Solution to Exercise 1.3. We solve the differential equations

$$\frac{dx}{dt} = 2, \quad \frac{du}{dt} = -u, \quad x(0) = h, \quad u(0) = f(h)$$

and obtain

$$x = 2t + h, \quad u = f(h)e^{-t}.$$

From the first equation we get $h = x - 2t$. We substitute that in the second equation and arrive at $u(x,t) = f\big(x - 2t\big)e^{-t}$.

Solution to Exercise 1.4. We solve the differential equations

$$\frac{dx}{dt} = x, \quad \frac{du}{dt} = 1, \quad x(0) = h, \quad u(0) = f(h)$$

and obtain

$$x = he^t, \quad u = f(h) + t.$$

From the first equation we get $h = xe^{-t}$. We substitute that in the second equation and arrive at $u(x,t) = f\left(xe^{-t}\right) + t$.

Solution to Exercise 1.5. We solve the differential equations

$$\frac{dx}{dt} = x, \quad \frac{du}{dt} = -2u, \quad x(0) = h, \quad u(0) = f(h)$$

and obtain

$$x = he^t, \quad u = f(h)e^{-2t}.$$

From the first equation we get $h = xe^{-t}$. We substitute that in the second equation and arrive at $u(x,t) = f\left(xe^{-t}\right)e^{-2t}$.

Solution to Exercise 1.8. We solve the differential equations

$$\frac{dx}{dt} = 2t, \quad \frac{du}{dt} = 2t, \quad x(0) = h, \quad u(0) = f(h)$$

and obtain

$$x = t^2 + h, \quad u = f(h) + t^2.$$

From the first equation we get $h = x - t^2$. We substitute that in the second equation and arrive at $u(x,t) = f(x - t^2) + t^2$.

Solution to Exercise 1.9. We solve the system of ODEs

$$\frac{dx}{dt} = -6t^2, \quad \frac{du}{dt} = -tu, \quad x(0) = h, \quad u(0) = f(h).$$

The first ODE yields $x = -2t^3 + h$, which we solve for h:

$$h = x + 2t^3.$$

In the second ODE we separate the variables as $\frac{du}{u} = -t\,dt$ then integrate to get $\ln u = -\frac{1}{2}t^2 + C$. Applying the initial condition yields $\ln f(x) = C$ and therefore $\ln u = -\frac{1}{2}t^2 + \ln f(x)$, which simplifies to

$$u = f(h)e^{-t^2/2}.$$

We conclude that

$$u(x,t) = f(x + 2t^3)e^{-t^2/2}.$$

Solution to Exercise 1.10. We solve the system of ODEs

$$\frac{dx}{dt} = -12t^2, \quad \frac{du}{dt} = -xu, \quad x(0) = h, \quad u(0) = f(h).$$

The first ODE yields $x = -4t^3 + h$, which we solve for h:

$$h = x + 4t^3.$$

We substitute the x calculated above into the second ODE

$$\frac{du}{dt} = -(-4t^3 + h)u$$

separate the variables

$$\frac{du}{u} = (4t^3 - h)\,dt,$$

and integrate:

$$\ln u = t^4 - ht + C.$$

Applying the initial condition yields $\ln f(x) = C$ and therefore $\ln u = t^4 - ht + \ln f(x)$, which simplifies to

$$u = f(h)e^{t^4 - ht}.$$

Plug in the previously calculated h to obtain

$$u(x,t) = f(x + 4t^3)e^{t^4 - (x+4t^3)t} = f(x + 4t^3)e^{-3t^4 - xt},$$

that is

$$u(x,t) = f(x + 4t^3)e^{-t(x+3t^3)}.$$

Solution to Exercise 1.13. We put the PDE in a form that matches (1.1a) by dividing it by u and rearranging the terms:

$$u_t - \frac{t+u}{u} u_x = 0.$$

This corresponds to $c(x,t,u) = -\frac{t+u}{u} = -1 - \frac{t}{u}$, $q(x,t,u) = 0$, and $f(x) = \frac{1}{1+x}$. The system of ODEs (1.3) takes the form

$$\frac{dx}{dt} = -1 - \frac{t}{u}, \tag{14.16a}$$

$$\frac{du}{dt} = 0, \tag{14.16b}$$

$$x(0) = h, \tag{14.16c}$$

$$u(0) = \frac{1}{1+h}. \tag{14.16d}$$

The ODE (14.16b) implies that u is a constant, let's say c_1. The initial condition (14.16d) says that $c_1 = \frac{1}{1+h}$.

