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A First Course in Partial Differential Equations with Animations

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First printing, whenever

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Preface

This electronic textbook is different from other conventional introductory textbooks on partial differential equations (PDEs) in that almost every problem analyzed in the textbook is accompanied with a movie-like animation that shows how the calculated solution evolves in time. That's very different from the traditional printed textbooks where the time-dependence of the solutions are left to the reader's imagination. Our hope is that this will help liven up the matter and lead to a deeper understanding and appreciation of the subject.

Adding animations has enabled us to add topics that are rarely if ever covered in conventional textbooks, such as advection of oil drops suspended in water flowing through a pipe with variable cross-section (Chapter 2), or reflection of waves in a piecewise homogeneous taut string, or interaction of waves with a point mass attached to that string (Chapter 6). These would be difficult to motivate without accompanying animations.

Another difference from the approach in prevailing textbooks is the early introduction of the Sturm–Liouville theory, which then enables us to solve a host of interesting problems with varying types of boundary conditions in the form of eigenfunction expansions.

Animations are an integral part of this textbook. Some essential ideas will be lost or remain obscure if you don't have the ability to view the animations.

How to play the animations

Animations are an integral part of this textbook. Some essential ideas will be lost or remain obscure if you don't have the ability to view the animations.

Animations are marked by links like this. Clicking on the link will open open the animation in a web browser. You may interact with the animation in various ways. You may play, pause, or manually step through the frames by dragging a slider.

You have the choice of reading this book on a stand-alone PDF viewer or within a Web browser. The manner of your interaction with the document will vary, depending on that choice.

- If reading on a stand-alone PDF viewer, clicking an animation link will open the animation in your computer's default browser. If the browser is already open, the animation will open a new tab.[1] Each animation will open a new tab. You may want to close a tab one you are done in order not to overwhelm your browser.

- If reading the book within a browser, left-clicking an animation link will replace the book view with the animation. Clicking the browser's "go back" button will return to the book but unfortunately it will go to the book's cover page, and thus you will lose your original place in the book.

  To avoid that issue, open the animation by *right-clicking* the link. Most browsers will pop up a menu which will offer a choice of "Open link in a new tab" or something similar.

[1] This assumes that your Web browser supports tabs. Most modern browsers do.

1
First Order PDEs

Our aim in this chapter is to solve the initial value problem

$$u_t + c(x,t,u)u_x = q(x,t,u) \qquad -\infty < x < \infty, \quad t > 0, \tag{1.1a}$$
$$u(x,0) = f(x). \qquad -\infty < x < \infty. \tag{1.1b}$$

for the unknown $u = u(x,t)$, where the functions c, q, and the *initial condition* f, are given.

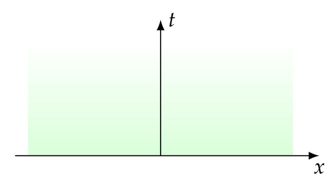


Figure 1.1: The initial condition (1.1b) prescribes the value of u along the x axis, that is, at $t = 0$. The PDE (1.1a) extrapolates that data into the half-space $t > 0$.

The domain of the function u is the upper-half of the xt plane in Cartesian coordinate system, partially tinted in Figure 1.1. The initial condition (1.1b) specifies the value of u at the bottom edge of that domain. It is the job of the PDE (1.1a) to extrapolate from that data into the upper-half plane and determine the values of u everywhere there. We will learn how that is done in Section 1.2.

Equation (1.1a) is called a *quasilinear partial differential equation* in recognition of the fact that although it is nonlinear, the derivatives u_t and u_x appear linearly in it. If the coefficient c does not depend on the unknown u, then the equation is said to be *semilinear*:

$$u_t + c(x,t)u_x = q(x,t,u).$$

If, in addition, q is linear in u, as in

$$u_t + c(x,t)u_x = \alpha(x,t)u + \beta(x,t),$$

then the equation is said to be *linear*. We will develop a procedure for solving the quasilinear initial value problem (1.1). That procedure will naturally apply to linear and semilinear equations as well since the latter are special cases of the full quasilinear case.

Toward that end, we introduce the idea of a *space-time diagram* which is an extremely useful mathematical construct that helps us gain insight into processes that evolve in time. We proceed with the analysis of the PDE afterward.

1.1 *The space-time diagram*

The gray strip at the bottom of the space-time diagram is meant to represent a one-lane highway. Four vehicles, marked A, B, C, D, move along it at various velocities. Vehicle A moves to the left with velocity -1. Vehicle B is stopped; it does not move at all. Vehicles C and D move to the right with velocities $1/2$ and 2.

Without an animation facility, it would be difficult, if not impossible, to convey the motion in a static one-dimensional drawing. The two-dimensional drawing in the xt plane, called the *space-time diagram*, however, can convey the motion quite readily. At any time t, a vehicle's position corresponds to a point in the space-time diagram. Over time, that point traces out a curve on the diagram, called the *vehicle's path in space-time*. Since our four vehicles are moving at constant velocities, their paths in space-time are straight lines. The paths would be curved if the velocities were other than constants.

Vehicle B is not moving, and therefore its path in space-time is a vertical line (constant x). Vehicle D is moving faster that vehicle C, therefore its path is inclined more toward the x axis as compared to that of C. In fact, the velocity is dx/dt, and therefore the slope of the path in space-time *relative to the time axis* equals the velocity.

1.2 *Solving a first order quasilinear PDE*

We now turn to the question of solving the initial value problem (1.1). Imagine, for the moment, that the solution $u(x,t)$ has been obtained somehow. Also imagine an as yet arbitrary path $x = x(t)$ in the space-time. Then $u(x(t),t)$, the value of the the solution along that path, is a function of the single variable t. We apply the chain rule of differentiation to determine the rate of change of that value

$$\frac{d}{dt}u(x(t),t) = u_x\big(x(t),t\big)\frac{d}{dt}x(t) + u_t\big(x(t),t\big). \tag{1.2}$$

Comparing this with (1.1a), we observe that if $x(t)$ is chosen so that $\frac{dx}{dt} = c(x,t,u)$, then (1.2) reduces to $\frac{du}{dt} = q(x,t,u)$, where $u(t) = u\big(x(t),t\big)$.[1] Let's write those two equations here together for convenient reference:

$$\frac{dx}{dt} = c(x,t,u), \tag{1.3a}$$

$$\frac{du}{dt} = q(x,t,u). \tag{1.3b}$$

This is a system of two first order (and generally nonlinear) coupled ODEs in the two unknowns $x(t)$ and $u(t)$. Let's say the path $x = x(t)$ starts out at $x = h$ at $t = 0$, that is, $x(0) = h$. Then from the initial

[1] Note the blatant abuse of notation here. In (1.1a), $u(x,t)$ is a function of *two* independent variables x and t, and x is independent of t, while here $x(t)$ depends on t, and $u(t)$ is a function of *one* variable. It's possible to avoid the double-meanings of u and x by introducing dedicated symbols for what we have clumsily called $x(t)$ and $u(t)$ here. That's done in this chapter's appendix (Section 1.4). The price you pay there is the significantly more verbose notation. Customarily one puts up with the clumsy $x(t)$ and $u(t)$ notation for the sake of brevity.

condition (1.1b) and $u(t) = u\big(x(t), t\big)$ we see that $u(0) = u\big(x(0), 0\big) = u(h, 0) = f(h)$. These supply the initial conditions

$$x(0) = h, \quad u(0) = f(h) \tag{1.3c}$$

to go along with the ODEs (1.3a) and (1.3b). Let

$$x = X(t, h), \quad u = U(t, h). \tag{1.4}$$

be the solution of the initial value problem (1.3a)–(1.3c). We eliminate h between the two equations (1.4)—typically by solving the first equation for h and inserting that h into the second equation, although that's not the only way—and thus arrive at an equation involving x, t, and u. We solve that to obtain u as a function of x and t, that is $u(x, t)$, which is the solution of the initial value problem (1.1).

Definition 1.1. The path $x = x(t)$ in space-time determined by the solution $x = X(t, h)$ of the initial value problem (1.3a)–(1.3c) is called a *characteristic curve* or simply a *characteristic* of the PDE (1.1a).

There is a continuum of characteristic curves, one per choice of the parameter h. Figure 1.2 shows the characteristic curve $x = x(t)$ that starts at $x = h$, and the curve $u(t) = u\big(x(t), t\big)$ that it traces on the solution surface $z = u(x, t)$.

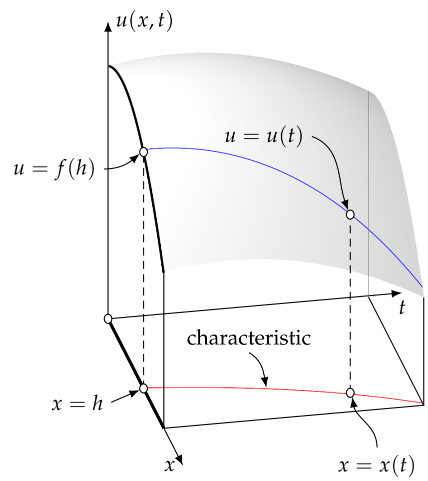


Figure 1.2: The surface is the graph of $z = u(x, t)$ of the solution of the initial value problem (1.1). The characteristic curve that starts out at $x = h$ is drawn in red. The curve $u(t) = u\big(x(t), t\big)$ that it traces on the surface is drawn in blue.

Remark 1.1. Equation (1.3a) says that the characteristic curves propagate with velocity $c(x, t, u)$. Equation (1.3b) says that the rate of change of u along a characteristic curve is $q(x, t, u)$. In the special case when $q = 0$, the value of the solution u remains constant along characteristics. That special case arises quite often in applications.

Remark 1.2. There are two major hurdles in implementing the solution method outlined in this section. First, the system of differential equations (1.3) may be difficult or impossible to solve analytically. Second, eliminating h between the two algebraic equations (1.4) may not be possible analytically. If all else fails, one may have to resort to numerical calculations to find an approximate solution to the PDE.

1.3 *Examples*

In the following subsections we present a variety of concrete cases of the initial value problem (1.1) and find their solutions. As pointed out earlier, not all such PDEs admit simple analytical solutions. Don't get a false impression—the ones treated in the following subsections have been hand-picked specifically for being solvable symbolically.

