Math 404, Spring 2025 **Exam #2** **Name:** ____________________

- Please make an effort to write neatly, and insert a few words where necessary to get your ideas across. It's difficult to understand (and evaluate) mathematics in the absence of guiding words. ***I will award up to 2 bonus points*** if I find your work *neat, well-documented and easy to read.*
- Consider using the two extra blank sheets at the end for scratch work before making a neat final copy.
- No books, notes, and electronic devices on this exam.
- Each of the four problems is worth 10 points.
- *Cheers!*

The wave equation:

$$u_{tt} = c^2 u_{xx}$$

The general solution:

$$u(x,t) = F(x+ct) + G(x-ct)$$

D'Alembert's solution of the initial value problem corresponding to the initial condition $u(x,0) = f(x)$, $u_t(x,0) = g(x)$:

$$u(x,t) = \frac{1}{2}\big[f(x+ct) + f(x-ct)\big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi.$$

The heat equation:

$$u_t = k u_{xx}$$

Solution on $-\infty < x < \infty$ when the initial condition is the Heaviside function:

$$u(x,t) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right]$$

Solution on $-\infty < x < \infty$ when the initial condition is $u(x,0) = f(x)$:

$$u(x,t) = \int_{-\infty}^{\infty} G(x-s,t) f(s)\,ds$$

The heat kernel:

$$G(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$$

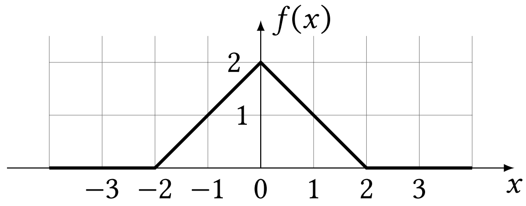
The definition of the erf function:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\,ds$$

1. Consider the following initial value problem for the function $u(x,t)$:

$$\begin{aligned} &u_{tt} = u_{xx} && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x) && -\infty < x < \infty, \\ &u_t(x,0) = 0 && -\infty < x < \infty, \end{aligned}$$

where f is as shown.



Sketch the graphs of $u(x,1)$, $u(x,2)$, and $u(x,3)$.

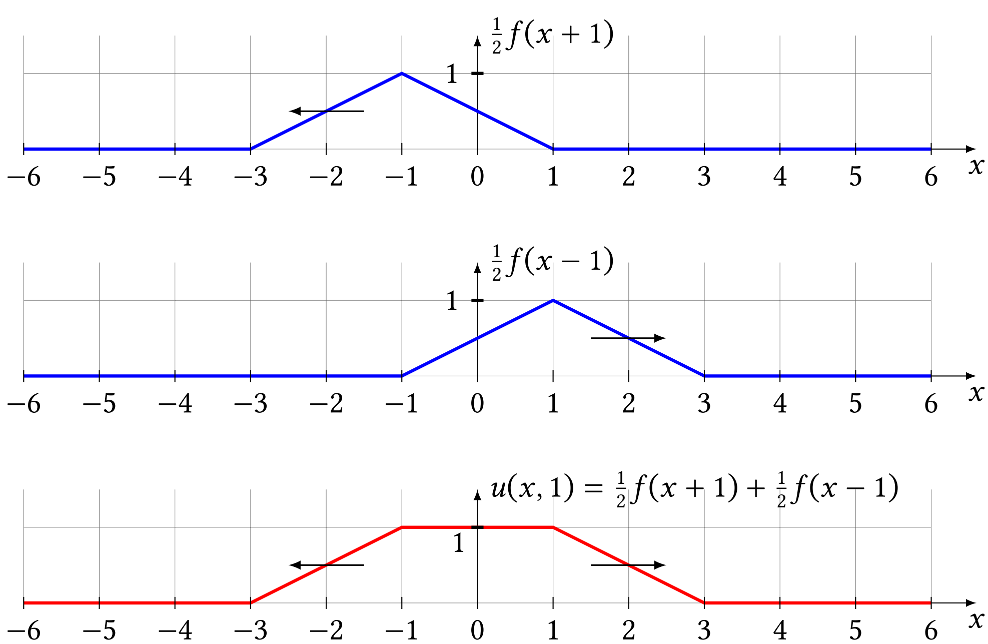
Solution: [***Like exercise 5.1***]

