Chapter 3

Solution to Exercise 3.1. From equation (3.4) and the given data we have $c(\rho) = 4 - \rho$. We see that $c(5) = -1$ and $c(3) = 1$. Therefore, the characteristics that originate on $x < 0$ and $x > 0$ propagate with velocities -1 and $+1$, respectively. The situation is depicted in regions 1 and 3 in Figure 14.3. In particular, the equations of the characteristics drawn in red are $x = \pm t$.

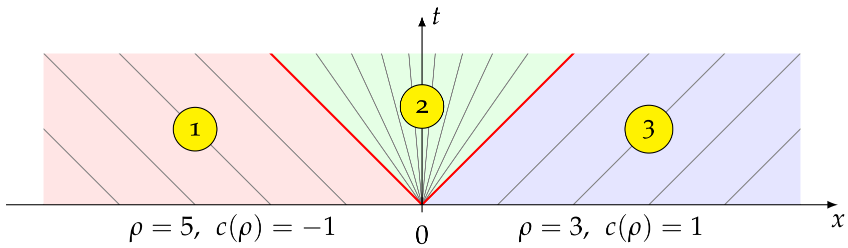


Figure 14.3: The space-time diagram for Exercise 1.

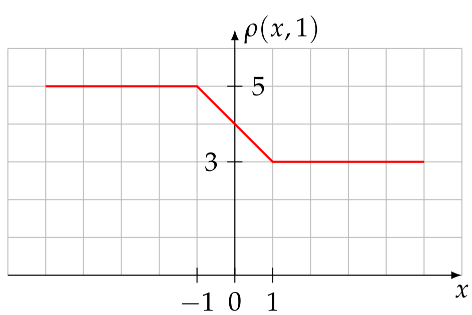
To calculate $\rho(x,t)$ in region 2 of that figure, note that the characteristic passing through (x,t) is $x = c(\rho)\,t = (4-\rho)\,t$. Solving for ρ we get $\rho = 4 - x/t$. We conclude that

$$\rho(x,t) = \begin{cases} 5 & \text{if } x < -t, \\ 4 - \frac{x}{t} & \text{if } -t < x < t, \\ 3 & \text{if } x > t. \end{cases}$$

In particular, at $t = 1$ we have

$$\rho(x,1) = \begin{cases} 5 & \text{if } x < -1, \\ 4 - x & \text{if } -1 < x < 1, \\ 3 & \text{if } x > 1. \end{cases}$$

Here is the graph of $\rho(x,1)$.



Solution to Exercise 3.3. As in the previous exercises, we have $c(\rho) = 4 - \rho$. Consequently $c(7) = -3$ and $c(5) = -1$, and therefore the characteristics that originate on $x < 0$ and $x > 0$ propagate with velocities -3 and -1, respectively. The situation is depicted in regions 1 and 3 in Figure 14.5. In particular, the equations of the characteristics drawn in red are $x = -3t$ and $x = -t$.

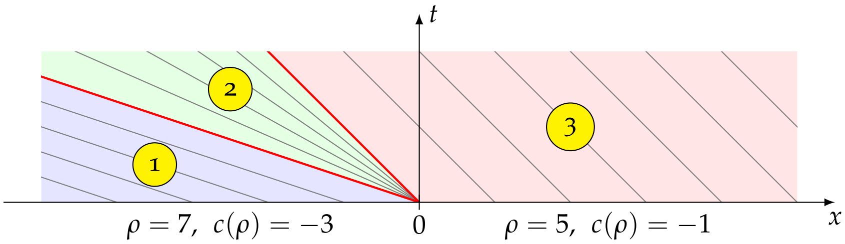


Figure 14.5: The space-time diagram for Exercise 3.