1.3.1 A very simple linear equation

Let us apply the Section 1.2's ideas to solve the linear initial value problem

$$u_t + cu_x = 0, \tag{1.5a}$$
$$u(x,0) = f(x), \tag{1.5b}$$

where c is a constant. The initial value problem (1.3)–(1.3c) reduces to

$$\frac{dx}{dt} = c, \quad \frac{du}{dt} = 0, \quad x(0) = h, \quad u(0) = f(h),$$

which immediately yields

$$x = ct + h, \quad u = f(h).$$

Following the procedure outlined in Section 1.2, we solve the first equation for h and obtain $h = x - ct$. Then we substitute that into the second equation and arrive at $u = f(x - ct)$, which is better written as

$$u(x,t) = f(x - ct). \tag{1.6}$$

This is the solution of the initial value problem (1.5). We observe that at any time t the solution is obtained by translating the initial data $f(x)$ by ct along the x axis, and that may be viewed as a wave that travels at velocity c.[2] The simple advection demonstrates this with $c = 1$ and

$$f(x) = \begin{cases} \cos^2\left(\frac{1}{2}\pi x\right) & \text{if } |x| < 1, \\ 0 & \text{otherwise} \end{cases} \tag{1.7}$$

[2] The wave travels in the increasing direction of the x axis when $c > 0$, and in the opposite direction when $c < 0$.

The characteristic curves in this simple case are the family of the parallel lines $x = ct + h$ corresponding to velocity c in space-time, as seen in the adjacent diagram. Since the right-hand side of the PDE (1.5a) is zero, the solution u remains constant along each characteristic. The value of that constant is determined by the value of the initial condition f at the point where the characteristic line intersects the x axis. One thinks of the characteristic lines as conduits through which the initial condition propagates in space-time.

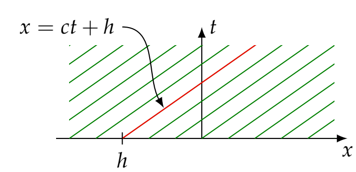


Figure 1.3: The characteristics of the PDE (1.5a) are the family of parallel lines $x = ct + h$. The solution u is constant along each characteristic line. Along the highlighted characteristic, drawn in red, that constant is $f(h)$.

1.3.2 A linear equation with variable coefficients

Let's solve the initial value problem

$$(1 + t)u_t + xtu_x = u, \tag{1.8a}$$
$$u(x,0) = 1 - x. \tag{1.8b}$$

We divide the PDE through by $1 + t$ in order to put it in the standard form (1.1a), and identify the equation's parts as $c(x,t,u) = xt/(1 +$

$t)$, $q(x,t,u) = u/(1+t)$. The initial value problem consisting of the equations (1.3a)–(1.3c) takes the form

$$\frac{dx}{dt} = \frac{xt}{1+t}, \quad \frac{du}{dt} = \frac{u}{1+t}, \quad x(0) = h, \quad u(0) = 1-h,$$

whose solution is[3]

$$x = \frac{he^t}{1+t}, \quad u = (1-h)(1+t).$$

[3] These ordinary differential equations may be solved through separation of variables or the integrating factor method.

Solving the first equation for h, we get $h = xe^{-t}(1+t)$. Substituting this into the second equation we arrive at

$$u(x,t) = (1+t)\Big[1 - xe^{-t}(1+t)\Big].$$

Have a look at the solution's animation in linear variable coefficients.

1.3.3 *A semilinear equation*

Here we solve the initial value problem

$$u_t + xtu_x = -u^2,$$
$$u(x,0) = x.$$

This corresponds to $c(x,t,u) = xt$ and $q(x,t,u) = -u^2$. The initial value problem(1.3) takes the form

$$\frac{dx}{dt} = xt, \quad \frac{du}{dt} = -u^2, \quad x(0) = h, \qquad u(0) = h,$$

whose solution is[4]

$$x = he^{t^2/2}, \quad u = \frac{1}{t+1/h}.$$

[4] These ordinary differential equations are easily solved through separation of variables.

Solving the first equation for h, we get $h = xe^{-t^2/2}$. Substituting this into the second equation and simplifying, we arrive at

$$u(x,t) = \frac{x}{xt + e^{t^2/2}}.$$

1.3.4 *A quasilinear equation*

Let's look at

$$u_t + uu_x = x,$$
$$u(x,0) = 2 - 3x.$$

This corresponds to $c(x,t,u) = u$, $q(x,t,u) = x$. The initial value problem(1.3) takes the form

$$\frac{dx}{dt} = u, \quad \frac{du}{dt} = x, \quad x(0) = h, \quad u(0) = 2 - 3h,$$

Each of the two ODEs involves the unknowns x and u, and therefore they cannot be solved individually; they need to be solve together as a system. One way of doing that is to eliminate u between the two ODEs. Toward that end, we differentiate the first ODE with resepct to t and get $\frac{d^2x}{dt^2} = \frac{du}{dt}$. We then substitute for $\frac{du}{dt}$ from the second ODE and arrive at $\frac{d^2x}{dt^2} = x$. We solve this second order linear ODE in the single unknown x through the well-known procedure[5] for such equations and obtain

$$x(t) = c_1e^t + c_2e^{-t},$$

where c_1 and c_2 are arbitrary constants. Considering that first ODE says that $u = dx/dt$, we differentiate the above and get

$$u(t) = c_1e^t - c_2e^{-t}.$$

We then apply the initial conditions

$$h = c_1 + c_2, \quad 2 - 3h = c_1 - c_2,$$

solve for c_1 and c_2, and obtain

$$c_1 = 1 - h, \quad c_2 = 2h - 1,$$

whence

$$x = (1 - h)e^t + (2h - 1)e^{-t},$$
$$u = (1 - h)e^t - (1 - 2h)e^{-t}.$$

Finally, to eliminate h between these two, we solve the first equation for h we get

$$h = \frac{1 - xe^t - e^{2t}}{2 - e^{2t}},$$

and substitute the result into the second, and arrive at the solution

$$u(x,t) = \frac{2x - 2e^t + xe^{2t}}{e^{2t} - 2}.$$

[5] Rewrite the equation in the compact form $x'' - x = 0$, solve the corresponding algebraic equation $r^2 - 1 = 0$, obtain roots $r = \pm 1$, and conclude that the solution is a linear combination of e^t and e^{-t}.

1.4 *Appendix: A rigorous version of Section 1.2's presentation*

Although the procedure presented in Section (1.2) leads to the correct result, its intermediate steps do not stand scrutiny. For one thing, the distinction between $u(x(t), t)$ and $u(t)$ is smeared; compare equation (1.2) and (1.3b). For another thing, equation (1.3a) treats x as a function of t, while the goal is to solve the initial value problem (1.1) where x and t are independent variables.

The sloppiness of that presentation may be removed at the expense of a somewhat cumbersome notation. We present the rigorous approach here which will serve to convince skeptics of the correctness of the approach. In practice, however, it is customary to follow Section 1.2's approach due to its notational simplicity.

Thus, we return to the initial value problem (1.1) and imagine an observer who moves along the x axis according to $x = \phi(t)$. We will specify ϕ later, but note that ϕ, whatever it is, determines the observer's path in the space-time. The observer's velocity is $\phi'(t)$. The observer wishes to control ϕ' so that it agrees with $c(x, t, u)$ along the path, that is

$$\phi'(t) = c\Big(\phi(t), t, u\big(\phi(t), t\big)\Big). \tag{1.9}$$

This cannot be done right away since u is not known ahead of the time, but we will soon find a way around that. Let's write

$$\psi(t) = u\big(\phi(t), t\big) \tag{1.10}$$

for the value of u that the observer would see at any time t. We apply the chain rule of differentiation to calculate

$$\begin{aligned}
\psi'(t) &= u_x\big(\phi(t), t\big)\phi'(t) + u_t\big(\phi(t), t\big) \\
&= u_x\big(\phi(t), t\big)\, c\Big(\phi(t), t, u\big(\phi(t), t\big)\Big) + u_t\big(\phi(t), t\big) \\
&\overset{\text{by (1.1a)}}{=} q\Big(\phi(t), t, u\big(\phi(t), t\big)\Big).
\end{aligned} \tag{1.11}$$

In view of the notation (1.10), we replace $u\big(\phi(t), t\big)$ by $\psi(t)$ in the equations (1.9) and (1.11), and arrive at a pair of coupled ODEs in ϕ and ψ:

$$\phi'(t) = c\big(\phi(t), t, \psi(t)\big), \tag{1.12a}$$

$$\psi'(t) = q\big(\phi(t), t, \psi(t)\big). \tag{1.12b}$$

Let's say the observer starts at $x = h$ at $t = 0$, that is, $\phi(0) = h$. Then from the initial condition (1.1b) and the definition (1.10) we see that $\psi(0) = u\big(\phi(0), 0\big) = u(h, 0) = f(h)$. These supply the initial conditions

$$\phi(0) = h, \quad \psi(0) = f(h) \tag{1.12c}$$

to go along with the system of ODEs (1.12a) and (1.12b). Let's write

$$\phi(t) = \Phi(t, h), \quad \psi(t) = \Psi(t, h) \tag{1.13}$$

for the solution of the initial value problem (1.12a)–(1.12c). We eliminate h between these two in whichever way is expedient. For instance, we may isolate h in the first equation[6] in (1.13) and obtain $h = H(x, t)$. Substituting this in the second equation of (1.13) we get $\psi(t) = \Psi\big(t, H(x, t)\big)$. This determines the value of ψ at any point (x, t) in space-time, which, according to (1.10), is the value of u at (x, t):

$$u(x, t) = \Psi\big(t, H(x, t)\big).$$

[6] Depending on the situation, it may be easier to begin with isolating h in the *second* equation.

In summary, the solution to the initial value problem (1.1) (a PDE) is obtained through solving the initial value problem (1.12a)–(1.12c) (a system of ODEs). The curves $x = \phi(t)$ in space-time are the *characteristic curves*, or simply *the characteristics* of the PDE. The PDE is transformed to a system of ODEs along the characteristics.

1.5 *Exercises*

1.1. (a) Solve the initial value problem $u_t + 2u_x = 0$, $u(x,0) = f(x)$, and (b) Sketch representative samples of the characteristic curves.

1.2. (a) Solve the initial value problem $u_t + xu_x = 0$, $u(x,0) = f(x)$, and (b) Sketch representative samples of the characteristic curves.

In each of the exercises 1.3 through 1.12, solve the given PDE for the unknown $u(x,t)$ subject an arbitrary initial condition $u(x,0) = f(x)$.

1.3. $u_t + 2u_x = -u$

1.4. $u_t + xu_x = 1$

1.5. $u_t + xu_x = -2u$

1.6. $u_t + 2xtu_x = 0$

1.7. $u_t + 2xtu_x = -u$

1.8. $u_t + 2tu_x = 2t$

1.9. $u_t - 6t^2u_x = -tu$

1.10. $u_t - 12t^2u_x = -xu$

1.11. $u_t + tu_x = x$

1.12. $u_t + (x-t)u_x = x$

In each of the exercises 1.13 through 1.18, solve the given initial value problem for the unknown $u(x,t)$.

