Solution to Exercise 8.5. We need to determine c so that $(f, g) = 0$. We have

$$(f, g) = \int_0^\pi f(x)g(x)\,dx$$
$$= \int_0^\pi (x + c)\sin x\,dx = \int_0^\pi x\sin x\,dx + c\int_0^\pi \sin x\,dx.$$

We evaluate the $x \sin x$ integral with Kronecker's method:[8]

[8] Or by integration by parts, if you prefer.

$$\int x\sin x\,dx = (x)\big(-\cos x\big) - (1)\big(-\sin x\big) = -x\cos x + \sin x,$$

and therefore

$$\int_0^\pi x\sin x\,dx = \big(-x\cos x + \sin x\big)\Big|_0^\pi = \pi.$$

The evaluation of the $\sin x$ integral is straightforward:

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -\cos\pi + \cos 0 = 2.$$

We conclude that $(f, g) = \pi + 2c$ and the orthogonality implies that $c = -\frac{\pi}{2}$.

Solution to Exercise 8.6. The three orthogonality conditions provide a set of three equations which we solve for the unknowns a, b, and c. Here are the details.

$$\begin{aligned}
(f,g) &= \int_0^1 f(x)g(x)\,dx = \int_0^1 (1)(x+a)\,dx \\
&= \int_0^1 (x+a)\,dx = \Big(\frac{1}{2}x^2 + ax\Big)\Big|_0^1 = \frac{1}{2} + a = 0, && (*) \\
(g,h) &= \int_0^1 g(x)h(x)\,dx = \int_0^1 (x+a)(x^2+bx+c)\,dx \\
&= \int_0^1 \Big(x^3 + (a+b)x^2 + (ab+c)x + ac\Big)\,dx \\
&= \Big(\frac{1}{4}x^4 + \frac{1}{3}(a+b)x^3 + \frac{1}{2}(ab+c)x^2 + acx\Big)\Big|_0^1 \\
&= \frac{1}{4} + \frac{1}{3}(a+b) + \frac{1}{2}(ab+c) + ac = 0, \\
(h,f) &= \int_0^1 h(x)f(x)\,dx = \int_0^1 (x^2+bx+c)(1)\,dx && (**) \\
&= \Big(\frac{1}{3}x^3 + \frac{1}{2}bx^2 + cx\Big)\Big|_0^1 \\
&= \frac{1}{3} + \frac{1}{2}b + c = 0. && (***)
\end{aligned}$$

From $(*)$ we get $a = -\frac{1}{2}$. Then $(**)$ reduces to $\frac{1}{12} + \frac{b}{12} = 0$, whence $b = -1$. Then $(***)$ reduces to $-\frac{1}{6} + c = 0$, whence $c = \frac{1}{6}$. We conclude that the following functions are mutually orthogonal:

$$f(x) = 1, \quad g(x) = x - \frac{1}{2}, \quad h(x) = x^2 - x + \frac{1}{6}.$$

Solution to Exercise 8.7. With the help of the trigonometric identity (8.33e) we calculate

$$\begin{aligned}\Phi_{m,n} &= \int_0^\pi \cos mx \cos nx \, dx \\ &= \frac{1}{2}\int_0^\pi \Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big]\, dx \\ &= \frac{1}{2}\left[\frac{1}{m-n}\sin\big((m-n)x\big) + \frac{1}{m+n}\sin\big((m+n)x\big)\right]\Bigg|_0^\pi \\ &= \frac{1}{2}\left[\frac{1}{m-n}\sin\big((m-n)\pi\big) + \frac{1}{m+n}\sin\big((m+n)\pi\big)\right].\end{aligned}$$

Since the sine of an integer multiple of π is zero, the expression above evaluates to zero.

Exception: The calculation above is *invalid* if $m = n$ due to the $m - n$ in the denominator. If $m = n$, we have

$$\begin{aligned}\Phi_{n,n} &= \int_0^\pi \cos^2 nx \, dx \\ &= \frac{1}{2}\int_0^\pi (1 + \cos 2nx)\, dx \\ &= \frac{1}{2}\left[x + \frac{1}{2n}\sin 2nx\right]\Bigg|_0^\pi \\ &= \frac{1}{2}\left[\pi + \frac{1}{2n}\sin 2n\pi\right].\end{aligned}$$

Since $\sin 2n\pi = 0$, we conclude that

$$\Phi_{n,n} = \frac{\pi}{2}.$$

Exception: The calculation above is *invalid* if $n = 0$ due to the n in the denominator. If $n = 0$ we get

$$\Phi_{0,0} = \int_0^\pi \cos^2 0 \, dx = \int_0^\pi 1 \, dx = x\Big|_0^\pi = \pi.$$

In summary, we we shown that

$$\Phi_{m,n} = \begin{cases} \pi & m = n = 0, \\ \frac{\pi}{2} & m = n \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Solution to Exercise 8.8. Following a procedure similar to that employed in Section 8.4, we fork the analysis of the eigenvalue problem into three branches.

