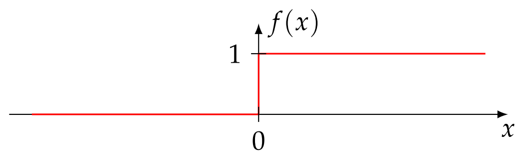
Chapter 4

Chapter 5

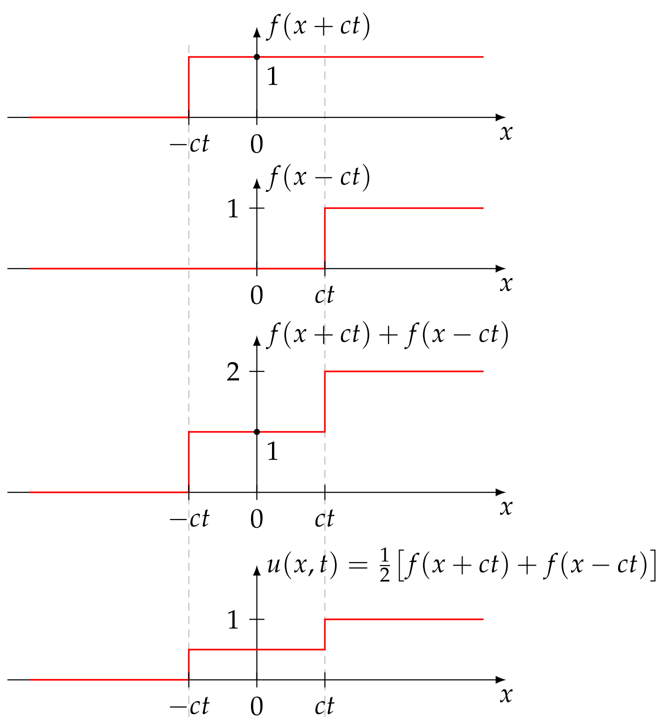
Solution to Exercise 5.1. The solution of (5.14) is given by d'Alembert's formula (5.17). Since $g = 0$ in this case, it reduces to

$$u(x,t) = \frac{1}{2}\big[f(x+ct) + f(x-ct)\big]$$

To sketch the solution, we begin with sketching the graph of the initial displacement f, in

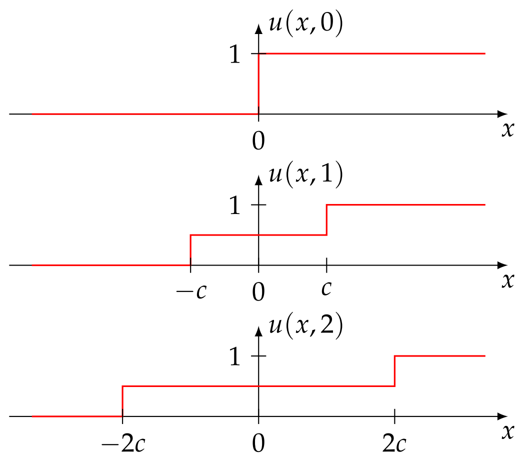


The graphs of $f(x+ct)$ and $f(x-ct)$ are obtained by translating the graph of $f(x)$ to the left and to the right by the amounts ct. The graph of $u(x,t)$ is obtained through the pointwise addition of those two graphs and then scaling the vertical dimension by $1/2$, as shown here:



The graph of $u(x,t)$ obtained above shows a snapshot of the solution at an *arbitrary* time $t \geq 0$. We are asked to sketch the solution at times

$t = 0$, $t = 1$, and $t = 2$. That's a matter of reproducing that graph for those values of t. We get



Remark: We see that that construction of the solution $u(x,t)$ and plotting its graphs involves no computation at all. That's the preferred way of doing it. If, however, you feel compelled to write down some equations, this may interest you.

