Solution to Exercise 7.5. Differentiating (7.21) with respect to x we get

$$u_x(x,t) = \frac{1}{2}\,\mathrm{erf}'\Big(\frac{x}{\sqrt{4kt}}\Big)\frac{1}{\sqrt{4kt}},$$

where erf′ is the derivative of erf. But recalling the definition of erf in (7.20), we have

$$\mathrm{erf}'(x) = \frac{2}{\sqrt{\pi}}e^{-x^2},$$

and therefore

$$\mathrm{erf}'\Big(\frac{x}{\sqrt{4kt}}\Big) = \frac{2}{\sqrt{\pi}}e^{-\frac{x^2}{4kt}}.$$

We conclude that

$$u_x(x,t) = \Big(\frac{1}{2}\Big)\Big(\frac{2}{\sqrt{\pi}}e^{-x^2/(4kt)}\Big)\Big(\frac{1}{\sqrt{4kt}}\Big) = \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}.$$

Solution to Exercise 7.6. We let $s = 1/t$ and look at the limit $s \to +\infty$:

$$G(x,t) = \frac{1}{\sqrt{4\pi k/s}} e^{-\frac{x^2 s}{4k}} = \frac{1}{4\pi k} \frac{\sqrt{s}}{e^{\frac{x^2 s}{4k}}}.$$

The numerator and denominator both go to infinity as $s \to +\infty$, so we apply l'Hôpital's rule to resolve the indeterminacy:

$$\begin{aligned}
\lim_{t \to 0^+} G(x,t) &= \frac{1}{4\pi k} \lim_{s \to +\infty} \frac{\sqrt{s}}{e^{\frac{x^2 s}{4k}}} \\
&= \frac{1}{4\pi k} \lim_{s \to +\infty} \frac{\frac{1}{2\sqrt{s}}}{\frac{x^2}{4k} e^{\frac{x^2 s}{4k}}} = \frac{1}{2\pi x^2} \lim_{s \to +\infty} \frac{1}{\sqrt{s} e^{\frac{x^2 s}{4k}}}.
\end{aligned}$$

In the rightmost expression, the denominator goes to infinity as $s \to +\infty$, and therefore the limit is zero.

Solution to Exercise 7.10. According to (7.31) we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds$$
$$= \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} e^{-\frac{(x-s)^2}{4kt}} e^{-s}\, ds.$$

We combine the exponents in the two exponential terms, and simplify the result through completing the square:

$$\frac{(x-s)^2}{4kt} + s = \frac{(x-s)^2 + 4kts}{4kt} = \frac{x^2 - 2xs + s^2 + 4kts}{4kt}$$
$$= \frac{s^2 - 2(x-2kt)s + x^2}{4kt} = \frac{\big[s-(x-2kt)\big]^2 - (x-2kt)^2 + x^2}{4kt}$$
$$= \frac{\big[s-(x-2kt)\big]^2 + 4kt(x-kt)}{4kt} = \Big(\frac{s-(x-2kt)}{\sqrt{4kt}}\Big)^2 + (x-kt).$$

Thus, we have

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_0^\infty e^{-\left(\frac{s-(x-2kt)}{\sqrt{4kt}}\right)^2} ds.$$

Our next task is evaluate the integral appearing above. We change the variable of integration from s to $r = \frac{s-(x-2kt)}{\sqrt{4kt}}$. Then

$$\int_0^\infty e^{-\left(\frac{s-(x-2kt)}{\sqrt{4kt}}\right)^2} ds = \int_{-\frac{x-2kt}{\sqrt{4kt}}}^{\infty} e^{-r^2} \sqrt{4kt}\, dr$$
$$= \sqrt{4kt} \left[\int_{-\frac{x-2kt}{\sqrt{4kt}}}^{0} e^{-r^2}\, dr + \int_0^\infty e^{-r^2}\, dr \right]$$
$$= \sqrt{4kt} \left[\int_0^{\frac{x-2kt}{\sqrt{4kt}}} e^{-r^2}\, dr + \int_0^\infty e^{-r^2}\, dr \right]$$
$$= \sqrt{4kt} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}\Big(\frac{x-2kt}{\sqrt{4kt}}\Big) + \frac{\sqrt{\pi}}{2} \right].$$

We conclude that

$$u(x,t) = \frac{1}{2} \left[1 + \operatorname{erf}\Big(\frac{x-2kt}{\sqrt{4kt}}\Big) \right] e^{kt-x}.$$

Solution to Exercise 7.15. Extend the initial condition $f(x)$ as an even function f_{ext} to the entire real line,

$$f_{\text{ext}} = \begin{cases} f(x) & \text{if } x > 0, \\ f(-x) & \text{if } x < 0, \end{cases}$$

and define

$$u(x,t) = \int_{-\infty}^{\infty} G(x-s,t) f_{\text{ext}}(s)\, ds.$$

According to (7.30), $u(x,t)$ solves the heat equation with initial data $f_{\text{ext}}(x)$. Therefore $u(x,t)$ satisfies both (7.32a) and (7.32b), since $f_{\text{ext}}(x)$ coincides with $f(x)$ on $x > 0$. It remains to verify that $u(x,t)$ satisfies the boundary condition $u_x(0,t) = 0$, so let's calculate

$$u_x(x,t) = \int_{-\infty}^{\infty} G_x(x-s,t) f_{\text{ext}}(s)\, ds \tag{$*$}$$

and evaluate the result at $x = 0$:

$$u_x(0,t) = \int_{-\infty}^{\infty} G_x(-s,t) f_{\text{ext}}(s)\, ds. \tag{$**$}$$