The ODE (14.16a) now takes the form

$$\frac{dx}{dt} = -1 - (1+h)t,$$

which upon integration yields

$$x = -t - \frac{1+h}{2}t^2 + c_2,$$

where c_2 is another arbitrary constant. The initial condition (14.16c) implies that $c_2 = h$, and therefore

$$x = -t - \frac{1+h}{2}t^2 + h.$$

Following Section 1.2's road map, we solve this for h

$$h = \frac{2x + 2t + t^2}{2 - t^2},$$

which we then the substitute into $f(h)$ to arrive at the solution $u(x,t)$. We get

$$u(x,t) = f\left(\frac{2x+2t+t^2}{2-t^2}\right) = \frac{1}{1 + \dfrac{2x+2t+t^2}{2-t^2}},$$

which simplifies to

$$u(x,t) = \frac{2-t^2}{2+2x+2t}.$$

Chapter 2

Solution to Exercise 2.1. The volume of a slice of thickness dx of Figure 2.1's tube is $A\,dx$, therefore the mass of smoke generated there is $Aq(x,t)\,dx$ per unit time. Therefore the mass of the smoke generated per unit time within the entire control volume is

$$\int_a^b q(x,t)A\,dx.$$

We conclude that the mass of smoke generated within the control volume between the times t_1 and t_2 is

$$\int_{t_1}^{t_2}\int_a^b q(x,t)A\,dx\,dt,$$

and the equation of balance of mass becomes

$$\int_a^b \rho(x,t_2)A\,dx - \int_a^b \rho(x,t_1)A\,dx = \int_{t_1}^{t_2} \phi(a,t)A\,dt - \int_{t_1}^{t_2} \phi(b,t)A\,dt + \int_{t_1}^{t_2}\int_a^b q(x,t)A\,dx\,dt.$$

We divide through by the common factor A and rearrange the terms into

$$\int_a^b \Big[\rho(x,t_2) - \rho(x,t_1)\Big]\,dx + \int_{t_1}^{t_2}\Big[\phi(b,t) - \phi(a,t)\Big]\,dt - \int_{t_1}^{t_2}\int_a^b q(x,t)\,dx\,dt = 0.$$

This is the counterpart of the equation (2.1) under the current scenario.

If $\rho(x,t)$ and $\phi(x,t)$ are sufficiently differentiable, then we may apply the Fundamental Theorem of Calculus as before and arrive at

$$\int_{t_1}^{t_2}\int_a^b \left[\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) - q(x,t)\right]\,dx\,dt = 0.$$

Then with an argument by contradiction as before, we conclude that

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) - q(x,t) = 0.$$

This agrees with (2.3).

Solution to Exercise 2.2. The volume of a slice of thickness dx of Figure 2.1's tube is $A\,dx$, therefore the mass of radioactive smoke contained in it is $\rho(x,t)A\,dx$, and consequently, the mass of the radioactive smoke in the entire control volume is

$$\int_a^b \rho(x,t)A\,dx,$$

and therefore the rate of loss of that mass due to radioactivity is

$$\frac{d}{dt}\int_a^b \rho(x,t)A\,dx = \int_a^b \frac{\partial}{\partial t}\rho(x,t)A\,dx = \int_a^b \beta\rho(x,t)A\,dx,$$

and thus, the loss of the radioactive mass during the time interval t_1 to t_2 is

$$\int_{t_1}^{t_2}\int_a^b \beta\rho(x,t)A\,dx.$$

Then the equation of balance of mass becomes

$$\int_a^b \rho(x,t_2)A\,dx - \int_a^b \rho(x,t_1)A\,dx$$
$$= \int_{t_1}^{t_2} \phi(a,t)A\,dt - \int_{t_1}^{t_2} \phi(b,t)A\,dt - \int_{t_1}^{t_2}\int_a^b \beta\rho(x,t)A\,dx\,dt.$$

We divide through by the common factor A and rearrange the terms into

$$\int_a^b \Big[\rho(x,t_2) - \rho(x,t_1)\Big]\,dx + \int_{t_1}^{t_2} \Big[\phi(b,t) - \phi(a,t)\Big]\,dt$$
$$+ \beta\int_{t_1}^{t_2}\int_a^b \rho(x,t)\,dx\,dt = 0.$$

If $\rho(x,t)$ and $\phi(x,t)$ are sufficiently differentiable, then we may apply the Fundamental Theorem of Calculus as before and arrive at

$$\int_{t_1}^{t_2}\int_a^b \left[\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) + \beta\rho(x,t)\right]\,dx\,dt = 0,$$

from which it follows that

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) = -\beta\rho(x,t).$$

Solution to Exercise 2.3. Following the technique introduced in Section 1.2, we solve the system of ODEs

$$\frac{dx}{dt} = c, \tag{14.17a}$$

$$\frac{d\rho}{dt} = -\beta\rho. \tag{14.17b}$$

The general solution of (14.17a) is $x = ct + A$, and therefore the characteristics are parallel lines corresponding to velocity c. Some of these lines intersect the x axis and some the t axis, separating the first quadrant into regions 1 and 2, exactly as depicted in Figure 2.3. The novelty in the current problem is that the value of ρ is not constant along the characteristics—it varies along the characteristics according to equation (14.17b) whose general solution is $\rho = Be^{-\beta t}$. We proceed to determine the unknowns A and B. These take different forms in the regions regions 1 and 2 of Figure 2.3 on page 20.