The solution of the IVP is given by d'Alembert's formula

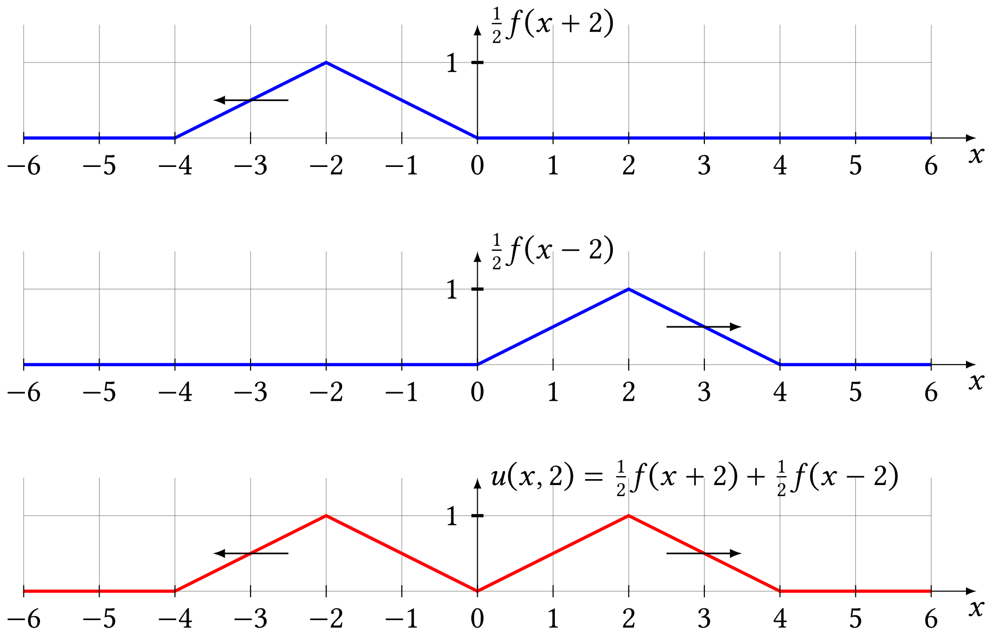
$$u(x,t) = \frac{1}{2} f(x+t) + \frac{1}{2} f(x-t).$$

Thus, the graph of f splits into two, and the two halves move to the left and to the right at the speed 1.

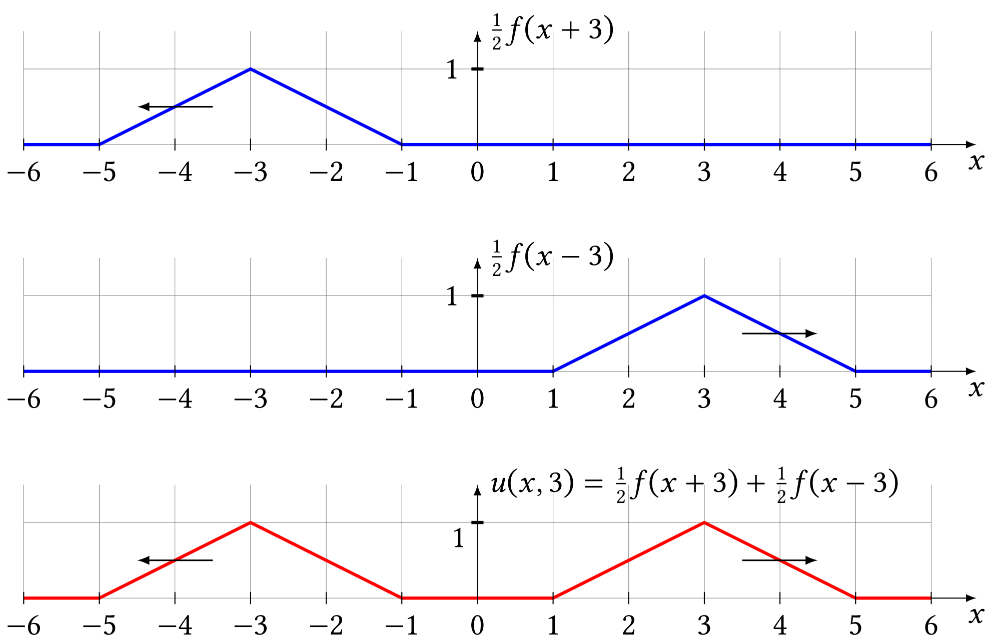
Solution at $t = 1$ shown in red



Solution at $t = 2$ shown in red



Solution at $t = 3$ shown in red



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Remark: It's possible to express $u(x,t)$ algebraically but that's not very informative. Nevertheless, let's see what it looks like. We have

$$f(x) = \begin{cases} 0 & x < -2 \\ 2+x & -2 \le x < 0 \\ 2-x & 0 \le x < 2 \\ 0 & x \ge 2 \end{cases}.$$

