Solution to Exercise 3.6. The initial density, $f(x)$, is given as a graph in this exercise. Let us observe that

$$f(x) = \begin{cases} 2 & \text{if } x < 0, \\ 2 + x & \text{if } 0 < x < 3, \\ 5 & \text{if } x > 3. \end{cases}$$

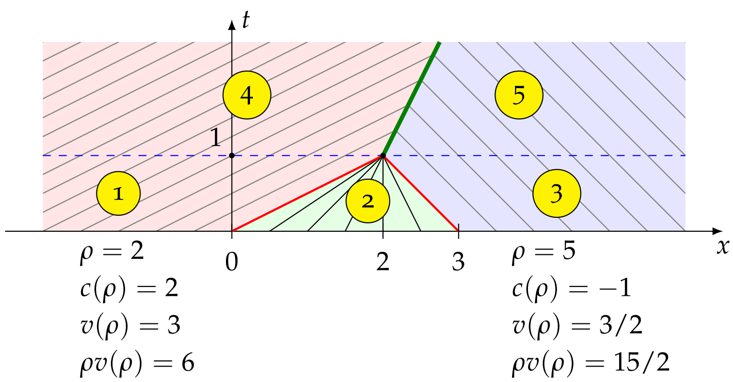
From equations (3.3) and (3.4), and the given data, we have

$$c(\rho) = 4 - \rho, \quad v(\rho) = 4 - \frac{1}{2}\rho.$$

Consequently,

$$c(2) = 2, \quad v(2) = 3, \quad c(5) = -1, \quad v(5) = \frac{3}{2}.$$

We conclude that the characteristics that originate in $x < 0$ propagate with velocity 2, and those that originate in $x > 3$ propagate with velocity -1. In particular, the equations of the characteristic from $x = 0$ and $x = 3$ are $x = 2t$ and $x = 3 - t$, respectively. These intersect at $t = 1$, $x = 2$ and give rise to a shock. Here is the corresponding space-time diagram:



We have $\rho = 2$ and $\rho v(\rho) = 6$ to the left of the shock, and $\rho = 5$ and $\rho v(\rho) = 15/2$ to the right of the shock. We calculate the velocity of the shock from the Rankine–Hugoniot jump condition (**??**):

$$v_{\text{shock}} = \frac{6 - 15/2}{2 - 5} = \frac{1}{2},$$

and therefore the equation of the shock is $x = \frac{1}{2}t + \frac{3}{2}$. This completely characterizes the regions 1–5 in the figure above. Since densities are constants along the characteristics, we know that the density in the regions 1 and 4 is 2 and the density in the regions 3 and 5 is 5. It remains to calculate the density in region 2.

The characteristics originating at $x = h$ in region 2 are given by $x = c(\rho)\,t + h = (4-\rho)\,t + h$. But

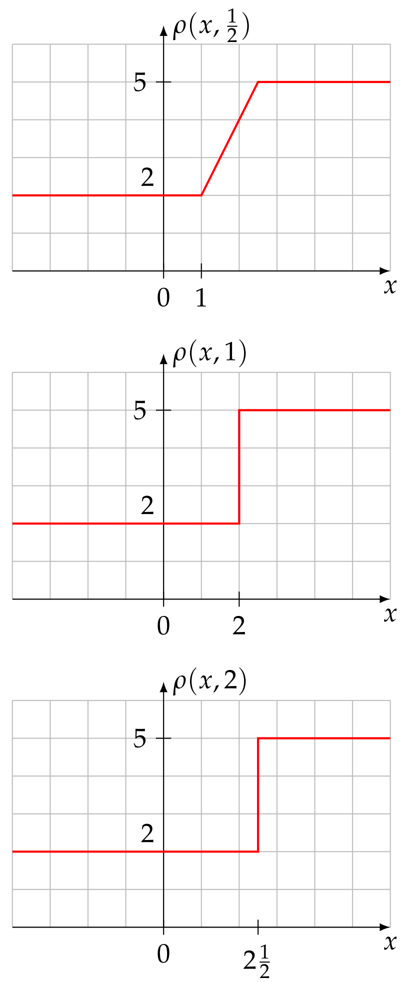
$$\rho(h,0) = f(h) = 2 + h,$$

from which we get $h = \rho - 2$. We conclude that $x = (4-\rho)\,t + (\rho - 2)$, whence $\rho = (x + 2 - 4t)/(1-t)$. We summarize our findings as:

$$\text{When } 0 \le t < 1: \qquad \rho(x,t) = \begin{cases} 2 & x < 2t, \\ \frac{x+2-4t}{1-t} & 2t < x < 3-t, \\ 5 & x > 3-t. \end{cases}$$

$$\text{When } t \ge 1: \qquad \rho(x,t) = \begin{cases} 2 & x < \frac{1}{2}t + \frac{3}{2}, \\ 5 & x > \frac{1}{2}t + \frac{3}{2}. \end{cases}$$

Here are the plots of $\rho(x,t)$ for $t = 1/2,\ 1,\ 2$.