We are given

$$f(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

Therefore

$$f(x+ct) = \begin{cases} 0 & x+ct < 0 \\ 1 & x+ct \geq 0 \end{cases}, \qquad f(x-ct) = \begin{cases} 0 & x-ct < 0 \\ 1 & x-ct \geq 0 \end{cases},$$

which is better expressed as

$$f(x+ct) = \begin{cases} 0 & x < -ct \\ 1 & x \geq -ct \end{cases}, \qquad f(x-ct) = \begin{cases} 0 & x < ct \\ 1 & x \geq ct \end{cases}.$$

These conditional statements distinguish among three disjoint subintervals of the x axis:

$$x < -ct, \quad -ct < x < ct, \quad x > ct.$$

When $x < -ct$, both conditional statements yield zero, and therefore the expression evaluates to zero. When $-ct < x < ct$, the first conditional statement yields 1 while the second conditional statement yields zero, and therefore the expression evaluates to 1. When $x > ct$, both conditional statements yield 1 and therefore the expression evaluates to 2. We conclude that

$$f(x+ct)+f(x-ct)=\begin{cases}0 & x<-ct,\\ 1 & -ct<x<ct,\\ 2 & x>ct,\end{cases}$$

and therefore

$$u(x,t)=\frac{1}{2}\big[f(x+ct)+f(x-ct)\big]=\begin{cases}0 & x<-ct,\\ 1/2 & -ct<x<ct,\\ 1 & x>ct,\end{cases}$$

which agrees with what we obtained earlier just by looking at the graphs.

Solution to Exercise 5.2. The solution of (5.14) is given by d'Alembert's formula (5.17). Since $f = 0$ in this case, it reduces to

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi)\, d\xi.$$

As in the example solved in Section 5.8, we define

$$G(x) = \int_0^x g(\xi)\, d\xi,$$

and express d'Alembert's solution as

$$u(x,t) = \frac{1}{2c} \big[G(x+ct) - G(x-ct) \big].$$

Considering that $g(x) = 0$ on $x \le 0$, we get $G(x) = \int_0^x 0\, dx = 0$ on $x \le 0$, and considering that $g(x) = 1$ on $x > 0$, we get $G(x) = \int_0^x 1\, dx = x$ on $x > 0$. Thus, altogether we have

$$G(x) = \begin{cases} 0 & x \le 0, \\ x & x > 0. \end{cases}$$

We evaluate the expression within the square brackets as follows:

$$G(x+ct) - G(x-ct) = \begin{cases} 0 & \text{if } x+ct < 0 \\ x+ct & \text{if } x+ct \ge 0 \end{cases} + \begin{cases} 0 & \text{if } x-ct < 0 \\ x-ct & \text{if } x-ct \ge 0 \end{cases},$$

or equivalently

$$G(x+ct) - G(x-ct) = \begin{cases} 0 & \text{if } x < -ct \\ x+ct & \text{if } x \ge -ct \end{cases} + \begin{cases} 0 & \text{if } x < ct \\ x-ct & \text{if } x \ge ct \end{cases}.$$

These conditional statements distinguish among three disjoint subintervals of the x axis:

$$x < -ct, \quad -ct < x < ct, \quad x > ct.$$

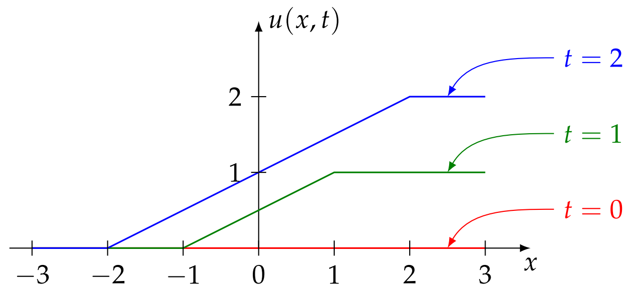
When $x < -ct$, both conditional statements yield zero, and therefore the expression evaluates to zero. When $-ct < x < ct$, the first conditional statement yields $x + ct$ while the second conditional statement yields zero, and therefore the expression evaluates to $x + ct$. When $x > ct$, the first conditional statement yields $x + ct$ and the second conditional statement yields $x - ct$, and therefore the expression evaluates to $2ct$. We conclude that

$$G(x+ct) - G(x-ct) = \begin{cases} 0 & \text{if } x < -ct, \\ x+ct & \text{if } -ct < x < ct, \\ 2ct & \text{if } x > ct, \end{cases}$$

and therefore the solution is

$$u(x,t) = \frac{1}{2c}\big[G(x+ct) - G(x-ct)\big] = \begin{cases} 0 & \text{if } x < -ct, \\ \frac{1}{2c}(x+ct) & \text{if } -ct < x < ct, \\ t & \text{if } x > ct. \end{cases}$$