Let's observe that G_x is an *odd function* of s. We see that by calculating G_x from its definition in (7.22):

$$G_x(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \Big[-\frac{2x}{4kt} \Big].$$

Alternatively, we may appeal to the general fact that the derivative and even function is an odd function. Either way, since $G_x(-s,t)$ is odd and $f_{\text{ext}}(s)$ is even, the integrand in ($**$) is odd, and therefore the integral evaluates to zero, which shows that $u(x,t)$ satisfies the boundary condition.

The solution $u(x,t)$ defined in ($*$) may be expressed solely in terms of the problem's data, f, by splitting the integration into the intervals $-\infty < s < 0$ and $0 < s < \infty$,

$$u(x,t) = \int_{-\infty}^{0} G(x-s,t) f_{\text{ext}}(s)\, ds + \int_{0}^{\infty} G(x-s,t) f_{\text{ext}}(s)\, ds,$$

and noting that

$$\int_{0}^{\infty} G(x-s,t) f_{\text{ext}}(s)\, ds = \int_{0}^{\infty} G(x-s,t) f(s)\, ds,$$

and

$$\int_{-\infty}^{0} G(x-s,t) f_{\text{ext}}(s)\, ds = \int_{-\infty}^{0} G(x-s,t) f(-s)\, ds$$
$$= \int_{0}^{\infty} G(x+s,t) f(s)\, ds,$$

Changing the variable of integration from $s \to -s$.

whereby we arrive at the solution

$$u(x,t) = \int_{0}^{\infty} \Big[G(x-s,t) + G(x+s,t) \Big] f(s)\, ds.$$

Solution to Exercise 7.16. Here are two different ways of solving this problem.

Method 1: Straightforward but long

Section 7.6 shows how to solve the more general IBVP (7.32), and arrives at the solution (7.34), which is

$$u(x,t) = \int_0^\infty \Big[G(x-s,t) - G(x+s,t)\Big] f(s)\,ds.$$

That solution holds for any initial value f. In the current exercise f is just a constant, and therefore the solution is

$$\begin{aligned} u(x,t) &= u_0 \int_0^\infty \Big[G(x-s,t) - G(x+s,t)\Big]\,ds \\ &= \frac{u_0}{\sqrt{4\pi kt}} \int_0^\infty \Big[e^{-\frac{(x-s)^2}{4kt}} - e^{-\frac{(x+s)^2}{4kt}}\Big]\,ds. \end{aligned}$$

The integral may be evaluated in terms of erf. In the case of the first term in the square brackets, we make a changes of variables from s to $r = -\frac{x-s}{\sqrt{4kt}}$, and therefore $dr = \frac{ds}{\sqrt{4kt}}$. Then

$$\begin{aligned} &\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x-s)^2}{4kt}}\,ds = \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4kt}}}^\infty e^{-r^2}\,dr \\ &= \frac{1}{2}\left[\frac{2}{\sqrt{\pi}} \int_{-\frac{x}{\sqrt{4kt}}}^0 e^{-r^2}\,dr + \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-r^2}\,dr\right] = \frac{1}{2}\left[1 + \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big)\right]. \end{aligned}$$

In the same way we obtain

$$\frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x+s)^2}{4kt}}\,ds = \frac{1}{2}\left[1 - \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big)\right].$$

We conclude that

$$u(x,t) = u_0 \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big).$$

Method 2: Clever and short

Section 7.3 shows how to solve the heat equation with the Heaviside function as initial data and arrives at the solution (7.21), which is

$$u(x,t) = \frac{1}{2}\left[1 + \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big)\right].$$

If the initial data is multiplied by a factor $2u_0$, then the solution gets mutiplied by that factor. Therefore, the solution corresponding to the intial value

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0, \\ 2u_0 & \text{if } x > 0, \end{cases} \qquad (*)$$

is

$$u(x,t) = u_0\left[1 + \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big)\right].$$

If we substract any constant from the intial data, then the solution is reduced by that constant. If we subtract u_0 from the initial data $(*)$, it changes to

$$u(x,0) = \begin{cases} -u_0 & \text{if } x < 0, \\ u_0 & \text{if } x > 0, \end{cases} \tag{$**$}$$

and the corresponding solution changes to

$$u(x,t) = u_0 \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big). \tag{$***$}$$

But the odd function $(**)$ agrees with the exercise's original initial data on $x > 0$, and the fact that it's odd, implies that $u(0,t) = 0$. We conclude that the restriction of $(***)$ to $x > 0$ is the exercise's solution.