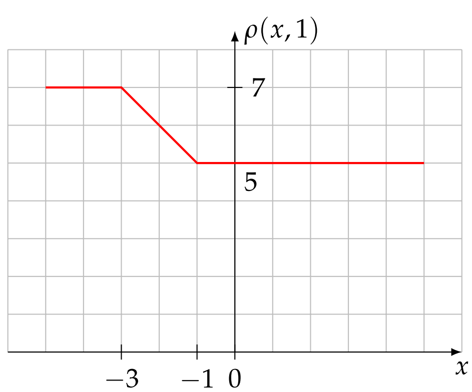
To calculate $\rho(x,t)$ in region 2 of that figure, note that the characteristic passing through (x,t) is $x = c(\rho)\,t = (4-\rho)\,t$. Solving for ρ we get $\rho = 4 - x/t$. We conclude that

$$\rho(x,t) = \begin{cases} 7 & \text{if } x < -3t, \\ 4 - \frac{x}{t} & \text{if } -3t < x < -t, \\ 5 & \text{if } x > -t. \end{cases}$$

In particular, at $t = 1$ we have

$$\rho(x,1) = \begin{cases} 7 & \text{if } x < -3, \\ 4 - x & \text{if } -3 < x < -1, \\ 5 & \text{if } x > -1. \end{cases}$$

Here is the graph of $\rho(x,1)$.



Solution to Exercise 3.5. The initial density, $f(x)$, is given as a graph in this exercise. Let us observe that

$$f(x) = \begin{cases} 7 & \text{if } x < 0, \\ 7 - 2x & \text{if } 0 < x < 1, \\ 5 & \text{if } x > 1. \end{cases}$$

As in the previous exercises, we have $c(\rho) = 4 - \rho$. Consequently $c(7) = -3$ and $c(5) = -1$, and therefore the characteristics that originate on $x < 0$ and $x > 1$ propagate with velocities -3 and -1, respectively. The situation is depicted in regions 1 and 3 in Figure 14.7. In particular, the equations of the characteristics drawn in red are $x = -3t$ and $x = 1 - t$.

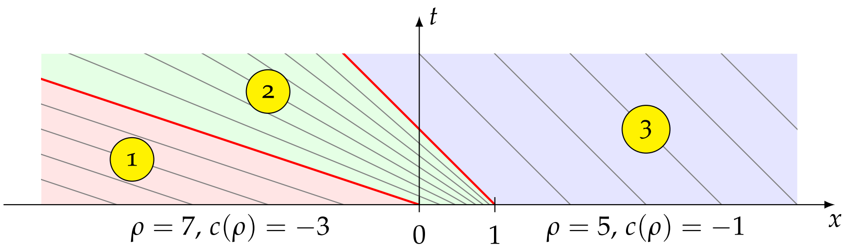


Figure 14.7: The space-time diagram for Exercise 5.

To calculate the density $\rho(x,t)$ within region 2, we look at a generic characteristic $x = c(\rho)\,t + \alpha = (4-\rho)\,t + \alpha$ that takes off from $x = \alpha$ at $t = 0$, where $0 < \alpha < 1$. The density at $x = \alpha$ is $\rho = f(\alpha) = 7 - 2\alpha$. Therefore $\alpha = \frac{1}{2}(7-\rho)$. It follows that $x = (4-\rho)\,t + \frac{1}{2}(7-\rho)$. We solve this for ρ and obtain

$$\rho = \frac{8t + 7 - 2x}{1 + 2t},$$

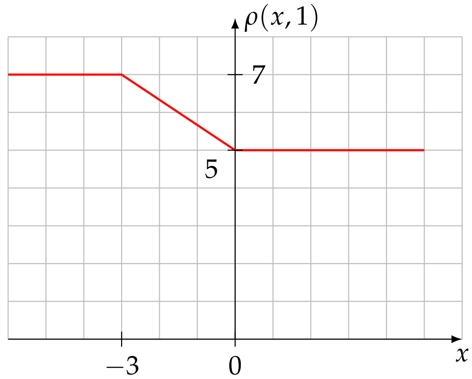
and therefore

$$\rho(x,t) = \begin{cases} 7 & \text{if } x < -3t, \\ \frac{8t+7-2x}{1+2t} & \text{if } -3t < x < 1 - t, \\ 5 & \text{if } x > 1 - t. \end{cases}$$

In particular, at $t = 1$ we have

$$\rho(x,1) = \begin{cases} 7 & \text{if } x < -3, \\ \frac{1}{3}(15 - 2x) & \text{if } -3 < x < 0, \\ 5 & \text{if } x > 0. \end{cases}$$

Here is the graph of $\rho(x,1)$.


$\rho(x,1)$
7
5
−3
0
x