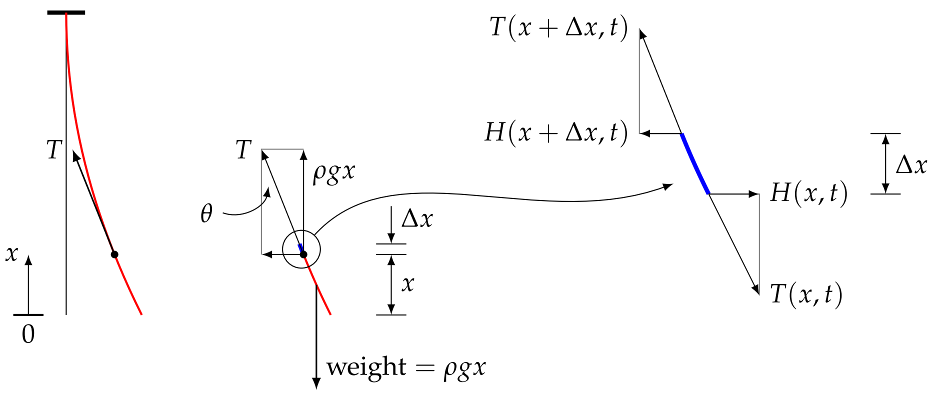
The diagram below shows the graphs of $u(x,t)$ at time $t = 0, 1, 2$. We have taken $c = 1$ for the purpose of plotting.



Solution to Exercise 5.4. We wish the initial displacement $f(x)$ to result in a right-traveling wave $u(x,t) = f(x - ct)$. In that case $u_t(x,t) = -cf'(x-ct)$, and therefore $u_t(x,0) = -cf'(x)$. We conclude that to get the solution $u(x,t) = f(x-ct)$, we need to provide the initial velocity

$$g(x) = u_t(x,0) = -cf'(x).$$

Solution to Exercise 5.5. The red curve in the figure below depicts a snapshot of the chain at generic time. The vertical x axis points up and the origin is at the chain's lowest point. We write $u(x,t)$ for the horizontal displacement of the chain's point at coordinate x at time t.



The magnitude of the tension, T, in the chain varies with x. To see that, consider a section of length x at the chain's free end, as shown in the diagram above. The weight of that segment is $\rho g x$, where g is the gravitational acceleration constant. The downward pull of the weight is resisted by the vertical component of the tensile force within the chain at the location x. The magnitude of that component is also $\rho g x$, making resultant of the vertical forces acting on the segment is zero. Thus, the chain does not experience acceleration in the vertical direction.

The horizontal component of the tensile force is nonzero, and it is what makes the loosely hanging chain sway back and forth once it is set into motion. The diagram shows the angle θ that the chain makes relative to the vertical. Observe that $\tan\theta$ is the slope of the chain's curve relative to the (vertical) x axis, and therefore $\tan\theta(x,t) = -u_x(x,t)$. The minus sign is because the direction of the x axis is reversed.

Furthermore, it is evident from the diagram that the ratio of the horizontal to vertical components of the tension is $\tan\theta$, and therefore the magnitude of the horizontal component, let's call it $H(x,t)$, is $\rho g x \tan\theta$, that is, $H(x,t) = -\rho g x u_x(x,t)$.

Applying Newton's law of motion to a small segment of length Δx of the chain at the location x we get

$$\begin{aligned}(\rho\Delta x)u_{tt}(x,t) &= H(x,t) - H(x+\Delta x,t)\\ &= -\rho g x u_x(x,t) + \rho g(x+\Delta x)u_x(x+\Delta x,t).\end{aligned}$$