Case 1: $\lambda < 0$. To enforce the negativity of λ, take $\lambda = -\gamma^2$, where $\gamma > 0$. Then the differential equation takes the form $y''(x) - \gamma^2 y(x) = 0$ whose general solution is

$$y(x) = a \cosh \gamma x + b \sinh \gamma x.$$

Applying the left boundary condition leads to $a = 0$, leaving us with

$$y(x) = b \sinh \gamma x, \tag{14.30}$$

and therefore $y'(x) = b\gamma \cosh \gamma x$. The boundary condition at $x = \ell$ implies that $b\gamma \cosh \gamma\ell = 0$. We don't want to take $b = 0$ because then (14.30) would reduce to the trivial solution, and since $\gamma > 0$ by assumption and the cosh function is never zero (see the marginal note on page 108), we reach a dead end and abandon the $\lambda < 0$ case.

Case 2: $\lambda = 0$. Then the differential equation takes the form $y''(x) = 0$ whose general solution is $y(x) = ax + b$. Applying the left boundary condition implies that $b = 0$, and therefore $y(x) = ax$. Then the boundary condition at $x = \ell$ implies that $a\ell = 0$. We are forced to take $a = 0$ and thus arrive at the trivial solution again. So we abandon the $\lambda = 0$ case.

Case 3: $\lambda > 0$. To enforce the positivity of λ, take $\lambda = \gamma^2$, where $\gamma > 0$. Then the differential equation takes the form $y''(x) + \gamma^2 y(x) = 0$ whose general solution is

$$y(x) = a \cos \gamma x + b \sin \gamma x.$$

Applying the left boundary condition leads to $a = 0$, leaving us with

$$y(x) = b \sin \gamma x, \tag{14.31}$$

and therefore $y'(x) = b\gamma \cos \gamma x$. The boundary condition at $x = \ell$ implies that $b\gamma \cos \gamma\ell = 0$. Since b cannot be zero—otherwise we will have a trivial solution—and since $\gamma > 0$, we are left with $\cos \gamma\ell = 0$ which is possible only if $\gamma\ell$ is an odd multiple of $\pi/2$, as in $(2n-1)\frac{\pi}{2}$. Thus, we arrive at infinitely many choices for γ:

$$\gamma_n = \frac{(2n-1)\pi}{2\ell}, \quad n = 1, 2 \ldots, \tag{14.32}$$

and since $\lambda = \gamma^2$, we have infinitely many choices for λ:

$$\lambda_n = \left(\frac{(2n-1)\pi}{2\ell}\right)^2, \quad n = 1, 2 \ldots. \tag{14.33}$$

Finally, (14.31) yields an eigenfunction $y_n(x)$ corresponding to each eigenvalue λ_n:

$$y_n(x) = \sin \gamma_n x = \sin \frac{(2n-1)\pi x}{2\ell}, \quad n = 1, 2 \dots . \tag{14.34}$$

To evaluate the inner products (y_m, y_n) of the eigenfunctions, we follow the method introduced in Section 8.5. Specifically, our mth and nth eigenvalues and eigenfunctions are solutions of

$$\begin{aligned} y_m'' + \lambda_m y_m &= 0, \qquad & y_m(0) &= 0, \qquad & y_m'(\ell) &= 0, \\ y_n'' + \lambda_n y_n &= 0, \qquad & y_n(0) &= 0, \qquad & y_n'(\ell) &= 0. \end{aligned}$$

A calculation identical to that in Section 8.5.2 leads to

$$\big(y_m'(x)y_n(x) - y_n'(x)y_m(x)\big)\Big|_0^\ell + (\lambda_m - \lambda_n)\int_0^\ell y_m(x)y_n(x)\,dx = 0.$$

The evaluation $\Big|_0^\ell$ yields zero because y_m and y_n are zero at $x = 0$ and y_m' and y_n' are zero at $x = \ell$. That leaves us with

$$(\lambda_m - \lambda_n)\int_0^\ell y_m(x)y_n(x)\,dx = 0.$$

If $m \neq n$, then $\lambda_m - \lambda_n \neq 0$, and therefore

$$\int_0^\ell y_m(x)y_n(x)\,dx = 0,$$

which confirms the case $m \neq n$ of this exercise's assertion. We handle the case $m = n$ through direct integration:

$$\begin{aligned} (y_n, y_n) &= \int_0^\ell \big|y_n(x)\big|^2\,dx = \int_0^\ell \sin^2 \gamma_n x\,dx \\ &= \frac{1}{2}\int_0^\ell [1 - \cos 2\gamma_n x]\,dx = \frac{1}{2}\left[x - \frac{1}{2\gamma_n}\sin 2\gamma_n x\right]_0^\ell . \end{aligned}$$

In view of (14.32) we have

$$\sin 2\gamma_n x\Big|_{x=\ell} = \sin 2\gamma_n \ell = \sin(2n-1)\pi = 0,$$

and thus we arrive at

$$(y_n, y_n) = \frac{1}{2}\ell.$$