**Math 404, Spring 2025** **Exam #1** **Name:** ____________________

- Please make an effort to write neatly, and insert a few words where necessary to get your ideas across. It's difficult to understand (and evaluate) mathematics in the absence of guiding words. ***I will award up to 2 bonus points*** if I find your work *neat, well-documented and easy to read.*
- Consider using the two extra blank sheets at the end for scratch work before making a neat final copy.
- No books, notes, and electronic devices on this exam.
- Each of the four problems is worth 10 points.
- *Cheers!*

**Traffic flow equations**

$$v(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)$$

$$c(\rho) = \frac{d}{d\rho}\Big(\rho\, v(\rho)\Big) = v_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right)$$

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0$$

$$v_{\text{shock}} = \frac{\text{jump in } \rho v}{\text{jump in } \rho}$$

1. Find the solution of the initial value problem

$$\begin{aligned} &u_t + uu_x + 1 = 0 && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = x && -\infty < x < \infty. \end{aligned}$$

*Solution:* [***Like exercise 1.13***]

We rearrange the PDE into the standard form

$$u_t + uu_x = -1, \qquad u(x,0) = f(x) = x,$$

and find the characteristics through solving the system of ODEs

$$\frac{dx}{dt} = u, \quad \frac{du}{dt} = -1,$$

with the initial conditions[1]

$$x(0) = h, \quad u(0) = f(h) = h.$$

From the second ODE we get $u = -t + C$ for some constant $C$. Applying the initial condition $u(0) = h$ yields $C = h$, and therefore

$$u = -t + h.$$

Substituting this $u$ into the first ODE, we see that $\frac{dx}{dt} = -t + h$, whence $x = -\frac{1}{2}t^2 + ht + C$ for some (other) constant $C$. The initial condition $x(0) = h$ tells us $C = h$, and therefore

$$x = -\frac{1}{2}t^2 + ht + h.$$

We isolate $h$ in this to get $(1+t)h = x + \frac{1}{2}t^2$, that is,

$$h = \frac{x + \frac{1}{2}t^2}{1+t} = \frac{2x + t^2}{2(1+t)}.$$

Plugging that into the previously calculated expression for $u$ we arrive at

$$u = -t + \frac{2x + t^2}{2(1+t)} = \frac{-2t - 2t^2 + 2x + t^2}{2(1+t)} = \frac{2x - 2t - t^2}{2(1+t)}.$$

[1] *Important!* A ODE without an initial condition is like the sound of one hand clapping.
*Did you write down* the initial conditions in your work?

2. Solve the initial boundary value problem for the density function $\rho(x,t)$:

$$\begin{aligned} &\rho_t + 2\rho_x = -\rho && x > 0, \quad t > 0, \\ &\rho(x,0) = \frac{1}{1+x^2} && x > 0, \\ &\rho(0,t) = \cos t && t > 0. \end{aligned}$$

*Solution:* [***Like exercise 2.3***]

Figure 1 depicts this problem's space-time diagram.[2] The solution is obtained through solving the system of ODEs

$$\frac{dx}{dt} = 2, \qquad \frac{d\rho}{dt} = -\rho,$$

which has the general solution

$$x = 2t + A, \qquad \rho = Be^{-t}, \tag{1}$$

and therefore the characteristics consist of parallel lines of slope 2 (relative to the $t$ axis) as seen in the figure. The initial values that go with these ODEs are are determined differently depending on whether we are in region 1 or region 2.

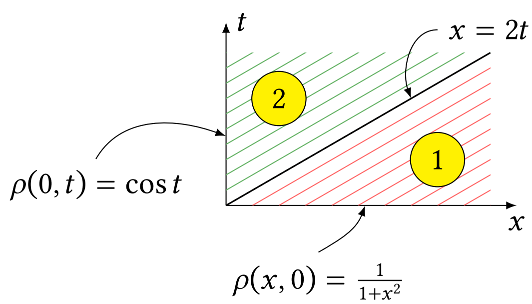



Figure 1: The space-time diagram of Problem 2.