The density ρ cancels from both sides, and we are left with

$$u_{tt}(x,t) = g\frac{(x+\Delta x)u_x(x+\Delta x,t) - x u_x(x,t)}{\Delta x}.$$

We let $\Delta x \to 0$ and arrive at the equation of motion

$$u_{tt} = g\big(xu_x\big)_x.$$

Remark 14.2. Going beyond the scope of this problem, we may look for separable solutions[4] of the PDE derived above. Setting $u(x,t) = X(x)T(t)$ we get

$$\big(xX'(x)\big)' + \lambda^2 X(x) = 0$$

whose general solution is expressed in terms of *Bessel functions* J_0 and Y_0 as

$$X(x) = c_1 J_0(2\lambda\sqrt{x}) + c_2 Y_0(2\lambda\sqrt{x}).$$

Since Y_0 blows up at $x = 0$, we let $c_2 = 0$. The boundary condition $u(L,t) = 0$ implies that $X(L) = 0$, and therefore

$$J_0(2\lambda\sqrt{L}) = 0.$$

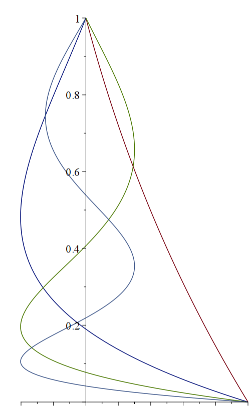
Let's write b_n, $n = 1, 2, \ldots$, for the zeros of the Bessel function J_0. Then we have infinitely many choices for λ, given by

$$\lambda_n = \frac{b_n}{2\sqrt{L}},$$

and the modal function

$$X_n(x) = J_0\Big(b_n\sqrt{\frac{x}{L}}\Big).$$

Here are the graphs of X_n for $n = 1, \ldots, 4$.

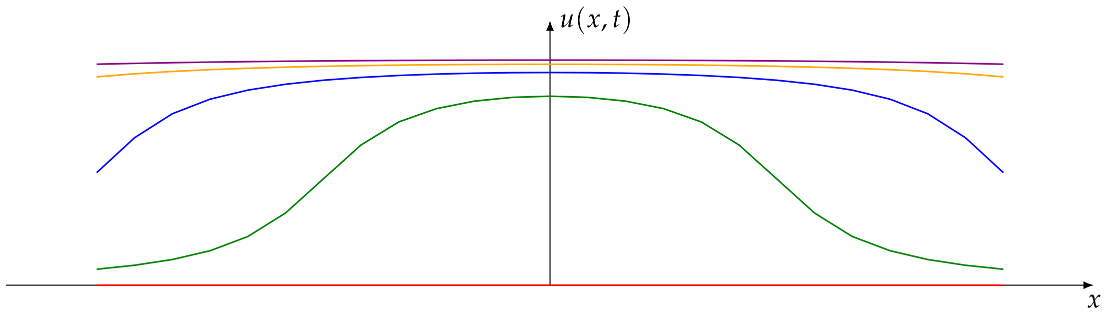


[4] You will learn about separation of variables in Chapter 9.

Solution to Exercise 5.7. Appealing again to d'Alembert's solution with $f(x) = 0$, $g(x) = 1/(1+x^2)$ we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}\big[f(x+ct) + f(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{1}{1+\xi^2}\,d\xi \\ &= \frac{1}{2c}\Big[\arctan(x+ct) - \arctan(x-ct)\Big]. \end{aligned}$$

Here are the graphs of u plotted at times $t = 0, 3, 6, 9, 12$ in red, green, blue, orange, purple. We have $\lim_{t\to\infty} u(x,t) = \frac{\pi}{2c}$ at any x.



Solution to Exercise 5.8. According to d'Alembert, we have

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\left[f(x+ct)+f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \\
&= \frac{1}{2c}\int_{x-ct}^{x+ct} \cos\xi\,d\xi \\
&= \frac{1}{2c}\sin\xi\Big|_{x-ct}^{x+ct} \\
&= \frac{1}{2c}\left[\sin(x+ct)-\sin(x-ct)\right] \\
&= \frac{1}{2c}\left[(\sin x\cos ct+\cos x\sin ct)-(\sin x\cos ct-\cos x\sin ct)\right] \\
&= \frac{1}{c}\sin ct\cos x.
\end{aligned}$$

Therefore, the solution $u(x,t)$ at any time t is $A\cos(x)$ where the amplitude is $A = \frac{1}{c}\sin t$.