Solution to Exercise 3.8. The initial density, $f(x)$, is given as a graph in this exercise. Let us observe that

$$f(x) = \begin{cases} 5 & \text{if } x < 0, \\ 5 + x & \text{if } 0 < x < 2, \\ 7 & \text{if } x > 2. \end{cases}$$

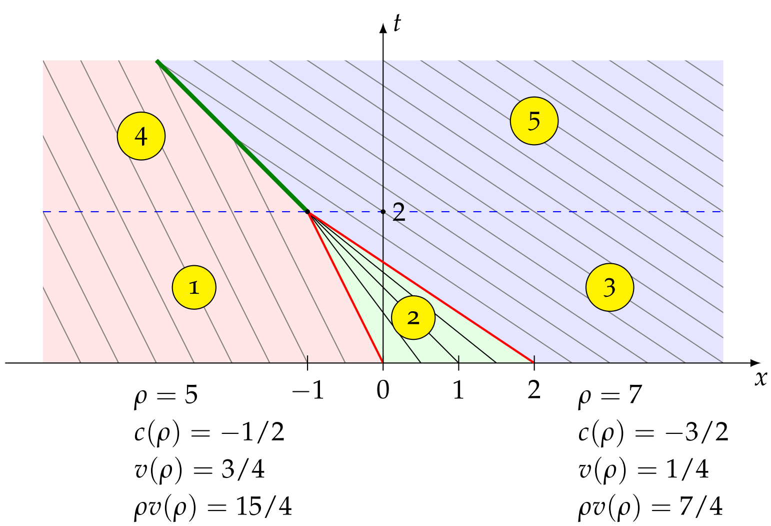
From equations (3.3) and (3.4), and the given data, we have

$$c(\rho) = 2 - \frac{1}{2}\rho, \quad v(\rho) = 2 - \frac{1}{4}\rho.$$

Consequently,

$$c(5) = -\frac{1}{2}, \quad v(5) = \frac{3}{4}, \quad c(7) = -\frac{3}{2}, \quad v(7) = \frac{1}{4}.$$

We conclude that the characteristics that originate in $x < 0$ propagate with velocity $-1/2$, and those that originate in $x > 2$ propagate with velocity $-3/2$. In particular, the equations of the characteristic from $x = 0$ and $x = 2$ are $x = -t/2$ and $x = 2 - 3t/2$, respectively. These intersect at $t = 2$, $x = -1$ and give rise to a shock. Here is the corresponding space-time diagram:



We have $\rho = 5$ and $\rho v(\rho) = 15/4$ to the left of the shock, and $\rho = 7$ and $\rho v(\rho) = 7/4$ to the right of the shock. We calculate the velocity of the shock from the Rankine–Hugoniot jump condition (**??**):

$$v_{\text{shock}} = \frac{15/4 - 7/4}{5 - 7} = -1,$$

and therefore the equation of the shock is $x = 1 - t$. This completely characterizes the regions 1–5 in the figure above. Since densities are constants along the characteristics, we know that the density in the regions 1 and 4 is 5 and the density in the regions 3 and 5 is 7. It remains to calculate the density in region 2.

The characteristics originating at $x = h$ in region 2 are given by $x = c(\rho)\,t + h = (2 - \frac{1}{2}\rho)\,t + h$. But

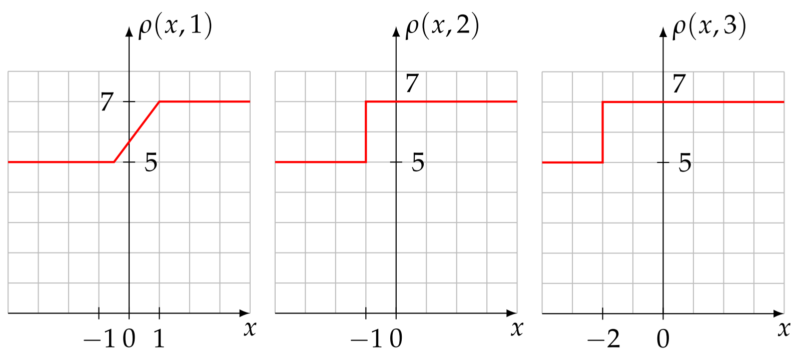
$$\rho(h, 0) = f(h) = 5 + h,$$

from which we get $h = \rho - 5$. We conclude that $x = (2 - \frac{1}{2}\rho)\,t + \rho - 5$, whence $\rho = (2x + 10 - 4t)/(2 - t)$. We summarize our findings as:

$$\text{When } 0 \le t < 2: \qquad \rho(x,t) = \begin{cases} 5 & x \le -\frac{1}{2}t, \\ \frac{2x+10-4t}{2-t} & -\frac{1}{2}t < x \le 2 - \frac{3}{2}t, \\ 7 & x > 2 - \frac{3}{2}t. \end{cases}$$

$$\text{When } t \ge 2: \qquad \rho(x,t) = \begin{cases} 5 & x < 1 - t, \\ 7 & x > 1 - t. \end{cases}$$

Here are the plots of $\rho(x, t)$ for $t = 1, 2, 3$.