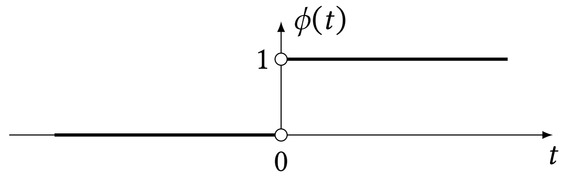
According to d'Alembert, the solution is $u(x,t) = \frac{1}{2}f(x+t) + \frac{1}{2}f(x-t)$, so we calculate

$$f(x+t) = \begin{cases} 0 & x+t < -2 \\ 2+x+t & -2 \le x+t < 0 \\ 2-x-t & 0 \le x+t < 2 \\ 0 & x+t \ge 2 \end{cases}, \qquad f(x-t) = \begin{cases} 0 & x-t < -2 \\ 2+x-t & -2 \le x-t < 0 \\ 2-x+t & 0 \le x-t < 2 \\ 0 & x-t \ge 2 \end{cases},$$

and conclude that

$$u(x,t) = \begin{cases} 0 & x+t < -2 \\ \frac{1}{2}(2+x+t) & -2 \le x+t < 0 \\ \frac{1}{2}(2-x-t) & 0 \le x+t < 2 \\ 0 & x+t \ge 2 \end{cases} + \begin{cases} 0 & x-t < -2 \\ \frac{1}{2}(2+x-t) & -2 \le x-t < 0 \\ \frac{1}{2}(2-x+t) & 0 \le x-t < 2 \\ 0 & x-t \ge 2 \end{cases}.$$

2. Consider a taut homogeneous rope stretched over $0 < x < \infty$ and initially at rest. At time $t = 0$ we quickly lift the endpoint $x = 0$ by one unit and keep it there. That is, the boundary condition is $u(0, t) = \phi(t)$ where ϕ is as in:



Find the rope's displacement $u(x, t)$ at all $x > 0$ and $t > 0$, and sketch the shape of the rope at some future time $t > 0$.

Solution: [***Like exercises 6.4 and 6.5***]

The general solution of the wave equation is

$$u(x, t) = F(x + ct) + G(x - ct).$$

In this problem there is no left-moving wave since we have no signals coming down from $+\infty$. So the solution is $u(x, t) = G(x - ct)$. Evaluating this at $t = 0$ we get

$$G(-ct) = u(0, t) = \begin{cases} 0 & t < 0, \\ 1 & t > 0. \end{cases}$$

Setting $-ct = \xi$ this becomes

$$G(\xi) = \begin{cases} 0 & -\frac{\xi}{c} < 0, \\ 1 & -\frac{\xi}{c} > 0, \end{cases}$$

which simplifies to (assuming $c > 0$)

$$G(\xi) = \begin{cases} 0 & \xi > 0, \\ 1 & \xi < 0, \end{cases}$$

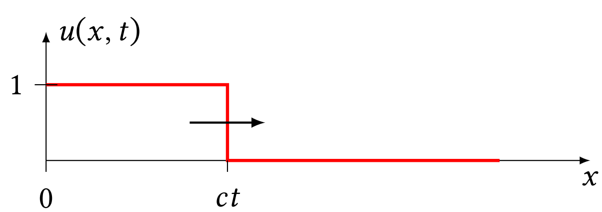
and therefore

$$u(x, t) = G(x - ct) = \begin{cases} 0 & x - ct > 0, \\ 1 & x - ct < 0, \end{cases}$$

or equivalently

$$u(x, t) = \begin{cases} 0 & x > ct, \\ 1 & x < ct. \end{cases}$$

Here is the shape of the rope at some time $t > 0$:

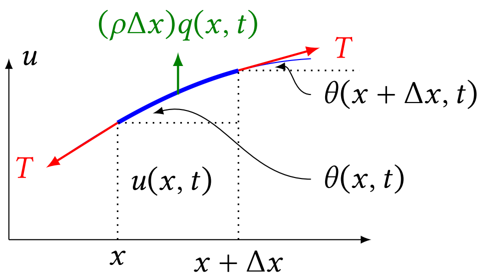


3. We place a taut steel wire in a magnetic field which applies a variable force of $q(x, t)$ per unit mass in the direction perpendicular to the wire. Find the PDE that describes the string's motion. Take ρ to be the mass per unit length of the wire, and T be the wire's tension.

Solution: [***Like exercise 5.5***]

We modify Section 5.1's calculations to account for the applied force. Thus, we apply Newton's law of motion to a small section of length Δx of the wire. The mass of that section is $\rho\Delta x$, and therefore the external force applied to it is $(\rho\Delta x)q(x, t)$.

The component of the vertical force at any location of the string is $T \sin\theta(x, t)$, where θ is the angle that the string makes with the x axis, as shown in the figure below.