In region 1, consider a characteristic that meets the $x$ axis at $x = h$. The density there is $\frac{1}{1+h^2}$, and therefore the initial conditions are[3]

$$x(0) = h, \qquad \rho(0) = \frac{1}{1+h^2}.$$

Substituting these into the general solution (1) we get $A = h$ and $B = \frac{1}{1+h^2}$, and therefore $x = 2t + h$, $\rho = \frac{1}{1+h^2}e^{-t}$. Eliminating $h$ between these two yields the density in region 1:

$$\rho = \frac{1}{1+(x-2t)^2}e^{-t}, \qquad x > 2t.$$

Now let's look at a characteristic in region 2 that meets the $t$ axis at some $t = \tau$. The density there is $\cos\tau$ and therefore the boundary conditions are

$$x(\tau) = 0, \qquad \rho(\tau) = \cos\tau.$$

[2] Did you draw the space-time diagram in your work? Attempting to solve this problem without that diagram is like searching for a black cat in a pitch dark room.

[3] See the footnote on the previous page.

Plugging these into the general solution (1) we get the system

$$0 = 2\tau + A, \qquad \cos\tau = Be^{-\tau},$$

which has the solution $A = -2\tau$, $B = e^{\tau}\cos\tau$, and therefore

$$x = 2t - 2\tau, \qquad \rho = \left(e^{\tau}\cos\tau\right)e^{-t} = e^{-(t-\tau)}\cos\tau.$$

To eliminate $\tau$ between these equations, we observe that the first equation implies that $\tau = t - \frac{x}{2}$ and $t - \tau = \frac{x}{2}$. We thus we arrive at the solution in region 2:

$$\rho = e^{-x/2}\cos\left(t - \frac{x}{2}\right), \qquad x < 2t.$$

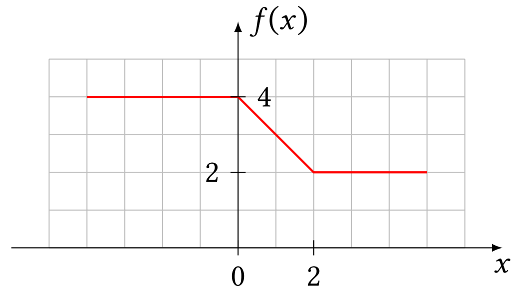
In summary, the solution of the IBVP is

$$\rho(x,t) = \begin{cases} \frac{1}{1+(x-2t)^2}e^{-t} & x > 2t \quad \text{(region 1)}, \\ e^{-x/2}\cos\left(t - \frac{x}{2}\right) & x < 2t \quad \text{(region 2)}. \end{cases}$$

3. (a) [8 pts] Solve the initial value problem of the traffic flow with the parameters

$$v_{\max} = 3, \qquad \rho_{\max} = 6,$$

and the initial condition $\rho(x,0) = f(x)$ as shown:




(b) [2 pts] Sketch the graph of $\rho(x,2)$.

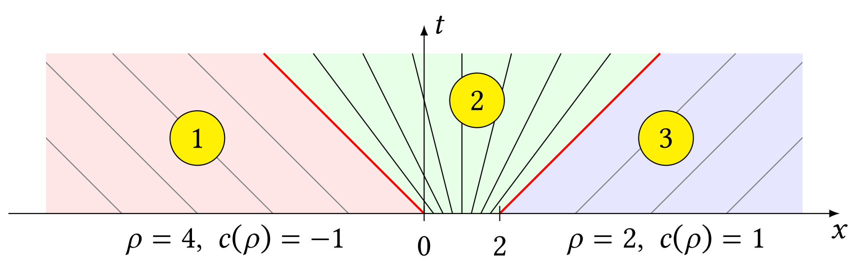
*Solution:* We have

$$c(\rho) = v_{\max}\left(1 - \frac{2\rho}{\rho_{\max}}\right) = 3\left(1 - \frac{2\rho}{6}\right) = 3 - \rho,$$

and in particular

$$c(4) = -1, \quad c(2) = 1.$$

This leads to the following space-time diagram.




