lambda_n^2}\left(k\lambda_n \sin\omega t - \omega\cos\omega t\right) + \frac{\omega J_n}{\omega^2 + k^2\lambda_n^2},$$

that is,

$$U_n(t) = \frac{\omega J_n}{\omega^2 + k^2\lambda_n^2}\left[e^{-k\lambda_n t} - \cos\omega t + \frac{k\lambda_n}{\omega}\sin\omega t\right],$$

which is identical to what was obtained in Example 10.5.

[16] We apply the integration formula $\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}\left(a\sin bt - b\cos bt\right)$ which is usually presented as an *integration by parts* exercise in most calculus textbooks.