Solution to Exercise 6.4. The rope is initially at rest, so the motion of the point $x = 0$ is

$$u(0,t) = \begin{cases} 0 & t < 0; \\ A \sin \omega t & t \geq 0. \end{cases}$$

The general solution of the wave equation is

$$u(x,t) = F(x+ct) + G(x-ct).$$

In this problem there is no left-moving wave since we have no signals coming down from $+\infty$. So the solution is $u(x,t) = G(x-ct)$. Evaluating this at $t = 0$ we get

$$G(-ct) = u(0,t) = \begin{cases} 0 & t < 0, \\ A \sin \omega t & t \geq 0. \end{cases}$$

Setting $-ct = \xi$ this becomes

$$G(\xi) = \begin{cases} 0 & -\frac{\xi}{c} < 0, \\ A \sin \omega \left(-\frac{\xi}{c}\omega\right) & -\frac{\xi}{c} \geq 0, \end{cases}$$

which simplifies to (assuming $c > 0$)

$$G(\xi) = \begin{cases} 0 & \xi > 0, \\ A \sin \omega \left(-\frac{\xi}{c}\right) & \xi \leq 0, \end{cases}$$

and therefore

$$u(x,t) = G(x-ct) = \begin{cases} 0 & x - ct > 0, \\ A \sin\left(\omega\left(-\frac{x-ct}{c}\right)\right) & x - ct \leq 0, \end{cases}$$

or equivalently

$$u(x,t) = \begin{cases} 0 & x > ct, \\ A \sin\left(\frac{\omega}{c}(ct - x)\right) & x \leq ct. \end{cases}$$

Solution to Exercise 6.5. As in the previous exercise, we have $u(x,t) = G(x-ct)$, and

$$G(-ct) = u(0,t) = \begin{cases} 0 & t < 0, \\ \phi(t) & t \geq 0. \end{cases}$$

Setting $-ct = \xi$ we get

$$G(\xi) = \begin{cases} 0 & \xi > 0, \\ \phi\Big(-\frac{\xi}{c}) & \xi \leq 0. \end{cases}$$

and therefore

$$u(x,t) = G(x-ct) = \begin{cases} 0 & x - ct > 0, \\ \phi\Big(-\frac{x-ct}{c}\Big) & x - ct \leq 0, \end{cases}$$

or equivalently

$$u(x,t) = \begin{cases} 0 & x > ct, \\ \phi\Big(\frac{ct-x}{c}\Big) & x \leq ct. \end{cases}$$

Solution to Exercise 6.6.

Solution 1: Motivated by the method of images introduced in Section 6.1, we introduce the image wave $f(ct-x)$ that travels along the negative toward $x=0$, and consider the solution candidate

$$u(x,t) = f(ct+x) + f(ct-x).$$

This certainly satisfies the wave equation since it is a special case of d'Alembert's general solution. It remains to verify that $u_x(0,t) = 0$. We have

$$u_x(x,t) = f'(ct+x) - f'(ct-x),$$

and therefore

$$u_x(0,t) = f'(ct) - f'(ct) = 0,$$

as required.

Solution 2: Upon interacting with the end support, the incoming wave $f(ct+x)$ gives rise to a reflected wave, let's say $g(ct-x)$, and therefore the overall motion of the string is

$$u(x,t) = f(ct+x) + g(ct-x), \tag{14.23}$$

and consequently,

$$u_x(x,t) = f'(ct+x) - g'(ct-x).$$

Evaluating the above at $x=0$ and accounting for $u_x(0,t) = 0$, we get

$$0 = f'(ct) - g'(ct).$$

This integrates to

$$\frac{1}{c}f(ct) - \frac{1}{c}g(ct) = K,$$

where K is the constant of integration. In particular, at $t=0$ we have $\frac{1}{c}f(0) - \frac{1}{c}g(0) = K$. We assume that the blip's support is in $x>0$, and therefore $f(0) = g(0) = 0$, and consequently $K=0$. We thus conclude that $f(ct) = g(ct)$ for all t, that is, $f=g$. Then from (14.23) we obtain

$$u(x,t) = f(ct+x) + f(ct-x).$$