Solution to Exercise 8.9. Following a procedure similar to that employed in Section 8.4, we fork the analysis of the eigenvalue problem into three branches.

Case 1: $\lambda < 0$. To enforce the negativity of λ, take $\lambda = -\gamma^2$, where $\gamma > 0$. Then the differential equation takes the form $y''(x) - \gamma^2 y(x) = 0$ whose general solution is

$$y(x) = a \cosh \gamma x + b \sinh \gamma x,$$

and therefore

$$y'(x) = a\gamma \sinh \gamma x + b\gamma \cosh \gamma x.$$

Applying the left boundary condition leads to $b\gamma = 0$. But $\gamma > 0$ by assumption. Therefore $b = 0$ and we arrive at

$$y(x) = a \cosh \gamma x, \quad y'(x) = a\gamma \sinh \gamma x.$$

The boundary condition at $x = \ell$ implies that $a\gamma \sinh \gamma\ell = 0$. We don't want to take $a = 0$ because then we will have a trivial solution. Since $\gamma > 0$ by assumption, we are led to $\sinh \gamma\ell = 0$, and therefore $\gamma\ell = 0$ (see the marginal note on page 108) which is impossible since neither γ nor ℓ is zero. So we abandon the $\lambda < 0$ case.

Case 2: $\lambda = 0$. Then the differential equation takes the form $y''(x) = 0$ whose general solution is $y(x) = ax + b$, and therefore $y'(x) = a$. The left boundary condition implies that $a = 0$, leaving us with $y(x) = b$, and therefore $y'(x) = 0$. This satisfies the boundary condition at $x = \ell$, and therefore $\lambda = 0$ is an eigenvalue. Call this λ_0. The corresponding eigenfunction is any (nonzero) constant function. We take $y_0(x) = 1$. In summary, we have found the following eigenvalue/eigenfunction pair:

$$\lambda_0 = 0, \quad y_0(x) = 1.$$

Case 3: $\lambda > 0$. To enforce the positivity of λ, take $\lambda = \gamma^2$, where $\gamma > 0$. Then the differential equation takes the form $y''(x) + \gamma^2 y(x) = 0$ whose general solution is

$$y(x) = a \cos \gamma x + b \sin \gamma x,$$

and therefore

$$y'(x) = -a\gamma \sin \gamma x + b\gamma \cos \gamma x.$$

Applying the left boundary condition leads to $b\gamma = 0$. Since $\gamma > 0$ by assumption, we let $b = 0$ and arrive at

$$y(x) = a \cos \gamma x, \quad y'(x) = -a\gamma \sin \gamma x.$$

Applying the boundary condition at $x = \ell$ we get $-a\gamma \sin \gamma\ell = 0$. Since a and γ are nonzero, we are left with $\sin \gamma\ell = 0$ and therefore $\gamma\ell$ is an integer multiple of π. We let

$$\gamma_n = \frac{n\pi}{\ell}, \quad n = 1, 2, \dots .$$

Then the eigenvalues and eigenfunctions are

$$\lambda_n = \gamma_n^2 = \left(\frac{n\pi}{\ell}\right)^2, \quad y_n(x) = \cos \gamma_n x = \cos \frac{n\pi x}{\ell}, \qquad n = 1, 2, \dots .$$

Note: The eigenvalue and eigenfunction pair λ_0 and $y_0(x)$ found in Case 2 can be subsumed in the result of Case 3 by letting the index n begin at $n = 0$. Thus, the complete set of the eigenvalues and eigenfunctions in this problem are given by

$$\lambda_n = \gamma_n^2 = \left(\frac{n\pi}{\ell}\right)^2, \quad y_n(x) = \cos \gamma_n x = \cos \frac{n\pi x}{\ell}, \qquad n = \mathbf{0}, 1, 2, \dots .$$

To evaluate the inner products (y_m, y_n) of the eigenfunctions, we follow the method introduced in Section 8.5. Specifically, our mth and nth eigenvalues and eigenfunctions are solutions of

$$\begin{aligned} y_m'' + \lambda_m y_m &= 0, & y_m'(0) &= 0, & y_m'(\ell) &= 0, \\ y_n'' + \lambda_n y_n &= 0, & y_n'(0) &= 0, & y_n'(\ell) &= 0. \end{aligned}$$

A calculation identical to that in Section 8.5.2 leads to

$$\big(y_m'(x)y_n(x) - y_n'(x)y_m(x)\big)\Big|_0^\ell + (\lambda_m - \lambda_n) \int_0^\ell y_m(x)y_n(x)\,dx = 0.$$

The evaluation $\Big|_0^\ell$ yields zero because y_m' and y_n' are zero at $x = 0$ and $x = \ell$. That leaves us with

$$(\lambda_m - \lambda_n) \int_0^\ell y_m(x)y_n(x)\,dx = 0.$$

If $m \neq n$, then $\lambda_m - \lambda_n \neq 0$, and therefore

$$\int_0^\ell y_m(x)y_n(x)\,dx = 0,$$

which confirms the case $m \neq n$ of this exercise's assertion. We handle the case $m = n$ through direct integration:

$$\begin{aligned} (y_n, y_n) = \int_0^\ell \big|y_n(x)\big|^2\,dx &= \int_0^\ell \cos^2 \gamma_n x\,dx \\ &= \frac{1}{2}\int_0^\ell [1 + \cos 2\gamma_n x]\,dx = \frac{1}{2}\left[x + \frac{1}{2\gamma_n}\sin 2\gamma_n x\right]_0^\ell . \end{aligned}$$