1.13. $(t+u)u_x - uu_t = 0, \quad u(x,0) = \dfrac{1}{1+x}$

1.14. $u_t + uu_x = 0, \quad u(x,0) = x$

1.15. $u_t + uu_x = t, \quad u(x,0) = x+1$

1.16. $u_t + uu_x = x, \quad u(x,0) = 1 - \dfrac{1}{2}x$

1.17. $u_t + uu_x = 4x, \quad u(x,0) = -x$

1.18. $u_t + 4uu_x = x, \quad u(x,0) = x$

2

Flux and density in one dimension

2.1 *Conservation of mass*

If you stand downwind of a smokestack, you will be exposed to the smoke that passes by you carried by the wind. The intensity of the smoke may vary in time, depending on the fluctuations of the amount that is spewed by the smokestack and the velocity and direction of the wind. The intensity would generally vary by your location as well. We begin by introducing some basic concepts that will help us make a quantitative model of the situation.

For simplicity, we adopt a "one-dimensional view" in the sense that we assume that the wind and smoke blow along a certain fixed direction in space, and that a single coordinate axis in that direction—the x axis—suffices for quantifying the interesting aspects of our observations.[1] We write $v(x,t)$ for the wind velocity at the location x at time t. We write $\rho(x,t)$ for the smoke's density, that is, it the mass of the smoke per unit volume, at x and t. We also introduce the *flux* $\phi(x,t)$ which indicates the mass of smoke passing per unit time through a unit area perpendicular to the x axis at time t.

[1] It is possible to formulate a true three-dimensional description of the flow, but that will take us beyond the objectives of this chapter, and indeed that of this book.

As you may surmise, the quantities v, ρ, and ϕ are intricately connected. We are going to investigate their interdependence.

Consider two arbitrarily picked observation stations $x = a$ and $x = b$ along the x axis and an imaginary cylindrical tube of cross-sectional area A that extends from $x = a$ to $x = b$, as in Figure 2.1.

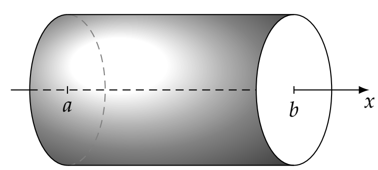


Figure 2.1: The one-dimensional flow along the x axis goes through an (imaginary) tube. The flow enters and leaves the tube's flat ends at $x = a$ and $x = b$, but nothing crosses the tube's curved surface. Bear in mind that this is *not* an actual physical tube; it is a mathematical construct that exists only in your head/imagination. Such a hypothetical region immersed within a flow, and whose size and shape may be adjusted as needed, is called a *control volume* in the fluid mechanics literature.

At any time t, the total mass of smoke within the tube is

$$\int_a^b \rho(x,t)A\,dx.$$

as $A\,dx$ is the volume of a slice of thickness dx of the cylinder. The change of the tube's smoke mass content from time t_1 to t_2 is

$$\int_a^b \rho(x,t_2)A\,dx - \int_a^b \rho(x,t_1)A\,dx.$$

That change is due to smoke flowing in and out of the sections at a

and b.[2] According to the definition of flux, smoke enters the cross-section at $x = a$ at the rate of $\phi(a,t)A$ per unit time, and leaves the cross section at $x = b$ at the rate of $\phi(b,t)A$ per unit time. Therefore during the time period $t_1 < t < t_2$, the net gain of smoke through crossing the tube's boundaries is

$$\int_{t_1}^{t_2} \phi(a,t)A\,dt - \int_{t_1}^{t_2} \phi(b,t)A\,dt.$$

We conclude that

$$\int_a^b \rho(x,t_2)A\,dx - \int_a^b \rho(x,t_1)A\,dx = \int_{t_1}^{t_2} \phi(a,t)A\,dt - \int_{t_1}^{t_2} \phi(b,t)A\,dt.$$

We divide through by the common factor A and rearrange the terms into

$$\int_a^b \Big[\rho(x,t_2) - \rho(x,t_1)\Big]\,dx + \int_{t_1}^{t_2} \Big[\phi(b,t) - \phi(a,t)\Big]\,dt = 0. \tag{2.1}$$

What we have obtained is the *equation of conservation of mass*. It asserts that the change in the smoke contents of the tube is exactly accounted for by what enters and leaves its ends. This is a fundamental observation which has significant repercussions in a broad range of applications in physics, engineering, and applied mathematics. The equation holds for any choices of a, b, t_1, and t_2. Although we derived it in the context of propagation of smoke, you can see that the idea generalize to much broader contexts. For instance, the conservation of the mass of air as it flows through a jet engine's turbine, or the size of an animal herd migrating through Sahara would be modeled in the same way. The principle applies to less tangible quantities as well. For instance, the conservation of momentum in the propagation of an acoustic wave, or conservation of thermal energy in heat conduction through a solid also satisfy (2.1) with the appropriate definitions of the density and flux.

Although the expression of conservation of mass in the form (2.1) is the most fundamental, and in a sense the most powerful, it is not what is most commonly used. What is commonly used, rather, is a versatile *consequence* of (2.1) obtained after making some mathematical (not physical!) assumptions. Specifically, let's assume that ρ and ϕ are differentiable and have continuous derivatives with respect to x and t.[3] Then, in view of the Fundamental Theorem of Calculus, we have

$$\rho(x,t_2) - \rho(x,t_1) = \int_{t_1}^{t_2} \frac{\partial}{\partial t}\rho(x,t)\,dt,$$
$$\phi(b,t) - \phi(a,t) = \int_a^b \frac{\partial}{\partial x}\phi(x,t)\,dx,$$

whereby (2.1) takes the form

$$\int_a^b \int_{t_1}^{t_2} \frac{\partial}{\partial t}\rho(x,t)\,dt\,dx + \int_{t_1}^{t_2} \int_a^b \frac{\partial}{\partial x}\phi(x,t)\,dx\,dt = 0.$$

[2] Due to our "one-dimensional" assumption, no smoke crosses the cylinder's curved boundary.

[3] Shock waves, which we will encounter in Chapter 3, certainly violate these continuity and differentiability assumptions, and force us to revert to (2.1) which then leads to the Rankine–Hugoniot shock condition.

We interchange the order of the integrals in the first term, and then combine the two terms into

$$\int_{t_1}^{t_2} \int_a^b \left[\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) \right] dx\, dt = 0. \tag{2.2}$$

Since this form of the equation of conservation of mass holds for arbitrary a, b, t_1, t_2, it follows that[4]

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) = 0. \tag{2.3}$$

[4] Here is a proof-by-contradiction of that assertion. Suppose the bracketed expression in (2.2) is nonzero, say positive, at a point (x_0, t_0) in the xt-plane. Then, by our assumption of the continuity, the bracketed expression is nonzero within some open neighborhood D of (x_0, t_0). Pick a, b, t_1, t_2 so that the rectangle $(a, b) \times (t_1, t_2)$ is centered at (x_0, t_0) and fits within D. Then (2.2) says that the integral of a positive quantity is zero, which is impossible.

The differential equation (2.3) is the most commonly occurring form of the expression of conservation law in fluid mechanics, gas dynamics, population dynamics, traffic flow, etc. One falls back to the original integral form (2.1) only when the continuity and differentiability conditions become untenable.

Remark 2.1. The equations of conservation of mass (2.1) and (2.3) admit generalizations under various scenarios.

Consider, for instance, a situation where smoke is issued not only by the smokestack, but that it is also generated everywhere in space, perhaps due to some unspecified chemical reactions. Let $q(x,t)$ be the mass of smoke generated per unit volume per unit time. This it can be shown that under this scenario, (2.1) changes to

$$\int_a^b \Big[\rho(x,t_2) - \rho(x,t_1)\Big]\, dx + \int_{t_1}^{t_2} \Big[\phi(b,t) - \phi(a,t)\Big]\, dt = \int_{t_1}^{t_2} \int_a^b q(x,t)\, dx\, dt, \tag{2.4}$$

and (2.3) changes to

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) = q(x,t). \tag{2.5}$$

We leave the demonstrations of these and other generalizations to exercises.

2.2 *Advection*

In the previous section we examined smoke carried by the wind and arrived at the conservation of mass equation but we have yet to account for the wind velocity, $v(x,t)$, which must play *some* role in the process. We address the latter in this section.

Let us look at a section at an arbitrary x of the tube of Figure 2.1. Smoke flows through that section at velocity $v(x,t)$, so the volume of the smoky air that passes through that section per unit time is $Av(x,t)$, where A is the tube's cross-sectional area as before. The mass of smoke within that volume is $\rho(x,t)Av(x,t)$, and therefore the mass per unit area passing through that section per unit time is $\rho(x,t)v(x,t)$. But "mass per unit area per unit time" is the smoke's flux, and therefore

$$\phi(x,t) = \rho(x,t)v(x,t). \tag{2.6}$$

This relationship between the flux, density, and velocity is called an *advection relationship*. It expresses our underlying assumption that the smoke is *advected* (Latin for "carried") by the wind.[5] Eliminating ϕ between (2.6) and (2.3), we obtain

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\Big(\rho(x,t)v(x,t)\Big) = 0. \tag{2.7}$$

which may be solved for $\rho(x,t)$ if $v(x,t)$ is prescribed. This is called the *advection equation*. In particular, if $v(x,t)$ is a constant, let's say c, then the advection equation reduces to

$$\frac{\partial}{\partial t}\rho(x,t) + c\frac{\partial}{\partial x}\rho(x,t) = 0. \tag{2.8}$$

We developed techniques for solving these (and actually more complex) equations in Chapter 1. Here we look at concrete cases that give rise to advection equations.

[5] Can smoke do anything other than be carried by the wind? Yes! Imagine an absolutely still air—no wind at all—and a mass of smoke suspended in it. If you wait long enough, the smoke will dissipate, spread out, diffuse. The mechanism for that is *not the wind*, but the random molecular motion of the smoke's particles. That process is called *diffusion* and it is the subject of Chapter 7. You can easily visualize diffusion by dropping a drop of ink in a glass of water and watch it spread.

2.3 *The advection of an oil-drop*

Consider a steady flow of water through a pipe, at velocity c.[6] We inject a drop of oil into the pipe and watch it as it gets carried along by the flow. In a grossly simplified model of the situation, we reduce the three-dimensional space to a one-dimensional line, and install a coordinate axis x along the pipe's longitudinal axis. In this formulation, the oil-drop's *linear density*, $\rho(x,t)$, is the mass of oil *per unit length* along the x axis.

[6] We are taking the liberty of presuming that "at velocity c" is a meaningful concept. In reality it is not. Water flowing through a pipe adheres to the pipe's wall and consequently it flows slowly near the wall, and faster near the pipe's central axis. For our purposes, you may want to think of "velocity c" as some sort of average taken over the pipe's cross-section.