Since the density is constant along the characteristics, the densities in regions 1 and 3 are $\rho = 4$ and $\rho = 2$, respectively. It remains to determine the density in region 2.

Here is the equation of the initial condition $f(x)$:

$$f(x) = \begin{cases} 4 & x \le 0, \\ 4 - x & 0 < x \le 2, \\ 2 & x > 2, \end{cases}$$

Characteristics that originate in the interval $0 < x < 2$ are given by $x = c(\rho)t + h = (3-\rho)t + h$. We isolate $h$ and get $h = x - (3-\rho)t$. But density is constant along that characteristic, therefore $\rho = 4 - h$, and therefore

$$\rho = 4 - \big[x - (3-\rho)t\big].$$

We solve this for $\rho$ and obtain

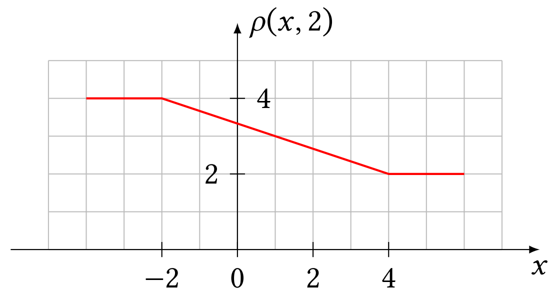
$$\rho = \frac{4 - x + 3t}{1 + t}.$$

We conclude that

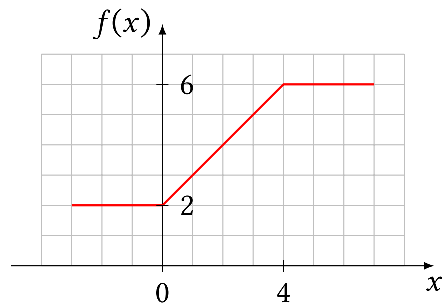
$$\rho(x,t) = \begin{cases} 4 & x \le -t, \\ \dfrac{4 - x + 3t}{1+t} & -t < x \le 2+t, \\ 2 & x > 2+t. \end{cases}$$

In particular, at $t = 2$ we have

$$\rho(x,2) = \begin{cases} 4 & x \le -2, \\ \dfrac{10 - x}{3} & -2 < x \le 4, \\ 2 & x > 4. \end{cases}$$



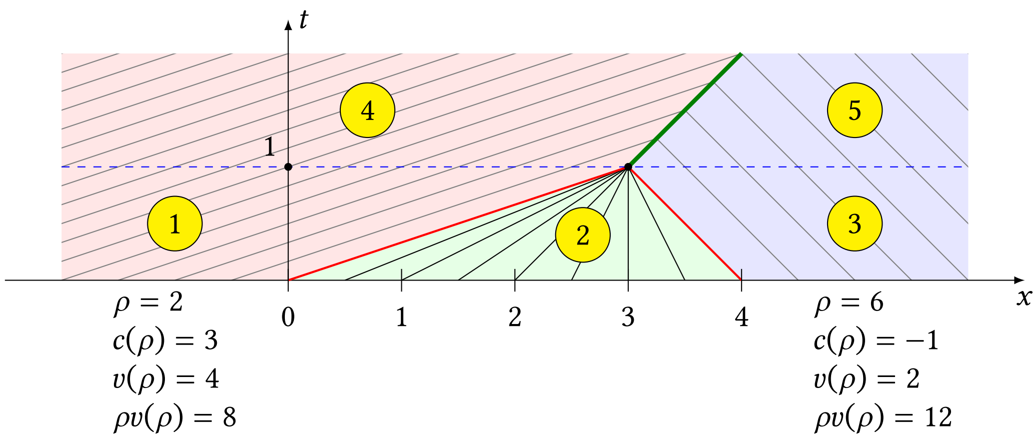
4. Solve the traffic flow problem with the parameters $v_{\max} = 5$, $\rho_{\max} = 10$ and the initial condition $\rho(x, 0) = f(x)$ as in the graph below.