The last step of the calculation above is valid only for $n > 0$ because $\gamma_0 = 0$. So if $n > 0$, we have

$$\sin 2\gamma_n x\Big|_{x=\ell} = \sin 2\gamma_n \ell = \sin n\pi = 0,$$

and thus we arrive at

$$(y_n, y_n) = \frac{1}{2}\ell, \quad \text{if } n > 0,$$

and if $n = 0$, we have $\gamma_0 = 0$ and therefore $y_0(x) = \cos 0 = 1$. Thus:

$$(y_0, y_0) = \int_0^\ell 1^2\,dx = x\Big|_0^\ell = \ell.$$

In summary, we have shown that

$$\int_0^\ell y_m(x)y_n(x)\,dx = \begin{cases} \ell & \text{if } m = n = 0, \\ \frac{1}{2}\ell & \text{if } m = n \neq 0, \\ 0 & \text{otherwise .} \end{cases}$$

Solution to Exercise 8.10. According to equations (8.13), (8.14), (8.15), (8.20), and (8.21), The Fourier sine series of a function f defined on the interval $0 < x < \ell$ is

$$f(x) = \sum_{n=1}^{\infty} b_n y_n(x),$$

where

$$\gamma_n = \frac{n\pi}{\ell}, \quad \lambda_n = \gamma_n^2, \quad y_n(x) = \sin\frac{n\pi x}{\ell}, \quad b_n = \frac{2}{\ell}(f, y_n).$$

In the current exercise we have $\ell = \pi$, and therefore $\gamma_n = n$ and

$$b_n = \frac{2}{\pi}\int_0^{\pi}(1)\sin nx\,dx = \frac{2}{\pi}\Big(-\frac{1}{n}\cos nx\Big)\Big|_0^{\pi} = \frac{2}{n\pi}\left[-\cos n\pi + 1\right] = \frac{2}{n\pi}\left[1-(-1)^n\right].$$

We conclude that

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx = \frac{4}{\pi}\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\frac{1}{n}\sin nx = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x.$$

Here are the first few terms:

$$f(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right].$$

Solution to Exercise 8.12. We have $\ell = 1$ and therefore according to (8.23) $\gamma_n = n\pi$, and according to (8.26) we have

$$\begin{aligned} b_n = 2(f, y_n) &= 2\int_0^1 \frac{1}{4} x(1-x)^2 \sin\gamma_n x\, dx \\ &= \int_0^1 \frac{1}{2}(x - 2x^2 + x^3)\sin\gamma_n x\, dx. \end{aligned}$$

We begin with evaluating[9] the indefinite integral:

$$\begin{aligned} &\int \frac{1}{2}(x-2x^2+x^3)\sin\gamma_n x\, dx \\ &\quad = \frac{1}{2}(x-2x^2+x^3)\left(-\frac{1}{\gamma_n}\cos\gamma_n x\right) - \frac{1}{2}(1-4x+3x^2)\left(-\frac{1}{\gamma_n^2}\sin\gamma_n x\right) \\ &\quad + \frac{1}{2}(-4+6x)\left(\frac{1}{\gamma_n^3}\cos\gamma_n x\right) - (3)\left(\frac{1}{\gamma_n^4}\sin\gamma_n x\right). \end{aligned}$$

[9] Here we apply Kronecker's method (see Appendix 8.10) to evaluate the integral but you may do it with any other integration method that you are comfortable with.

Evaluating the result at $x = 0$ yields $-\frac{2}{\gamma_n^3}$, and at $x = 1$ yields $\frac{1}{\gamma_n^3}\cos\gamma_n - \frac{3}{\gamma_n^4}\sin\gamma_n$, and therefore

$$b_n = \frac{1}{\gamma_n^3}\cos\gamma_n - \frac{3}{\gamma_n^4}\sin\gamma_n + \frac{2}{\gamma_n^3}.$$

Considering that $\gamma_n = n\pi$, we have $\sin\gamma_n = 0$ and $\cos\gamma_n = (-1)^n$, and therefore

$$b_n = \frac{1}{\gamma_n^3}\left[2 + (-1)^n\right] = \frac{1}{\pi^3}\cdot\frac{2+(-1)^n}{n^3}.$$

We conclude that

$$f(x) = \frac{1}{\pi^3}\sum_{n=1}^{\infty}\frac{2+(-1)^n}{n^3}\sin n\pi x.$$

The expression $2 + (-1)^n$ alternates between 1 and 3. Here are the first five terms:

$$f(x) = \frac{1}{\pi^3}\left[\sin\pi x + \frac{3}{2^3}\sin 2\pi x + \frac{1}{3^3}\sin 3\pi x + \frac{3}{4^3}\sin 4\pi x + \frac{1}{5^3}\sin 5\pi x + \cdots\right]$$