Assuming θ is small, $\sin\theta$ may be replaced by $\tan\theta$ which is equal to u_x. Thus, Newton's law takes the form

$$(\rho\Delta x)u_{tt}(x, t) = T\big[u_x(x + \Delta x, t) - u_x(x, t)\big] + (\rho\Delta x)q(x, t).$$

We divide through by Δx

$$\rho u_{tt}(x, t) = T\frac{u_x(x + \Delta x, t) - u_x(x, t)}{\Delta x} + \rho q(x, t).$$

and let $\Delta x \to 0$. We thus arrive at the string's equation of motion

$$\rho u_{tt} = T u_{xx} + \rho q.$$

4. Solve the initial value problem for the heat equation

$$\begin{aligned} u_t &= k u_{xx} && -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x) && -\infty < x < \infty, \end{aligned}$$

where $f(x) = \begin{cases} 1 & \text{if } |x| < a \\ 0 & \text{otherwise} \end{cases}$ and a is a positive constant.

Express the solution in terms of the erf function.

Solution: [***Like exercise 7.10***]

The solution to the initial value problem of the heat equation is given on the cover sheet as

$$u(x,t) = \int_{-\infty}^{\infty} G(x-s,t) f(s)\, ds,$$

where G is the heat kernel. Substituting for the given f this becomes

$$u(x,t) = \int_{-a}^{a} G(x-s,t)\, ds.$$

and therefore

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-a}^{a} e^{-\frac{(x-s)^2}{4kt}}\, ds,$$

To evaluate the integral, we change the variable of integration from s to ξ where $\xi = \dfrac{x-s}{\sqrt{4kt}}$. Observe that

- $d\xi = -\dfrac{1}{\sqrt{4kt}}\, ds$, that is $ds = -\sqrt{4kt}\, d\xi$;
- when $s = -a$ we have $\xi = \frac{x+a}{\sqrt{4kt}}$;
- when $s = a$ we have $\xi = \frac{x-a}{\sqrt{4kt}}$;

and therefore the expression for $u(x,t)$ changes to

$$u(x,t) = -\frac{\sqrt{4kt}}{\sqrt{4\pi k t}} \int_{\frac{x+a}{\sqrt{4kt}}}^{\frac{x-a}{\sqrt{4kt}}} e^{-\xi^2}\, ds = -\frac{1}{\sqrt{\pi}} \int_{\frac{x+a}{\sqrt{4kt}}}^{\frac{x-a}{\sqrt{4kt}}} e^{-\xi^2}\, ds = \frac{1}{\sqrt{\pi}} \int_{\frac{x-a}{\sqrt{4kt}}}^{\frac{x+a}{\sqrt{4kt}}} e^{-\xi^2}\, ds.$$

Note the swapping of the integral's limits and the disappearance of the minus sign in front of it in the last step above.

The integral that we have arrived at looks very much related to the definition of erf. To make it look even closer to erf, we break it into two pieces:

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{\frac{x-a}{\sqrt{4kt}}}^{0} e^{-\xi^2}\, ds + \frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+a}{\sqrt{4kt}}} e^{-\xi^2}\, ds.$$

We evaluate the two integrals separately, and then add up the results. We have

$$\frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x+a}{\sqrt{4kt}}} e^{-\xi^2}\, ds = \frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x+a}{\sqrt{4kt}}} e^{-\xi^2}\, ds = \frac{1}{2} \operatorname{erf}\left(\frac{x+a}{\sqrt{4kt}}\right),$$

$$\frac{1}{\sqrt{\pi}} \int_{\frac{x-a}{\sqrt{4kt}}}^{0} e^{-\xi^2}\, ds = -\frac{1}{\sqrt{\pi}} \int_{0}^{\frac{x-a}{\sqrt{4kt}}} e^{-\xi^2}\, ds = -\frac{1}{2} \cdot \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x-a}{\sqrt{4kt}}} e^{-\xi^2}\, ds = -\frac{1}{2} \operatorname{erf}\left(\frac{x-a}{\sqrt{4kt}}\right),$$

and therefore

$$u(x,t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+a}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-a}{\sqrt{4kt}}\right)\right].$$

Remark: This problem may be solved through a shortcut as follows. We know that the solution of the initial value problem

$$u_t = k u_{xx}, \quad u(x,0) = H(x) \qquad [H \text{ is the Heaviside function}]$$

is

$$u(x,t) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right].$$

Therefore, the solution of

$$u_t = k u_{xx}, \quad u(x,0) = H(x \pm a)$$

is

$$u(x,t) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x \pm a}{\sqrt{4kt}}\right)\right].$$

Considering that our initial condition f may be expressed as

$$f(x) = H(x+a) - H(x-a),$$

we conclude that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x+a}{\sqrt{4kt}}\right)\right] - \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-a}{\sqrt{4kt}}\right)\right] \\ &= \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+a}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-a}{\sqrt{4kt}}\right)\right]. \end{aligned}$$