*Solution:* From the formulas on the cover page we have:

$$v(\rho) = 5\left(1 - \frac{\rho}{10}\right) = 5 - \frac{1}{2}\rho, \qquad c(\rho) = 5\left(1 - \frac{2\rho}{10}\right) = 5 - \rho.$$

We build the space-time diagram as usual:




The characteristic velocities in regions 1 and 3 are 3 and $-1$, respectively, and therefore the equations of the characteristics originating at $x = 0$ and $x = 4$ (drawn in red) are $x = 3t$ and $x = 4 - t$, respectively. Those characteristics intersect at $x = 3$, $t = 1$, at which point a shock develops.

We split the analysis into the time ranges $0 < t < 1$ (before the shock) and $t > 1$ (after the shock). This results in the five distinct regions depicted in the diagram. Regions 1, 2, and 3 are given as

$$\text{region } 1 = \Big\{(x, t) \,:\, x < 3t\Big\},$$
$$\text{region } 2 = \Big\{(x, t) \,:\, 3t < x < 4 - t\Big\},$$
$$\text{region } 3 = \Big\{(x, t) \,:\, x > 4 - t\Big\}.$$

Specifications for the regions 4 and 5 will be obtained later after we find the equation of the shock that separates them.

The density $\rho(x, t)$ in the region 1 (and also 4) is inherited from that on the negative $x$ axis, and therefore it is a constant 2. The density $\rho(x, t)$ in the regions 3 (and also 5) is inherited from that on the $x > 4$ interval and therefore it is a constant 6.

To calculate the density in region 2, we translate the graph of the initial condition $f(x)$ into a formula:

$$f(x) = \begin{cases} 2 & \text{if } x < 0, \\ 2 + x & \text{if } 0 < x < 4, \\ 6 & \text{if } x > 4, \end{cases} \tag{$*$}$$

We recall that the characteristic that takes off from the $x$ axis at $x = h$ is given by $x = c(\rho)t + h$, that is, $x = (5 - \rho) + h$. But from $(*)$ we see that the density at $x = h$ is $2 + h$. The density remains constant along the characteristic, and therefore the $\rho = 2 + h$. We conclude that $h = \rho - 2$, and therefore $x = (5 - \rho)t + \rho - 2$. We solve this for $\rho$ and obtain $\rho = (x + 2 - 5t)/(1 - t)$. We conclude that

$$\rho(x,t) = \begin{cases} 2 & \text{in regions 1 and 4,} \\ \dfrac{x + 2 - 5t}{1 - t} & \text{in region 2,} \\ 6 & \text{in region 3 and 5.} \end{cases} \tag{$**$}$$

To complete the analysis, we need to characterize the regions 4 and 5. The velocity of the shock is given by the Rankine–Hugoniot condition[4]:

$$v_{\text{shock}} = \frac{\text{jump in } v\rho(v)}{\text{jump in } \rho} = \frac{(6)(2) - (2)(4)}{6 - 2} = 1.$$

The shock starts out at $x = 3$, $t = 1$, and therefore its equation is $x - 3 = \frac{1}{2}(t - 1)$, which simplifies to $x = t + 2$. We conclude that the regions 4 and 5 are given as

$$\text{region } 4 = \Big\{ (x,t) \, : \, x < t + 2 \Big\},$$
$$\text{region } 5 = \Big\{ (x,t) \, : \, x > t + 2 \Big\}.$$

This, along with $(**)$ determines $\rho(x,t)$ everywhere.

[4]Alternatively, the velocity of the shock may be calculated from the result of one of our homework problems where we showed that the shock velocity is the average of the characteristic velocities on its two sides. In the current case those velocities are 3 and $-1$, and therefore the average is 1.