Chapter 9

Solution to Exercise 9.1. This is a special case of the IBVP (9.1) whose solution is given in (9.9). The values of γ_n and λ_n are available in (9.8). Since $k = 1$ and $\ell = 1$ in this special case, we have

$$\gamma_n = n\pi, \quad \lambda_n = n^2\pi^2.$$

We evaluate the coefficients b_n from (9.10):

$$b_n = 2\int_0^1 \sin \pi x \sin n\pi x \, dx.$$

However, according to (8.7), we have

$$\int_0^1 \sin \pi x \sin n\pi x \, dx = \int_0^1 X_1(x) X_n(x) \, dx = \begin{cases} 0 & \text{if } n \neq 1, \\ \frac{1}{2} & \text{if } n = 1. \end{cases}$$

Consequently, all b_n are zero except for b_1 which is 1, and the solution (9.9) reduces to a single term

$$u(x,t) = e^{-\pi^2 t} \sin \pi x.$$

An alternative approach: We have seen that the solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\lambda_n t} \sin \gamma_n x.$$

Evaluate that at $t = 0$, set $u(x,0) = \sin \pi x$ and $\gamma_n = n\pi$. We get

$$\sin \pi x = \sum_{n=1}^{\infty} b_n \sin n\pi x = b_1 \sin \pi x + b_2 \sin 2\pi x + b_3 \sin 3\pi x + \cdots,$$

from which it follows that $b_1 = 1$ and the rest of the b_n are zeros.

Solution to Exercise 9.3. This is a special case of the IBVP (9.1) whose solution is given in (9.9). The values of γ_n and λ_n are available in (9.8). Since $k = 1$ and $\ell = \pi$ in this special case, we have

$$\gamma_n = n, \quad \lambda_n = n^2.$$

We evaluate the coefficients b_n from (9.10) with help from the identity (8.33g) in Table 8.1 on page 120:

$$\begin{aligned}
b_n &= \frac{2}{\pi} \int_0^\pi \cos \frac{x}{2} \sin nx \, dx \\
&= \frac{1}{\pi} \int_0^\pi \left[\sin\Big(n + \frac{1}{2}\Big)x + \sin\Big(n - \frac{1}{2}\Big)x \right] dx \\
&= \frac{1}{\pi} \left[-\frac{1}{n+\frac{1}{2}} \cos\Big(n + \frac{1}{2}\Big)x - \frac{1}{n-\frac{1}{2}} \cos\Big(n - \frac{1}{2}\Big)x \right]_0^\pi \\
&= -\frac{1}{\pi} \left[\frac{2}{2n+1} \cos\Big((2n+1)\Big)\frac{x}{2} + \frac{2}{2n-1} \cos\Big((2n-1)\Big)\frac{x}{2} \right]_0^\pi .
\end{aligned}$$

Since the cosine of an odd multiple of $\pi/2$ is zero, the square bracket evaluates to zero at $x = \pi/2$, and thus arrive at

$$b_n = \frac{1}{\pi} \left[\frac{2}{2n+1} + \frac{2}{2n-1} \right] = \frac{8}{\pi} \cdot \frac{n}{4n^2 - 1}.$$

We conclude that

$$u(x,t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n^2-1} e^{-n^2 t} \sin nx$$

Solution to Exercise 9.5. This is a special case of the IBVP (9.14) whose solution is given in (9.21). The values of γ_n and λ_n are available in (9.20). Since $c = 1$ and $\ell = 1$ in this special case, we have

$$\gamma_n = n\pi, \quad \lambda_n = n^2\pi^2.$$

We are given $f(x) = 0$ and $g(x) = \sin \pi x$. Therefore we may evaluate the coefficients α_n and β_n from (9.23). Since f is zero, all α_n are zero. As to the β_n, we have

$$\beta_n = \frac{2}{n\pi} \int_0^1 \sin \pi x \sin n\pi x \, dx.$$

However, according to (8.7), we have

$$\int_0^1 \sin \pi x \sin n\pi x \, dx = \int_0^1 X_1(x) X_n(x) \, dx = \begin{cases} 0 & \text{if } n \neq 1, \\ \frac{1}{2} & \text{if } n = 1. \end{cases}$$

Consequently, all α_n are zero except for α_1 which is $1/\pi$, and the solution (9.21) reduces to a single term

$$u(x,t) = \frac{1}{\pi} \sin \pi t \sin \pi x.$$

Solution to Exercise 9.8. This is a special case of the IBVP (9.14) whose solution is given in (9.21). The values of γ_n and λ_n are available in (9.20). Since $c = 1$ and $\ell = \pi$ in this special case, we have

$$\gamma_n = n, \quad \lambda_n = n^2.$$

We are given $f(x) = 0$ and $g(x) = \cos \frac{x}{2}$. Therefore we may evaluate the coefficients α_n and β_n from (9.23). Since f is zero, all α_n are zero. As to the β_n, we evaluate it with help from the identity (8.33g) in Table 8.1 on page 120:

$$\begin{aligned}
\beta_n &= \frac{2}{n\pi} \int_0^\pi \cos \frac{x}{2} \sin nx \, dx \\
&= \frac{1}{n\pi} \int_0^\pi \Big[\sin\Big(n + \frac{1}{2}\Big)x + \sin\Big(n - \frac{1}{2}\Big)x \Big] \, dx \\
&= \frac{1}{n\pi} \Bigg[-\frac{1}{n + \frac{1}{2}} \cos\Big(n + \frac{1}{2}\Big)x - \frac{1}{n - \frac{1}{2}} \cos\Big(n - \frac{1}{2}\Big)x \Bigg]_0^\pi \\
&= -\frac{1}{n\pi} \left[\frac{2}{2n+1} \cos\Big((2n+1)\Big)\frac{x}{2} + \frac{2}{2n-1} \cos\Big((2n-1)\Big)\frac{x}{2} \right]_0^\pi .
\end{aligned}$$

Since the cosine of an odd multiple of $\pi/2$ is zero, the square bracket evaluates to zero at $x = \pi/2$, and thus arrive at

$$\beta_n = \frac{1}{n\pi} \left[\frac{2}{2n+1} + \frac{2}{2n-1} \right] = \frac{8}{\pi} \cdot \frac{1}{4n^2 - 1}.$$

We conclude that

$$u(x,t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \sin nt \sin nx.$$