Let us calculate the oil-drop's linear density, assuming that it has the shape of a ball of radius R, and that its (conventional) density per unit volume is σ. Figure 2.2 shows the oil-drop positioned at $x = 0$, and a section of thickness dx perpendicular to the x axis, at an arbitrary coordinate location x. We see from the geometry of diagram that the radius of that cross-section is $\sqrt{R^2 - x^2}$, and therefore the mass of the slice is $\pi\sigma(R^2 - x^2)\,dx$. This shows that the mass per unit length at the location x is $\pi\sigma(R^2 - x^2)$ provided that $-R < x < R$. We extend the density by zero to $-\infty < x < \infty$ because there is no oil outside of $|x| < R$. Thus, for the initial condition of our advection problem we take:

$$f(x) = \begin{cases} \pi\sigma(R^2 - x^2) & \text{if } |x| < R, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

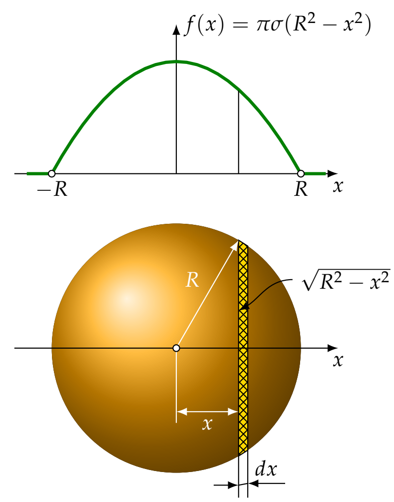


Figure 2.2: The graph of the linear density function of a spherical oil-drop is a parabola.

The conservation of oil-drop's mass is expressed through (2.8) since water and oil move at a constant velocity c, Thus, we are led to the initial value problem

$$\frac{\partial}{\partial t}\rho(x,t) + c\frac{\partial}{\partial x}\rho(x,t) = 0,$$
$$\rho(x,0) = f(x),$$

were $f(x)$ is defined above. We solved exactly this initial value problem in Section 1.3.1 and obtained the solution

$$\rho(x,t) = f(x - ct).$$

Not surprisingly, this says that the oil-drop injected into the water pipe is carried at the water velocity c effectively as a solid object, without redistribution of density. Go to density of advected oil drop to see an animation of the solution.

2.4 *Advection in a semi-infinite domain*

Let us look at the advection of a substance of density $\rho(x,t)$ over the semi-infinite domain $x > 0$, assuming that the flow velocity c is constant. Specifying the initial condition $\rho(x,0)$ is insufficient in this case since the solution in later times is also affected by what happens at the boundary at $x = 0$. That requires specifying a boundary condition such as $\rho(0,t) = \eta(t)$ at the $x = 0$ boundary, as in

$$\frac{\partial}{\partial t}\rho(x,t) + c\frac{\partial}{\partial x}\rho(x,t) = 0 \qquad x > 0, \quad t > 0, \tag{2.10a}$$
$$\rho(x,0) = f(x) \qquad x > 0, \tag{2.10b}$$
$$\rho(0,t) = \eta(t) \qquad t > 0. \tag{2.10c}$$

This is unlike the IVPs that we have encountered thus far in that the spatial domain is is not the entire x axis. The problem's statement here involves an initial condition (2.10b), and a *boundary condition* (2.10c). That makes the problem into an *initial boundary value problem*, or an IBVP for short.[7] To cope with this new situation, we return to the origins of our investigation by revisiting the system of ODEs (1.3) and noting that there was no assumption on the extent of the x axis at that stage.

[7] Note the distinction: An *initial condition* refers to the values of the unknown function at *initial time*, which is typically $t = 0$. A *boundary condition* refers to the values of the unknown function at the *domain's boundary*, which is $x = 0$ in the current problem. So remember this: *initial* refers to time, while *boundary* refers to space.

Considering that the coefficient c is a constant in the current setting, and that $q = 0$, the system of ODEs take the form

$$\frac{dx}{dt} = c, \quad \frac{d\rho}{dt} = 0,$$

whose general solution is

$$x = ct + A, \quad \rho = B, \tag{2.11}$$

where the integration constants A and B are to be determined. Note that regardless of the values of A and B, the expression $x = ct + A$ indicates that the characteristics are straight lines that propagate at velocity c, and $\rho = B$ indicates that the density ρ is constant along each characteristic. The characteristics are depicted as red and green lines in the adjacent diagram.

The red characteristics originate at the x axis and pick up their density data from the prescription $f(x)$. The green characteristics originate at the t axis and pick up their density data from the prescription $\eta(t)$. The characteristic $x = ct$ (drawn in black) separates the two modes of behavior, dividing the solution domain into the regions 1 and 2 as marked in the figure. We solve for the density $\rho(x,t)$ separately in the two regions and then patch together the parts.

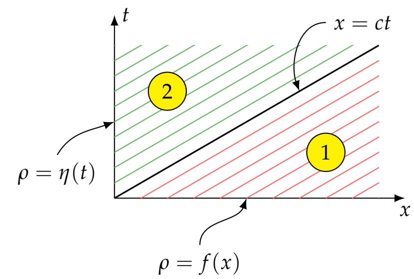


Figure 2.3: The characteristic $x = ct$ separates the regions 1 and 2.

Solving in region 1: Consider a characteristic in region 1 that originates at $x = h$, $t = 0$ for some $h > 0$. Considering that the density there is $f(h)$, we are led to the initial conditions

$$x(0) = h, \quad \rho(0) = f(h).$$

Substituting these into the general solution (2.11) we obtain $A = h$ and $B = f(h)$, and consequently

$$x = ct + h, \quad \rho = f(h).$$

As usual, we eliminate h between these two equations by solving the first one for h and substituting the result in the second one. We arrive at $\rho = f(x - ct)$.

Solving in region 2: Consider a characteristic in region 2 that originates at $x = 0$, $t = \tau$ for some $\tau > 0$. Considering that the density there is $\eta(h)$, we are led to the initial conditions

$$x(\tau) = 0, \quad \rho(\tau) = \eta(\tau).$$

Substituting these into the general solution (2.11) we obtain $A = -ct$ and $B = \eta(\tau)$, and consequently

$$x = ct - c\tau, \quad \rho = \eta(\tau).$$

To eliminate τ between these two equations, we solve the first one for τ and obtain $\tau = -\frac{1}{c}(x - ct)$. Plugging this into the second equation we arrive at $\rho = \eta\Big(-\frac{1}{c}(x - ct\Big)$.

We conclude the solution of the IBVP (2.10) is

$$\rho(x,t) = \begin{cases} f(x - ct) & \text{in region 1, that is, } x > ct, \\ \eta\Big(-\frac{1}{c}(x - ct\Big) & \text{in region 2, that is, } x < ct. \end{cases} \tag{2.12}$$

Remark 2.2. The solution (2.12) can be expressed concisely as

$$\rho(x,t) = \psi(x - ct), \quad x > 0,\ t > 0, \tag{2.13}$$

where

$$\psi(s) = \begin{cases} f(s) & s > 0, \\ \eta\left(-\frac{1}{c}s\right) & s < 0. \end{cases}$$

In (2.13) we see immediately that the graph of the density at any time is obtained by translating the graph of ψ along the x axis by ct.

Example 2.1. Solve the initial boundary value problem

$$\begin{aligned} &\frac{\partial}{\partial t}\rho(x,t) + 2\frac{\partial}{\partial x}\rho(x,t) = 0 && x > 0, \quad t > 0, \\ &\rho(x,0) = e^{-x} && x > 0, \\ &\rho(0,t) = \cos t && t > 0. \end{aligned}$$

Solution. We substitute the given data into (2.12) and obtain

$$\rho(x,t) = \begin{cases} e^{-(x-2t)} & x > 2t, \\ \cos\left(-\frac{1}{2}(x-2t)\right) & x < 2t. \end{cases}$$

Alternatively, we may define

$$\psi(s) = \begin{cases} e^{-s} & s > 0, \\ \cos\left(-\frac{1}{2}s\right) & s < 0 \end{cases}$$

and write the solution as $\rho(x,t) = \psi(x - 2t)$. The graph of ψ is shown in Figure 2.4. On the positive s axis the graph drops down to zero as $s \to +\infty$. On the negative s axis the graph is that of the ordinary cosine function which oscillates forever. As remarked above, the graph of the solution $\rho(x,t)$ at any time t is obtained by translating the graph of ψ to the right by $2t$ along the x axis. With that information, you should be able to make sense of the animation in advection in a semi-infinite domain. □

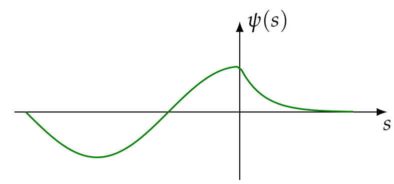


Figure 2.4: The function ψ is defined for $-\infty < s < \infty$ but here is plotted only on the range $-3\pi < s < 5$.

2.5 *Advection through a constriction*

Here we repeat Section 2.3's scenario of oil-drop in a water pipe, but now we assume that the pipe's cross-sectional area a changes abruptly at $x = 0$:

$$a(x) = \begin{cases} a_1 & x < 0, \\ a_2 & x > 0. \end{cases}$$

The pipe's *throughput*, that is, the volume of fluid[8] passing per unit time through a cross-section, is $P = a(x)c(x)$, where $c(x)$ is the fluid's

[8] In what follows, *fluid* refers to the pipe's contents which consists of a drop of oil immersed in water.

velocity. The throughput is constant (independent of x) since what flows through a section x_1 should emerge at another section x_2, for any choice of x_1 and x_2.[9] It follows that the velocities c_1 and c_2 of the fluid in the $x < 0$ and $x > 0$ regions are related through $P = a_1 c_1 = a_2 c_2$, that is, $c_1 = P/a_1$ and $c_2 = P/a_2$, which incidentally shows that the fluid flows faster in the narrower part of the pipe. We set

[9] We are assuming that oil and water are incompressible. If they could be compressed, the volume emerging from x_2 would not necessarily be the same as the volume that enters x_1.

$$c(x) = \begin{cases} c_1 & x < 0, \\ c_2 & x > 0, \end{cases}$$

With that notation, the advection equation (2.7) takes the form

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\Big(c(x)\rho\Big) = 0, \tag{2.14a}$$

$$\rho(x,0) = f(x), \tag{2.14b}$$

where $\rho = \rho(x,t)$ and $f(x)$ are the oil's linear density at time t and time 0, respectively.

Equation (2.14a) raises a serious concern. How do we make sense of the derivative with respect to x since $c(x)$ is discontinuous? The answer lies in the discussion in Section 2.1 that led from the integral form of the conservation of mass equation (2.1) to the differential forms in (2.3) and then to (2.7). In the current situation, the velocity field is discontinuous, therefore the reduction to those differential forms is *not justified*. We will have to proceed cautiously.