Solution to Exercise 3.9. Solving (3.4) for ρ we obtain

$$\rho = \frac{1}{2}\rho_{\max}\left[1 - \frac{c}{v_{\max}}\right],$$

and in particular,

$$\frac{\rho}{\rho_{\max}} = \frac{1}{2}\left[1 - \frac{c}{v_{\max}}\right].$$

Then, in view of (3.3), we have

$$v(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right) = v_{\max}\left(1 - \frac{1}{2}\left[1 - \frac{c}{v_{\max}}\right]\right) = \frac{1}{2}v_{\max}\left[1 + \frac{c}{v_{\max}}\right].$$

We then calculate

$$\rho v(\rho) = \left(\frac{1}{2}\rho_{\max}\left[1 - \frac{c}{v_{\max}}\right]\right)\left(\frac{1}{2}v_{\max}\left[1 + \frac{c}{v_{\max}}\right]\right) = \frac{1}{4}\rho_{\max}v_{\max}\left[1 - \left(\frac{c}{v_{\max}}\right)^2\right].$$

Let's use indices 1 and 2 to indicate quantities to the left and to the right of the shock. We have

$$\rho_2 - \rho_1 = \frac{1}{2}\rho_{\max}\left[\left(1 - \frac{c_2}{v_{\max}}\right) - \left(1 - \frac{c_1}{v_{\max}}\right)\right] = \frac{1}{2}\cdot\frac{\rho_{\max}}{v_{\max}}\left(c_1 - c_2\right).$$

We also have

$$\rho_2 v(\rho_2) - \rho_1 v(\rho_1) = \frac{1}{4}\rho_{\max}v_{\max}\left[\left(1 - \left(\frac{c_2}{v_{\max}}\right)^2\right) - \left(1 - \left(\frac{c_1}{v_{\max}}\right)^2\right)\right] = \frac{1}{4}\cdot\frac{\rho_{\max}}{v_{\max}}\left[c_1^2 - c_2^2\right].$$

We conclude that

$$v_{\text{shock}} = \frac{\rho_2 v(\rho_2) - \rho_1 v(\rho_1)}{\rho_2 - \rho_1} = \frac{1}{2}\cdot\frac{c_1^2 - c_2^2}{c_1 - c_2} = \frac{1}{2}\left(c_1 + c_2\right).$$

Solution to Exercise 3.10. With the equation $v(\rho)$ given in this exercise, equation (3.1) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[\rho v_{\max}\left(1 - \left(\frac{\rho}{\rho_{\max}}\right)^2\right)\right] = 0.$$

or equivalently,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[v_{\max}\left(\rho - \frac{\rho^3}{\rho_{\max}^2}\right)\right] = 0.$$

By the chain rule of differentiation we obtain

$$\frac{\partial \rho}{\partial t} + v_{\max}\left(1 - \frac{3\rho^2}{\rho_{\max}^2}\right)\frac{\partial \rho}{\partial x} = 0.$$

We let

$$c(\rho) = v_{\max}\left(1 - \frac{3\rho^2}{\rho_{\max}^2}\right).$$

This replaces Section 3.1's equation (3.4).

We wish to solve the initial value problem

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0,$$
$$\rho(x,0) = f(x),$$

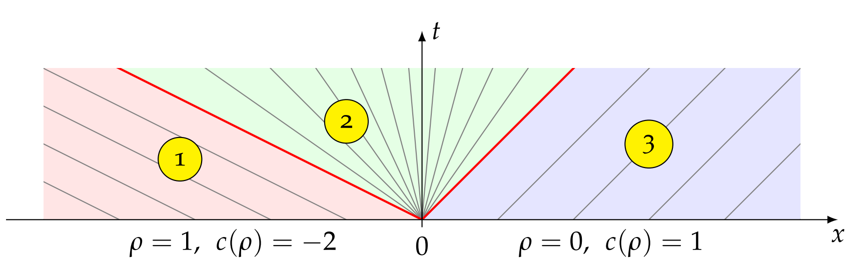
where

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

With $v_{\max} = 1$ and $\rho_{\max} = 1$, the expression for $c(\rho)$ becomes

$$c(\rho) = 1 - 3\rho^2.$$

We see that $c(0) = 1$. Therefore, the characteristics that originate on the positive x axis propagate with velocity 1. We also see that $c(1) = -2$. Therefore, the characteristics that originate on the negative x axis propagate with velocity -2. Here is the corresponding space-time diagram.



Density in the regions 1 and 3 is 1 and 0, respectively. The density in the rarefaction region 2 is calculated in the usual way—density is constant along the characteristics $x = c(\rho)t$, that is, along $x = (1 - 3\rho^2)t$. We solve this for ρ and obtain $\rho = \pm\sqrt{\frac{1}{3}\left(1 - \frac{x}{t}\right)}$. We discard the negative solution since density cannot be negative, and conclude that

$$\rho(x,t) = \begin{cases} 1 & x < -2t, \\ \sqrt{\frac{1}{3}\left(1 - \frac{x}{t}\right)} & -2t < x < t, \\ 0 & x > t. \end{cases}$$

Here is a plot of the solution at time $t = 1$: