Solution to Exercise 6.7. As in Section 6.5, we look for a motion of the form

$$u(x,t) = f(ct+x) + g(ct-x) \quad 0 < x < \infty,$$

where f is given and g is to be determined. We write $U(t)$ for the ring's displacement

$$U(t) = u(0,t) = f(ct) + g(ct), \tag{14.26}$$

and note that

$$U'(t) = cf'(ct) + cg'(ct),$$

and therefore

$$g'(ct) = \frac{1}{c}U'(t) - f'(ct). \tag{14.27}$$

According to (5.4), the magnitude of the vertical (i.e., in the direction of the pole) component of the tensile force exerted by the string on the ring is $Tu_x(0,t)$, and therefore Newton's law of motion applied to the ring is expressed as

$$mU''(t) = Tu_x(0,t).$$

But since $u_x(x,t) = f'(ct+x) - g'(ct-x)$, we have $u_x(0,t) = f'(ct) - g'(ct)$. It follows that

$$mU''(t) = T\big[f'(ct) - g'(ct)\big].$$

Substituting for g' from (14.27) we get

$$mU''(t) = T\Big[f'(ct) - \Big(\frac{1}{c}U'(t) - f'(ct)\Big)\Big],$$

which we rearrange into

$$U''(t) + \frac{T}{mc}U'(t) = \frac{2T}{m}f'(ct).$$

All three terms in this ODE appear as derivatives, so we integrate once with respect to t and obtain

$$U'(t) + \frac{T}{mc}U(t) = \frac{2T}{mc}f(ct) + K.$$

The ring is not moving at $t = 0$, and therefore $U(0) = 0$ and $U'(0) = 0$. Furthermore, the blip has not arrived yet at the origin at $t = 0$, therefore $f(0) = 0$. Evaluating the equation above according to this data yields $K = 0$, and thus we conclude that

$$U'(t) + \frac{T}{mc}U(t) = \frac{2T}{mc}f(ct). \tag{14.28}$$

To solve this first order ODE, we multiply it through by the integrating factor $e^{\frac{T}{mc}t}$ and group the left-hand side terms

$$\frac{d}{dt}\left[e^{\frac{T}{mc}t}U(t)\right] = \frac{2T}{mc}e^{\frac{T}{mc}t}f(ct),$$

and then integrate to get

$$e^{\frac{T}{mc}t}U(t) = \frac{2T}{mc}\int_0^t e^{\frac{T}{mc}\tau} f(c\tau)\, d\tau + C.$$

Applying initial condition $U(0) = 0$ gives $C = 0$. We may further simplify the integral through the change of variables $c\tau = \xi$, as in

$$\int_0^t e^{\frac{T}{mc}\tau} f(c\tau)\, d\tau = \int_0^{ct} e^{\frac{T}{mc^2}\xi} f(\xi) \left(\frac{1}{c} d\xi\right) = \frac{1}{c}\int_0^{ct} e^{\frac{T}{mc^2}\xi} f(\xi)\, d\xi,$$

and arrive at

$$U(t) = \frac{2T}{mc^2}\int_0^{ct} e^{-\frac{T}{mc}(t-\frac{\xi}{c})} f(\xi)\, d\xi. \tag{14.29}$$

Having thus determined U, the profile of the reflected wave may be calculated by setting $\xi = ct$ in (14.26):

$$g(\xi) = U\left(\frac{\xi}{c}\right) - f(\xi).$$

Solution to Exercise 6.8. Recall the solution (14.29) obtained in the previous exercise:

$$U(t) = \frac{2T}{mc^2} \int_0^{ct} e^{-\frac{T}{mc}(t-\frac{\xi}{c})} f(\xi)\, d\xi.$$

We split that into two parts as follows:

$$U(t) = \frac{2T}{mc^2} \int_0^{a} e^{-\frac{T}{mc}(t-\frac{\xi}{c})} f(\xi)\, d\xi + \frac{2T}{mc^2} \int_a^{ct} e^{-\frac{T}{mc}(t-\frac{\xi}{c})} f(\xi)\, d\xi$$

The first integral evaluates to zero because f is zero over the interval $[0, a]$. As to the second integral, consider two cases where $ct > a$ or $ct < a$.

If $ct < a$, then the integration takes place over the interval $[ct, a]$ where f is zero, so in that case the result is zero. If $ct > a$, then the integration takes place over the interval $[a, ct]$ where f is 1, so in that case the integral evaluates to

$$\frac{2T}{mc^2} \int_a^{ct} e^{-\frac{T}{mc}(t-\frac{\xi}{c})}\, d\xi = 2\Big(1 - e^{-\frac{T}{mc^2}(ct-a)}\Big).$$

We conclude that

$$U(t) = \begin{cases} 0 & \text{if } ct < a \\ 2\Big(1 - e^{-\frac{T}{mc^2}(ct-a)}\Big) & \text{if } ct > a \end{cases}$$

Solution to Exercise 6.11. We retain the formulation and notation of Section 6.5, but equation (6.16) is now replaced by the balance of forces:

$$kU(t) = T\Big[\Big(f'(ct) - g'(ct)\Big) - h'(ct)\Big].$$

We substitute for $h'(ct)$ and $g'(ct)$ from (6.14) as before:

$$U(t) = T\Big[f'(ct) - \Big(\frac{1}{c}U'(t) - f'(ct)\Big) - \frac{1}{c}U'(t)\Big],$$

and rearrange the result into

$$U'(t) + \frac{ck}{2T}U(t) = cf'(ct).$$

To solve this linear first order ODE, we multiply it by the integrating factor $e^{\frac{ck}{2T}t}$

$$\frac{d}{dt}\Big(e^{\frac{ck}{2T}t}U(t)\Big) = ce^{\frac{ck}{2T}t}f'(ct),$$

and integrate. Accounting for $U(0) = 0$, we arrive at

$$e^{\frac{ck}{2T}t}U(t) = c\int_0^t e^{\frac{ck}{2T}\tau}f'(c\tau)\,d\tau.$$

The integral may be simplified by changing the variable of integration from τ to $\xi = c\tau$, whereby

$$e^{\frac{ck}{2T}t}U(t) = \int_0^{ct} e^{\frac{k}{2T}\xi}f'(\xi)\,d\xi,$$

and therefore

$$U(t) = \int_0^{ct} e^{-\frac{k}{2T}(ct-\xi)}f'(\xi)\,d\xi.$$

The integral can be further simplified through integration by parts:

$$\begin{aligned} e^{\frac{ck}{2T}t}U(t) &= e^{\frac{k}{2T}\xi}f(\xi)\Big|_0^{ct} - \frac{k}{2T}\int_0^{ct} e^{\frac{k}{2T}\xi}f(\xi)\,d\xi. \\ &= e^{\frac{ck}{2T}t}f(\xi)\Big|_0^{ct} - \frac{k}{2T}\int_0^{ct} e^{\frac{k}{2T}\xi}f(\xi)\,d\xi. \end{aligned}$$

Since $f(0) = 0$, this reduces to

$$\begin{aligned} U(t) &= f(ct) - \frac{k}{2T}e^{-\frac{ck}{2T}t}\int_0^{ct} e^{\frac{k}{2T}\xi}f(\xi)\,d\xi \\ &= f(ct) - \frac{k}{2T}\int_0^{ct} e^{\frac{k}{2T}(ct-\xi)}f(\xi)\,d\xi. \end{aligned}$$

Having thus determined $U(t)$, the reflected and transmitted wave profiles may be obtained from (6.19).