In region 1 the characteristics meet the x axis where the density ρ is known to be $f(x)$. Let's look at the characteristic that meets the x axis at $x = h$. This corresponds to the initial conditions $x(0) = h$ and $\rho(0) = f(h)$ of the initial value problem (14.17) which yields the solution

$$x = ct + h, \quad \rho = f(h)e^{-\beta t}.$$

Eliminating h between these to leads to

$$\rho = f(x - ct)e^{-\beta t},$$

which is the solution of the IBVP in the region 1.

In region 2 the characteristics meet the t axis where the density ρ is known to be $\eta(t)$. Let's look at the characteristic that meets the t axis at $t = \tau$. This corresponds to the initial conditions $x(\tau) = 0$ and $\rho(\tau) = \eta(\tau)$ of the initial value problem (14.17) which yields the solution

$$x = c(t - \tau), \quad \rho = \eta(\tau)e^{-\beta(t-\tau)}.$$

Eliminating τ between these to leads to

$$\rho = \eta\Big(t - \frac{x}{c}\Big)e^{-\beta\frac{x}{c}},$$

which is the solution of the IBVP in the region 2. We conclude that

$$\rho(x,t) = \begin{cases} f(x - ct)e^{-\beta t} & \text{in region 1, that is, } x > ct, \\ \eta\Big(t - \frac{x}{c}\Big)e^{-\beta\frac{x}{c}} & \text{in region 2, that is, } x < ct. \end{cases}$$

Solution to Exercise 2.4. We mimic every step of the derivation of the equation of conservation of mass (2.3) but remain alert that the cross-sectional area is not a constant.

At any time t, the total mass of smoke within the tube is

$$\int_a^b \rho(x,t)A(x)\,dx.$$

The change of that mass content between times t_1 and t_2 is

$$\int_a^b \rho(x,t_2)A(x)\,dx - \int_a^b \rho(x,t_1)A(x)\,dx.$$

That change is due to smoke flowing in and out of the sections at a and b. According to the definition of flux, smoke enters the cross-section at $x = a$ at the rate of $\phi(a,t)A(a)$ per unit time, and leaves the cross section at $x = b$ at the rate of $\phi(b,t)A(b)$ per unit time. Therefore during the time period $t_1 < t < t_2$, the net gain of smoke through crossing the tube's boundaries is

$$\int_{t_1}^{t_2} \phi(a,t)A(a)\,dt - \int_{t_1}^{t_2} \phi(b,t)A(b)\,dt.$$

We conclude that

$$\int_a^b \rho(x,t_2)A(x)\,dx - \int_a^b \rho(x,t_1)A(x)\,dx \\ = \int_{t_1}^{t_2} \phi(a,t)A(a)\,dt - \int_{t_1}^{t_2} \phi(b,t)A(b)\,dt,$$

which we rearrange that into

$$\int_a^b A(x)\Big[\rho(x,t_2) - \rho(x,t_1)\Big]\,dx + \int_{t_1}^{t_2} \Big[A(b)\phi(b,t) - A(a)\phi(a,t)\Big]\,dt = 0.$$

Now, assuming that all functions are sufficiently differentiable, and in view of the Fundamental Theorem of Calculus, we have

$$A(x)\Big[\rho(x,t_2) - \rho(x,t_1)\Big] = \int_{t_1}^{t_2} A(x)\frac{\partial}{\partial t}\rho(x,t)\,dt,$$
$$A(b)\phi(b,t) - A(a)\phi(a,t) = \int_a^b \frac{\partial}{\partial x}\Big(A(x)\phi(x,t)\Big)\,dx,$$

whereby the previous equation takes the form

$$\int_a^b\int_{t_1}^{t_2} A(x)\frac{\partial}{\partial t}\rho(x,t)\,dt\,dx + \int_{t_1}^{t_2}\int_a^b \frac{\partial}{\partial x}\Big(A(x)\phi(x,t)\Big)\,dx\,dt = 0.$$

We interchange the order of the integrals on the right and then combine the terms into

$$\int_{t_1}^{t_2}\int_a^b \left[A(x)\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\Big(A(x)\phi(x,t)\Big)\right]\,dx\,dt = 0.$$

The expression within the square brackets is zero for the same reason as before, and therefore

$$A(x)\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\Big(A(x)\phi(x,t)\Big) = 0.$$

We differentiate the product term

$$A(x)\frac{\partial}{\partial t}\rho(x,t) + A'(x)\phi(x,t) + A(x)\frac{\partial}{\partial x}\phi(x,t) = 0,$$

where $A'(x)$ is the derivative of $A(x)$, and conclude that

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) = -\frac{A'(x)}{A(x)}\phi(x,t).$$

Note that this reduces to (2.3) if $A(x)$ is constant.