We note that equation (2.14a) is quite well-behaved on the half-axes $x < 0$ and $x > 0$, as we have

$$\frac{\partial \rho}{\partial t} + c_1 \frac{\partial \rho}{\partial x} = 0 \qquad \text{if } x < 0,$$

$$\frac{\partial \rho}{\partial t} + c_2 \frac{\partial \rho}{\partial x} = 0 \qquad \text{if } x > 0.$$

The data propagates along the characteristics with speeds c_1 where $x < 0$, and c_2 where $x > 0$, as usual. These are shown as families of parallel lines in Figure 2.5. Characteristics in $x < 0$ (region 1 in the figure) pick up their data from the x axis where ρ is known to be $f(x)$, and therefore the solution within region 1 is

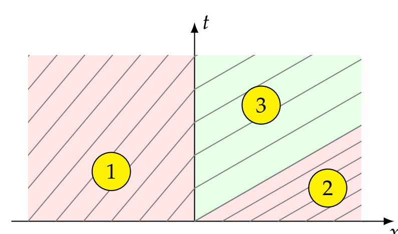


Figure 2.5: The characteristics in regions 1 and 2 pick up their data from the x axis, while those in region 3 pick up their data from the t axis..

$$\rho(x,t) = f(x - c_1 t), \quad (x,t) \text{ in region 1.} \tag{2.15}$$

Characteristics in region 2, delimited from the above by the line $x = c_2 t$, also pick up their data from the x axis, and therefore the solution within region 2 is

$$\rho(x,t) = f(x - c_2 t), \quad (x,t) \text{ in region 2.} \tag{2.16}$$

Characteristics in region 3, however, pick up their data from the t axis, and that data needs to be computed. That is done by enforcing the conservation of mass across the t axis; something that equation (2.14) fails to do due to the discontinuity of $c(x)$. Here is how we proceed.

The density of oil in region 3 is $\rho(x,t) = g(x - c_2 t)$ where the function g is to be determined. In particular, the density just to the right of the t axis is $\rho(0^+, t) = g(-c_2 t)$. The density just to the left of the t axis is seen from from (2.16) to be $\rho(0^-, t) = f(-c_1 t)$. The mass of oil arriving per unit time at $x = 0$ is $c_1 \rho(0^-, t)$, see (2.6), while the mass of oil per leaving unit time at $x = 0$ is $c_2 \rho(0^+, t)$. These quantities should be equal, that is, $c_1 \rho(0^-, t) = c_2 \rho(0^+, t)$ since the mass is conserved, and there is nothing at $x = 0$ where the oil can hide. It follows that

$$c_1 f(-c_1 t) = c_2 g(-c_2 t).$$

To determine g, we introduce $\xi = -c_2 t$. Then $t = -\xi/c_2$ and the equation above takes the form $c_1 f\left(\frac{c_1}{c_2}\xi\right) = c_2 g(\xi)$, and therefore

$$g(\xi) = \frac{c_1}{c_2} f\left(\frac{c_1}{c_2}\xi\right).$$

Thus we arrive at the solution of the initial value problem (2.14):

$$\rho(x,t) = \begin{cases} f(x - c_1 t) & \text{in region 1,} \\ f(x - c_2 t) & \text{in region 2,} \\ \frac{c_1}{c_2} f\left(\frac{c_1}{c_2}(x - c_2 t)\right) & \text{in region 3.} \end{cases}$$

This solution is animated in oil drop through a stepped tube where the two pipe diameters are in the ratio of $\sqrt{2}$, and therefore their cross-sectional areas are in the ratio of 2. Consequently the speed of the flow in the narrower pipe is twice that of the wider one. The linear density profile is taken according to (2.9) which corresponds to a spherical oil-drop in the wider section of the tube. The animation oil drop 3D presents a three-dimensional view.

2.6 *Exercises*

2.1. Derive the equations (2.4) and (2.5).

2.2. Suppose that the smoke studied in Section (2.1) is radioactive with a decay constant of β, that is, the density decays according to $\frac{\partial \rho}{\partial t} = -\beta\rho$. Modify the equation of conservation of mass (2.3) to model the density $\rho(x,t)$ of the radioactive smoke.

2.3. Solve the initial boundary value problem

$$\begin{aligned} &\frac{\partial}{\partial t}\rho(x,t) + c\frac{\partial}{\partial x}\rho(x,t) = -\beta\rho(x,t) && x > 0, \quad t > 0, \\ &\rho(x,0) = f(x) && x > 0, \\ &\rho(0,t) = \eta(t) && t > 0, \end{aligned}$$

where $\rho = \rho(x,t)$ and c is a positive constant.

2.4. Repeat the calculations of Section 2.1 but allow for the cross-sectional area $A(x)$ of the tube to vary smoothly with x. How does that change the equation of conservation of mass (2.3)?

2.5. Find the solution $\rho(x,t)$ of the initial value problem (2.14) when

$$c(x) = \begin{cases} c_1 & x < 0, \\ c_2 & 0 < x < L, \\ c_3 & x > L. \end{cases}$$

Assume that c_1, c_2, c_3, and L are positive.

3
Traffic flow

In this chapter we look at the Lighthill–Whitham–Richards mathematical model for traffic flow[15, 18]. The model's analysis is a direct application of Chapter 1's method of solving quasilinear PDEs.

3.1 *The derivation of the traffic flow equation*

Consider a one-lane highway, and for simplicity, suppose that there are no entrances or exits along the stretch under the study.[1] We designate locations along the highway through the coordinate x, and write t for time. The main players in our mathematical model are the *traffic velocity* $v(x,t)$, the *traffic density* $\rho(x,t)$, and the *traffic flux* $\phi(x,t)$.

[1] Accounting for entrances and exits is not difficult but we will forgo that for the sake of keeping this chapter's treatment as simple as possible. Once you have learned the basics, you may consider including entrances and exits as an interesting and instructive project.

The traffic velocity, v, is readily observable. You may watch the traffic by standing by the side of the highway and gain quite an accurate picture of how fast it is moving. By "it" we don't mean the speed of an individual car. Rather, we are referring to the speed of cars in general as they go past your station. During the rush hour in major urban areas, for instance, the traffic velocity tends to be slow.

As the notation $v(x,t)$ indicates, it is expected that the traffic velocity to depend on the observer's location x and the time t. For instance, if your observation station is near a traffic light, then then your observed traffic velocity will fluctuate as the light alternates between green and red. That shows that v depends on t. If you move to a location some good distance away from the traffic light (that is, you change your x), then it is likely that you will observe a more or less steady flow of traffic there. That shows that v depends on x. For the same reasons, the traffic density ρ and the traffic flux ϕ, also depend on x and t.

The traffic density, ρ, is a measure of the number of cars per unit length of the highway. You can get a good picture of it if you imagine you were hovering over the highway in a helicopter. The more crowded parts of the highway will have higher traffic density (more cars per unit length). You may quantify that by counting the number of cars per, let's say, every 500 feet. In our idealized model we take $\rho(x,t)$ to be *defined everywhere*, that is, at all locations x along the highway. In a congested area ρ would be large. In a lightly traveled highway ρ would be small. In an empty highway ρ would be zero.

The traffic flux, ϕ, is the number of cars that go past your observation station per unit time. Traffic stopped at a red light has nonzero density but the velocity and flux are zero there. In a lightly traveled patch of a dessert highway, the velocity can be high, but the flux can be low—only a few cars will be passing your observation station per unit time, albeit at high speed.

Chapter 2's equations of conservation of mass, that is, (2.3), (2.6), and (2.7) correspond, in the current context, to a principle of *conservation of cars*—the cars just don't teleport away from the highway or teleport into it. We duplicate equation (2.7) here for ease of reference:

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\Big(\rho(x,t)v(x,t)\Big) = 0,$$

or more succinctly,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0. \tag{3.1}$$

That conservation equation is insufficient on its own since it is one equation in the two unknowns ρ and v. To complete it, we need to supply the equivalent of what is called a *constitutive equation* in continuum mechanics, that is, an empirical relation between the density ρ and the velocity v.

A crucial observation is that *the traffic density determines the traffic velocity.* To see that, think of the extreme case of essentially zero density traffic. Then the cars travel without interference from each other. In an idealized world, they will travel at the maximum legal speed, say $v_{\max}$. At the other extreme, when the highway is crowded with bumper-to-bumper traffic—let's call the corresponding density $\rho_{\max}$—the velocity would in effect be zero. We conclude that the velocity varies from $v_{\max}$ to zero as the density goes from zero to $\rho_{\max}$, that is, $v = \sigma(\rho)$ for some monotonically decreasing function σ such that $\sigma(0) = v_{\max}$ and $\sigma(\rho_{\max}) = 0$. That's something like the hypothetical curve shown in the adjacent graph. With $v = \sigma(\rho)$, equation (3.1) becomes

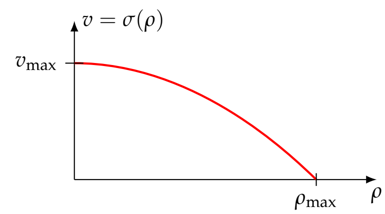


Figure 3.1: The velocity v is some monotonically decreasing function of density with $v(0) = v_{\max}$ and $v(\rho_{\max}) = 0$. The shape of the curve may be determined empirically, in principle. Just stand by the side of the highway, measure or estimate the traffic density ρ and velocity v at your location and mark the corresponding (ρ, v) point on a graph paper. Wait a little for the traffic density and velocity to change, repeat the measurement, and add a new point to the graph. Keep on doing that until you gain a good sense of the shape of the graph.

It is reasonable to expect different such graphs depending on the driving conditions, such as daytime or nighttime, clear weather or rain or snow, lane closures due to road repairs or an accident, etc.

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\Big(\rho\sigma(\rho)\Big) = 0,$$

and then, by the chain rule of differentiation,

$$\frac{\partial \rho}{\partial t} + \left[\frac{d}{d\rho}\Big(\rho\sigma(\rho)\Big)\right]\frac{\partial \rho}{\partial x} = 0.$$

We define

$$c(\rho) = \frac{d}{d\rho}\Big(\rho\sigma(\rho)\Big), \tag{3.2}$$

and rewrite the equation as

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0. \tag{3.3}$$

This first order PDE, in the single unknown $\rho(x,t)$, is the main working model of our traffic flow.

The coefficient $c(\rho)$ in (3.3) depends on the choice of σ in the constitutive equation $v = \sigma(\rho)$. For the purpose of this study we take σ to be the linear function whose graph is shown in Figure 3.2, and whose equation is

$$v = \sigma(\rho) = v_{\max}\Big(1 - \frac{\rho}{\rho_{\max}}\Big). \tag{3.4}$$

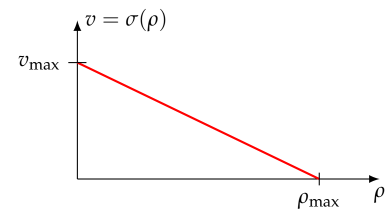


Figure 3.2: The function $v = \sigma(\rho)$ models traffic velocity versus density. See Exercise 3.10 for a study of of the case of quadratic $\sigma(\rho)$.

Therefore

$$c(\rho) = \frac{d}{d\rho}\Big(\rho\sigma(\rho)\Big) = \frac{d}{d\rho}\bigg(\rho v_{\max}\Big(1 - \frac{\rho}{\rho_{\max}}\Big)\bigg) = v_{\max}\frac{d}{d\rho}\Big(\rho - \frac{\rho^2}{\rho_{\max}}\Big),$$

that is

$$c(\rho) = v_{\max}\Big(1 - \frac{2\rho}{\rho_{\max}}\Big). \tag{3.5}$$

3.2 *The initial value problem and the characteristics*

We wish to analyze and understand the initial value problem associated with (3.3), that is

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad -\infty < x < \infty, \quad t > 0, \tag{3.6a}$$

$$\rho(x,0) = f(x) \qquad -\infty < x < \infty, \tag{3.6b}$$

where $c(\rho)$ is defined in (3.5), and $f(x)$ is a given traffic density at time $t = 0$. The solution is obtained through the procedure outlined in Chapter 1. The equivalent of that chapter's equations (1.3) in the current case are the ODEs

$$\frac{dx}{dt} = c(\rho), \quad \frac{d\rho}{dt} = 0, \quad x(0) = h, \quad \rho(0) = f(h).$$

The second ODE says ρ is constant, and since $\rho(0) = f(h)$, we have

$$\rho = f(h) \quad \text{for all } t. \tag{3.7}$$

As a result, the right-hand side of the first ODE is just a constant, and therefore

$$x = c(\rho)t + h, \tag{3.8}$$

or, inserting the value of ρ from (3.7),

$$x = c\big(f(h)\big)t + h. \tag{3.9}$$

That's the equation of the characteristic that originates at $x = h$ at $t = 0$. Ideally we would like to solve (3.9) for h and insert the result in (3.7) to obtain an expression for $\rho(x,t)$. Unfortunately there is no symbolic solution to (3.9) in general since h is hiding in the guts of the unspecified function f. So we take a different tack.

Equation (3.9) says that the characteristic curve that starts out at $x = h$ at $t = 0$ is a straight line, and the speed of the characteristic, $c\big(f(h)\big)$, is known since c and f are given. That provides enough information to determine characteristics graphically, as illustrated in Figure 3.3. Since ρ remains constant along the characteristic, and since ρ is given at $t = 0$, we may determine $\rho(x,t)$ anywhere that can be reached through such characteristic lines, and even obtain an explicit expression for ρ in terms of x and t, as illustrated in the examples that follow.

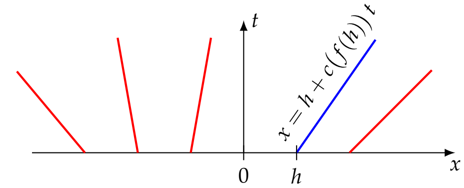


Figure 3.3: Sample characteristic lines.

3.3 *Rarefaction waves*

Figure 3.3 shows the basic idea of determining the density $\rho(x,t)$ at any point (x,t) in space-time. We determine the characteristic line that goes through the point (x,t), determine the intersection of that line with the x axis, and read off the density there from the prescribed initial condition ρ_0.

That strategy works provided that the characteristic lines "fan out" as in that figure, resulting in a unique characteristic line through any point (x,t). The expanding set of characteristics results in a *rarefaction wave*[2] in the traffic as we will see in the examples that follow. On the other hand, if the characteristics converge and intersect, then we get more than one characteristic through an intersection point, and that results in an ambiguous reading of the density. Such collision of characteristics give rise to *shock waves*[3] which we will explore in a subsequent section.

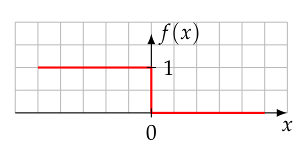
[2] The word "rarefaction wave" is inherited from gas dynamics. The phenomenon associated with the expanding fan of characteristics results in the thinning, or *rarefaction*, of a gas.

[3] This term is also inherited from gas dynamics.

In the rest of the current section we provide detailed solutions of three illustrative examples of the use of characteristics to analyze the initial value problem (3.6). In all cases we take $\sigma(\rho)$ and $c(\rho)$ as in (3.4) and (3.5). All three examples correspond to rarefaction waves.

Example 3.1. Consider the special case of the initial value problem (3.6) where $v_{\max} = 1$, $\rho_{\max} = 1$, and $f(x)$ is the step function as in

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$



Find the traffic density $\rho(x,t)$.

Solution. This corresponds to a traffic that begins to move after the red light turns green. From equation (3.5) and the given data, we have

$c(\rho) = 1 - 2\rho$. We see that $c(0) = 1$ and $c(1) = -1$. Having in mind that according to (3.8) the characteristics propagate at speed $c(\rho)$, and that according to (3.7) ρ is constant along a characteristic, we conclude that characteristics that originate at the positive x axis propagate at velocity 1 and characteristics that originate on the at the negative x axis propagate at velocity -1. This leads to the space-time diagram is shown in Figure 3.4. We see that the diagram is divided into three distinct regions. Within region 1 we have $\rho(x,t) = 1$ since density is constant along the characteristics and each characteristic originates on the negative x axis where the density is 1. For the same reason, within region 3 we have $\rho(x,t) = 0$. The characteristics $x = -t$ and $x = t$, drawn in red, separate the three regions. This, region 1 corresponds to $x < -t$, region 2 corresponds to $-t < x < t$, and region 3 corresponds to $x > t$.

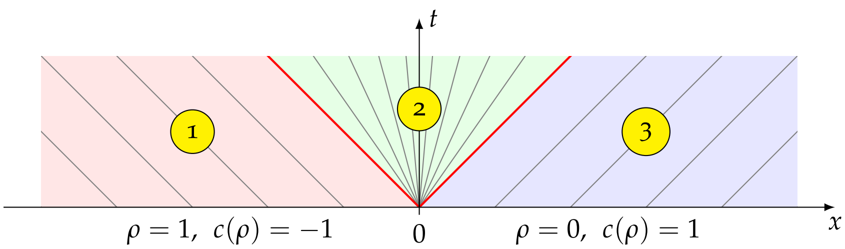


Figure 3.4: The space-time diagram of Example 3.1.

Within the "fan" region 2 we calculate $\rho(x,t)$ as follows. Pick any point (x,t) within that region. The equation of characteristic through it is $x = c(\rho)t$, where ρ is the (constant) density along that characteristic.[4] Since $c(\rho) = 1 - 2\rho$, the equation of the characteristic takes the form $x = (1 - 2\rho)t$. We solve this for ρ and obtain $\rho = \frac{1}{2}(1 - \frac{x}{t})$. We conclude that

$$\rho(x,t) = \begin{cases} 1 & \text{if } x < -t, \\ \frac{1}{2}(1 - \frac{x}{t}) & \text{if } -t < x < t, \\ 0 & \text{if } x > t. \end{cases} \tag{3.10}$$

[4] Here we are appealing to equation (3.8).

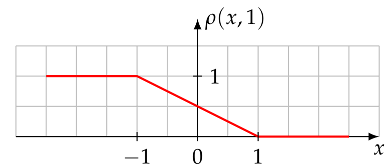
Remark: The solution expressed in (3.10) does not stand under scrutiny. If it is the solution of the initial value problem (3.6), shouldn't $\rho(x,0) = f(x)$? Yes, it should, but (3.10) is undefined at $t = 0$ since it involves a division by t. To remedy the situation, we need to be explicit about the need for special treatment of the $t = 0$ case. Thus, a pedantically correct expression of the solution would be

$$\rho(x,t) = \begin{cases} f(x) & \text{if } t = 0, \\ \begin{cases} 1 & \text{if } x < -t, \\ \frac{1}{2}(1 - \frac{x}{t}) & \text{if } -t < x < t, \\ 0 & \text{if } x > t. \end{cases} & \text{if } t > 0. \end{cases} \tag{3.11}$$

Normally people prefer the simpler but questionable representation (3.10) over the precise but cumbersome version (3.11), with the understanding that (3.10) is meant for $t > 0$. We will retain that tacit understanding when presenting the solutions to this section's remaining examples. Beware, however, that a computer program is generally not as forgiving as people, and may balk at accepting (3.10). You may want to use (3.11) in that case. □

Having obtained the solution $\rho(x,t)$ of the problem, we evaluate the density at any time t. For instance, at $t = 1$ we have

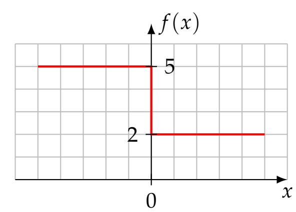
$$\rho(x,1) = \begin{cases} 1 & \text{if } x < -1, \\ \frac{1}{2}(1-x) & \text{if } -1 < x < 1, \\ 0 & \text{if } x > 1. \end{cases}$$



The animation in Example 1 shows the time evolution of the density $\rho(x,t)$ over the period $0 \le t \le 3$.

Example 3.2. Solve the initial value problem (3.6) with the data $v_{\max} = 1$, $\rho_{\max} = 8$, and

$$f(x) = \begin{cases} 5 & \text{if } x < 0, \\ 2 & \text{if } x > 0, \end{cases}$$



Solution. From the given data we have $c(\rho) = 1 - \frac{1}{4}\rho$. We see that $c(2) = 1/2$ and $c(5) = -1/4$. Therefore, characteristics that originate at the positive x axis propagate at velocity $1/2$. and the characteristics that originate at the negative x axis propagate at velocity $-1/4$. The corresponding space-time diagram is shown in Figure 3.5. As in the previous example, the diagram is divided into three regions. Within region 1 we have $\rho(x,t) = 5$. Within region 3 we have $\rho(x,t) = 2$. The characteristics $x = -\frac{1}{4}t$ and $x = \frac{1}{2}t$, drawn in red, separate the three regions. Thus, region 1 corresponds to $x < -\frac{1}{4}t$, region 2 corresponds to $-\frac{1}{4}t < x < \frac{1}{2}t$, and region 3 corresponds to $x > \frac{1}{2}t$.

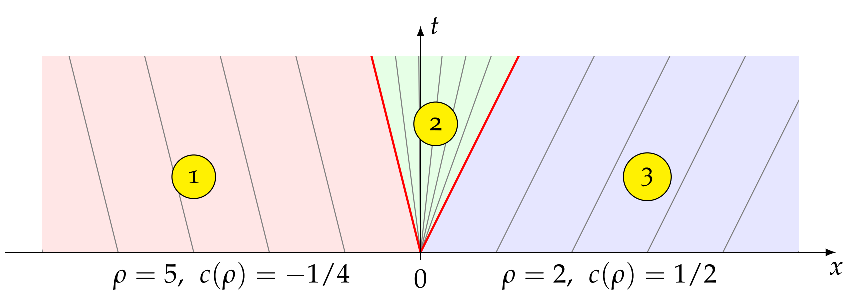


Figure 3.5: The space-time diagram of Example 3.2.

We calculate the density $\rho(x,t)$ in the fan region as we did in the previous example. The equations of characteristics in that region are

$x = c(\rho)\,t = \left(1 - \frac{1}{4}\rho\right) t$. We solve this for ρ and obtain $\rho = 4\left(1 - \frac{x}{t}\right)$. We conclude that

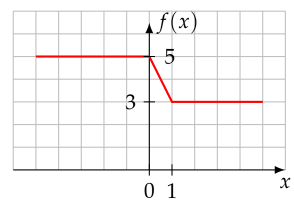
$$\rho(x,t) = \begin{cases} 5 & \text{if } x < -\frac{1}{4}t, \\ 4\left(1 - \frac{x}{t}\right) & \text{if } -\frac{1}{4}t < x < \frac{1}{2}t \\ 2 & \text{if } x > \frac{1}{2}t. \end{cases}$$

The animation in Example 2 shows the time evolution of the density $\rho(x,t)$ over the period $0 \le t \le 8$.

□

Example 3.3. Solve the initial value problem (3.6) with the data $v_{\max} = 4$, $\rho_{\max} = 8$, and

$$f(x) = \begin{cases} 5 & \text{if } x < 0, \\ 5 - 2x & \text{if } 0 < x < 1, \\ 3 & \text{if } x > 1. \end{cases}$$



Solution. From the given data we have $c(\rho) = 4 - \rho$. We see that $c(3) = 1$ and $c(5) = -1$. Therefore, characteristics that originate in $x > 1$ propagate at velocity 1 and characteristics that originate in $x < 0$ propagate at velocity -1 The corresponding space-time diagram is shown in Figure 3.6. As in the previous examples, the diagram is divided into three regions. Within region 1 we have $\rho(x,t) = 5$. Within region 3 we have $\rho(x,t) = 3$. The characteristics $x = -t$ and $x = t + 1$, drawn in red, separate the three regions. Thus, region 1 corresponds to $x < -t$, region 2 corresponds to $-t < x < t + 1$, and region 3 corresponds to $x > t + 1$.

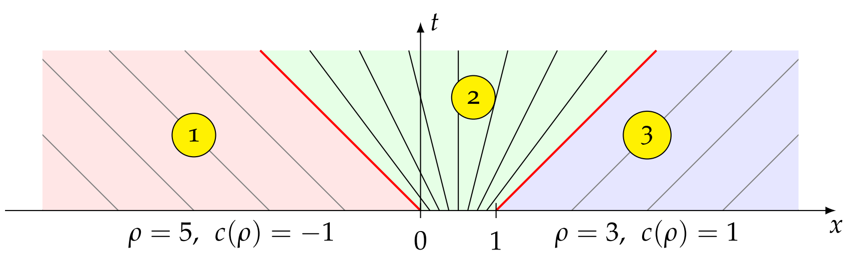


Figure 3.6: The space-time diagram of Example 3.3.

To calculate the density $\rho(x,t)$ within region 2, we look at a generic characteristic $x = c(\rho)t + h = (4 - \rho)\,t + h$ that takes off from $x = h$ at $t = 0$, where $0 < h < 1$. The density at $x = h$ is $\rho = f(h) = 5 - 2h$. Therefore $h = \frac{1}{2}(5 - \rho)$. It follows that $x = (4 - \rho)t + \frac{1}{2}(5 - \rho)$. We solve this for ρ and obtain

$$\rho = \frac{8t + 5 - 2x}{1 + 2t},$$

and therefore

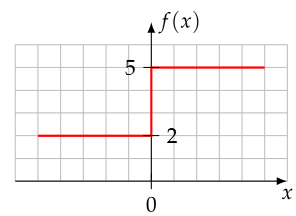
$$\rho(x,t) = \begin{cases} 5 & \text{if } x < -t, \\ \frac{8t+5-2x}{1+2t} & \text{if } -t < x < t+1, \\ 3 & \text{if } x > t+1. \end{cases}$$

The animation in Example 3 shows the time evolution of the density $\rho(x,t)$ over the period $0 \le t \le 2.5$. □

3.4 *Shock waves*

The previous section's examples consisted of rarefaction waves, manifested by sets of characteristic lines that diverge with increasing t. In this section we study situations where the characteristic lines converge, resulting in *shock waves*. These are best illustrated through examples.

Example 3.4. Consider the initial value problem for the traffic flow, equations (3.6), where $v_{\text{max}} = 3$, $\rho_{\text{max}} = 6$, and $f(x)$ is as in



$$f(x) = \begin{cases} 2 & \text{if } x < 0, \\ 5 & \text{if } x > 0. \end{cases}$$

Find the traffic density $\rho(x,t)$.

Solution. Based on the given data, we have $c(\rho) = 3 - \rho$. We see that $c(2) = 1$ and $c(5) = -2$. Therefore, characteristics that originate in the negative x axis propagate at velocity 1 and characteristics that originate in the positive x axis propagate at velocity -2. The space-time diagram of Figure 3.7 (top) shows the resulting characteristics.

The problem that confronts us is that wherever the characteristics intersect, they carry conflicting information from their initial data, making the value of the density at the intersection indeterminate. The inconsistency/clash is resolved by introducing a line of demarcation, called a *shock*, shown as a heavy green line in Figure 3.7 (bottom). Characteristics from the left and from the right impinge upon the shock, bringing with them conflicting density information. As a result, the density is indeterminate at the shock and discontinuous across it; the density jumps from 2 on the left of the shock to 5 on the right.

The questions that remain to be resolved are: What is the the equation of the shock? Why is the shock drawn as a straight line in the diagram and not a curve? How do we reconcile the discontinuous density with the law of conservation of cars?

To work out the details, let us pick a point P along the shock and let v_{shock} be the shock's velocity at that point. Let us write ρ_{left} and v_{left} for the traffic density and velocity just to the left of P. Similarly, let ρ_{right} and v_{right} be the traffic density and velocity just to the right of P.

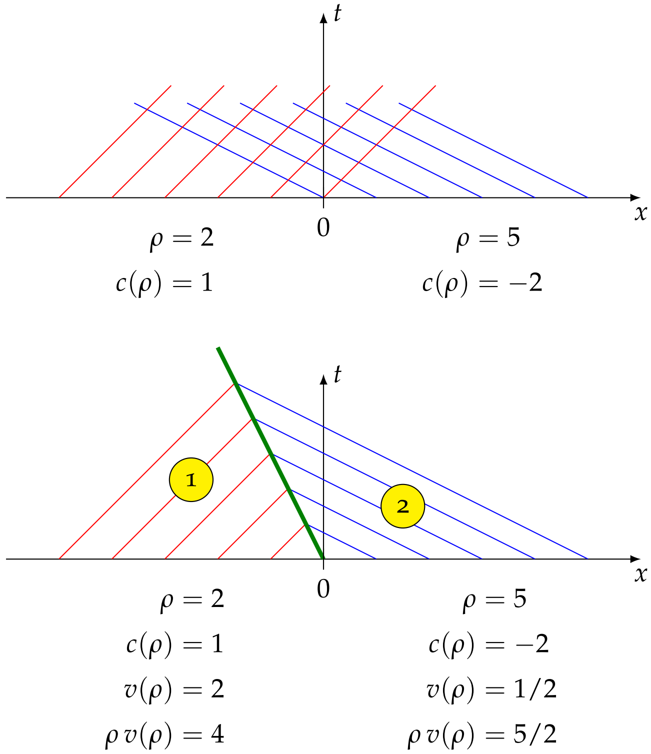


Figure 3.7: At the top: The left-moving and right-moving characteristics clash. At the bottom: The shock forms a barrier between the clashing characteristics.

The traffic approaches P from the left at velocity $v_{\text{left}} - v_{\text{shock}}$, and it moves away from P at velocity $v_{\text{right}} - v_{\text{shock}}$. Recalling the discussion that led to equation (2.6) (page 18), the number of cars arriving at P per unit time is $\rho_{\text{left}}(v_{\text{left}} - v_{\text{shock}})$, and the number of cars pulling away from P per unit time is $\rho_{\text{right}}(v_{\text{right}} - v_{\text{shock}})$. We conclude that

$$\rho_{\text{left}}(v_{\text{left}} - v_{\text{shock}}) = \rho_{\text{right}}(v_{\text{right}} - v_{\text{shock}}),$$

which we solve for v_{shock} and obtain

$$v_{\text{shock}} = \frac{\rho_{\text{right}}\, v_{\text{right}} - \rho_{\text{left}}\, v_{\text{left}}}{\rho_{\text{right}} - \rho_{\text{left}}}. \tag{3.12}$$

This is called the *Rankine–Hugoniot jump condition*[5] and was first derived in the context of gas dynamics. (That was before there were automobiles, highways, and traffic as we know it.) The equation may be expressed succinctly in the more easily remembered from

$$v_{\text{shock}} = \frac{\text{the jump in } (\rho v)}{\text{the jump in } \rho}. \tag{3.13}$$

[5] Named after Rankine [17] and Hugoniot [11, 12]. See Salas [19] for a detailed historical account.

Let us apply (3.12) to determine the shock velocity in our problem. Since $v_{\max} = 3$ and $\rho_{\max} = 6$, equation (3.4) which relates traffic density to traffic velocity is

$$v = 3\Big(1 - \frac{\rho}{6}\Big) = 3 - \frac{1}{2}\rho.$$

We have $\rho_{\text{left}} = 2$, and therefore $v_{\text{left}} = 2$. We also have $\rho_{\text{right}} = 5$, and therefore $v_{\text{right}} = 1/2$. Substituting these into (3.12) we get

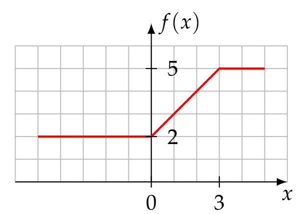
$$v_{\text{shock}} = \frac{(5)(1/2) - (2)(2)}{5 - 2} = -\frac{1}{2}.$$

We conclude that the shock propagates at the velocity $-1/2$, which incidentally shows that the shock's graph is a straight line since its speed is constant. The shock originates at $(0, 0)$, therefore its equation is $x = -\frac{1}{2}t$. Thus, referring to Figure 3.7, we arrive at the following solution to our initial value problem:

$$\rho(x,t) = \begin{cases} 2 & \text{in } x < -\frac{1}{2}t \quad \text{(that is, in region 1)}, \\ 5 & \text{in } x > -\frac{1}{2}t \quad \text{(that is, in region 2)}. \end{cases} \tag{3.14}$$

The animation in Example 4 shows the time evolution of the density $\rho(x,t)$ over the period $0 \le t \le 6$. □

Example 3.5. Let $v_{\max} = 3$ and $\rho_{\max} = 6$ as before, and take



$$f(x) = \begin{cases} 2 & \text{if } x < 0, \\ 2 + x & \text{if } 0 < x < 3, \\ 5 & \text{if } x > 3. \end{cases}$$

Find the traffic density $\rho(x,t)$.

Solution. As in the previous section, we have $c(\rho) = 3 - \rho$. We see that $c(2) = 1$ and $c(5) = -2$. Therefore, characteristics that originate on the negative x axis propagate at velocity 1, and characteristics that originate in $x > 3$ propagate at velocity -2. A preliminary version of the corresponding space-time diagram is shown in Figure 3.8. We see that the characteristics that originate at $x = 0$ and $x = 3$ (drawn in red) intersect at some time t_0 marked with the dotted blue line. The one coming from $x = 0$ carries with it the signal that $\rho = 2$. The one coming from $x = 3$ carries with it the signal that $\rho = 5$. This creates a conflict at the intersection and results in a shock.

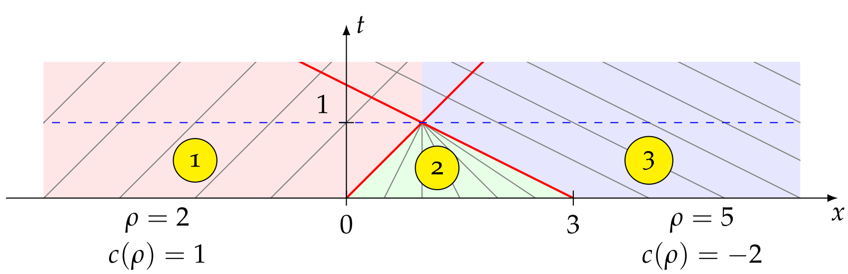


Figure 3.8: A preliminary sketch of the space-time diagram. A shock forms when the red characteristic lines meet. The diagram is *not valid* above the dotted blue line.

The conflict is not merely at the point where the red characteristics intersect. *All left-moving and right-moving characteristics intersect above* *the dotted blue line*, bringing with them conflicting signals. We will address this issue further down, but for now let us complete the analysis of the solution within the time interval $0 < t < t_0$, before the first conflict develops.

We begin by noting that the equation of the characteristic that originates at $x = 0$ is $x = t$, while the equation of the characteristic that originates at $x = 3$ is $x = 3 - 2t$. These characteristics intersect at $x = 1$, $t = 1$, and therefore $t_0 = 1$. The regions 1, 2, and 3 are characterized as

$$\begin{aligned}\text{region } 1 &= \big\{(x,t) : t < 1, \quad x < t\big\},\\ \text{region } 2 &= \big\{(x,t) : t < 1, \quad t < x < 3 - 2t\big\},\\ \text{region } 3 &= \big\{(x,t) : t < 1, \quad x > 3 - 2t\big\}.\end{aligned}$$

As in the previous examples, density $\rho(x,t)$ is 2 in region 1, and it is 5 in region 3. We calculate the density within region 2 as before, by considering the characteristic line $x = c(\rho)\,t + h$ that originates at $x = h$, $t = 0$, where ρ is the (constant) density along that characteristic. We know that $c(\rho) = 3 - \rho$, and that the density at $x = h$ is $\rho = f(h) = 2 + h$, whence $h = \rho - 2$. Therefore the equation of the characteristic is $x = (3 - \rho)\,t + \rho - 2$. Solving this for ρ, we obtain:

$$\rho = \frac{x - 3t + 2}{1 - t}.$$

We conclude that

$$\rho(x,t) = \begin{cases} 2 & \text{in region 1,} \\ \frac{x-3t+2}{1-t} & \text{in region 2,} \\ 5 & \text{in region 3.} \end{cases} \tag{3.15}$$

Now we turn to the situation above $t = t_0$. The conflict there is resolved by introducing the shock shown in Figure 3.9. Characteristics from the left and from the right impinge upon the shock, bringing with them conflicting density information from their respective initial data. As a result, the density is discontinuous across the shock. Specifically, the density is 2 in region 4, and it is 5 in region 5. The density jumps from 2 to 5 across the shock.

The shock velocity is determined by the Rankine–Hugoniot jump condition, equation (3.12). Since $v_{\max} = 3$ and $\rho_{\max} = 6$, equation (3.4) which relates traffic density to traffic velocity is

$$v = 3\Big(1 - \frac{\rho}{6}\Big) = 3 - \frac{1}{2}\rho.$$

We have $\rho_{\text{left}} = 2$, and therefore $v_{\text{left}} = 2$. We also have $\rho_{\text{right}} = 5$, and therefore $v_{\text{right}} = 1/2$. Then from (3.12) we get

$$v_{\text{shock}} = \frac{(5)(1/2) - (2)(2)}{5 - 2} = -\frac{1}{2}.$$

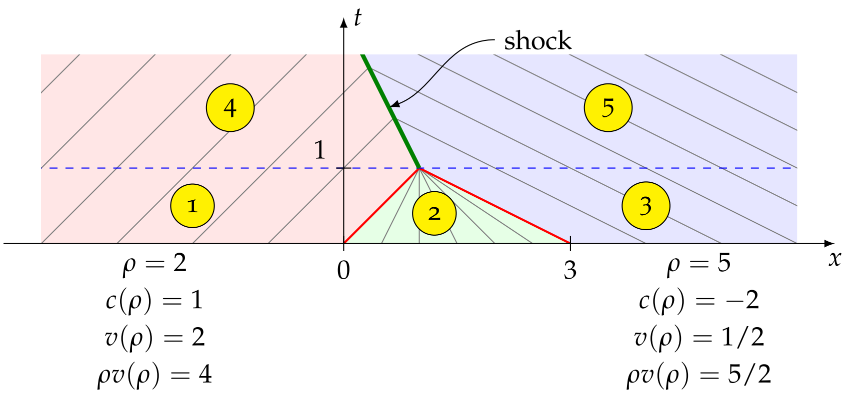


Figure 3.9: The completed space-time diagram is valid both below and above the dotted blue line. The shock that forms above that line marks the interface between two regions of differing densities.

We conclude that the shock propagates at velocity $-1/2$, which incidentally shows that the shock is a straight line since its speed is a constant. The shock originates at $(1, 1)$, therefore its equation is $x = -\frac{1}{2}(t-1) + 1$, which simplifies to $x = -\frac{1}{2}t + \frac{3}{2}$. It follows that

$$\text{region } 4 = \Big\{(x,t) : t \geq 1, \quad x < -\frac{1}{2}t + \frac{3}{2}\Big\},$$
$$\text{region } 5 = \Big\{(x,t) : t \geq 1, \quad x > -\frac{1}{2}t + \frac{3}{2}\Big\},$$

and consequently,

$$\rho(x,t) = \begin{cases} 2 & \text{in region } 4\,, \\ 5 & \text{in region } 5\,. \end{cases} \tag{3.16}$$

This, together with (3.15), presents a complete solution to our initial value problem. Figure 3.10 shows the graphs of $\rho(x,t)$ at times $t = 1/2$ and $t = 5$. The animation in Example 5 shows the time evolution of the density $\rho(x,t)$ over the period $0 \leq t \leq 9$. □

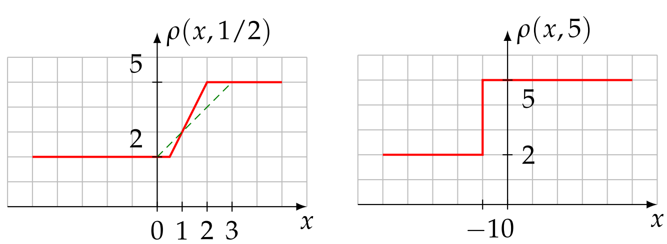
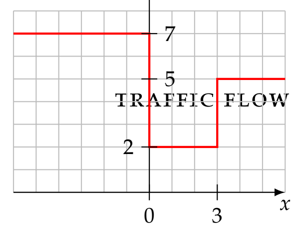


Figure 3.10: Graphs of the density $\rho(x,t)$ of the solution of Example 3's initial value problem, plotted at $t = 1/2$ and $t = 5$. The slanted and dashed gray line in the diagram on the left depicts the initial the initial value $\rho(x, 0)$.

Example 3.6. Let $v_{\max} = 4$, $\rho_{\max} = 8$, and

$$f(x) = \begin{cases} 7 & \text{if } x < 0, \\ 2 & \text{if } 0 < x < 3, \\ 5 & \text{if } x > 3. \end{cases}$$



Find the traffic density $\rho(x,t)$.

Solution. The Rankine–Hugoniot jump condition in equation (3.13) implies, among other things, that if the shock separates two regions of uniform density and velocity in the space-time diagram, then the shock velocity is a constant, and consequently, the shock appears as a straight line in that diagram. In this section we analyze a situation where the conditions are non-uniform on one side of the shock. As a result, the shock velocity is not constant and the shock appears as a bowed line in the space-time diagram.

Here is what we have for $v(\rho)$ and $c(\rho)$ based on (3.4) and (3.5):

$$v(\rho) = 4 - \frac{1}{2}\rho, \quad c(\rho) = 4 - \rho, \tag{3.17}$$

The discontinuities at $x = 0$ and $x = 3$ of f give rise to a rarefaction wave originating at $x = 0$ and a shock wave originating at $x = 3$. Figure 3.11(a) depicts our initial attempt at sketching the characteristics.

To complete that diagram, let's begin with calculating the velocity of the shock that originates at $x = 3$. To the left of the shock we have, referring to equations (3.4) and (3.5), $\rho = 2$, $c(\rho) = 2$, $v(\rho) = 3$, and therefore $\rho\, v(\rho) = 6$. To the right of the shock we have, $\rho = 5$, $c(\rho) = -1$, $v(\rho) = 3/2$, and therefore $\rho\, v(\rho) = 15/2$. Then we obtain the shock's velocity from the the Rankine–Hugoniot jump condition (3.13) as

$$v_{\text{shock}} = \frac{15/2 - 6}{5 - 2} = \frac{1}{2},$$

and therefore the shock propagates as $x = 3 + \frac{1}{2}t$. This is shown as the thick green line in Figure 3.11(b).

That calculation of the shock is based on the interaction of the characteristics in regions 2 and 3 of Figure 3.11(b). The uppermost characteristic in region 2 is the one that emanates from the origin, and is given by $x = 2t$. That characteristic intersects the shock $x = 3 + \frac{1}{2}t$ at $x = 4$, and $t = 2$. For $t > 2$ the shock is formed by the interaction of the characteristics from regions 3 and 4. That's the part of the diagram that we turn to next.

The characteristics in the rarefaction region 4 are $x = c(\rho)\,t = (4 - \rho)\,t$, and therefore $\rho = 4 - \frac{x}{t}$. Consider a point P with coordinates (x,t) along the shock that separates the regions 3 and 4. Just to the left of P the density is $\rho_{\text{left}} = 4 - \frac{x}{t}$, therefore the traffic velocity, as calculated from (3.17) is

$$v_{\text{left}} = 4 - \frac{1}{2}\rho_{\text{left}} = 4 - \frac{1}{2}\Big(4 - \frac{x}{t}\Big) = 2 + \frac{x}{2t}.$$

The density and velocity just to the right of P are those in the homogeneous region 3, that is

$$\rho_{\text{right}} = 5, \quad v_{\text{right}} = \frac{3}{2}.$$

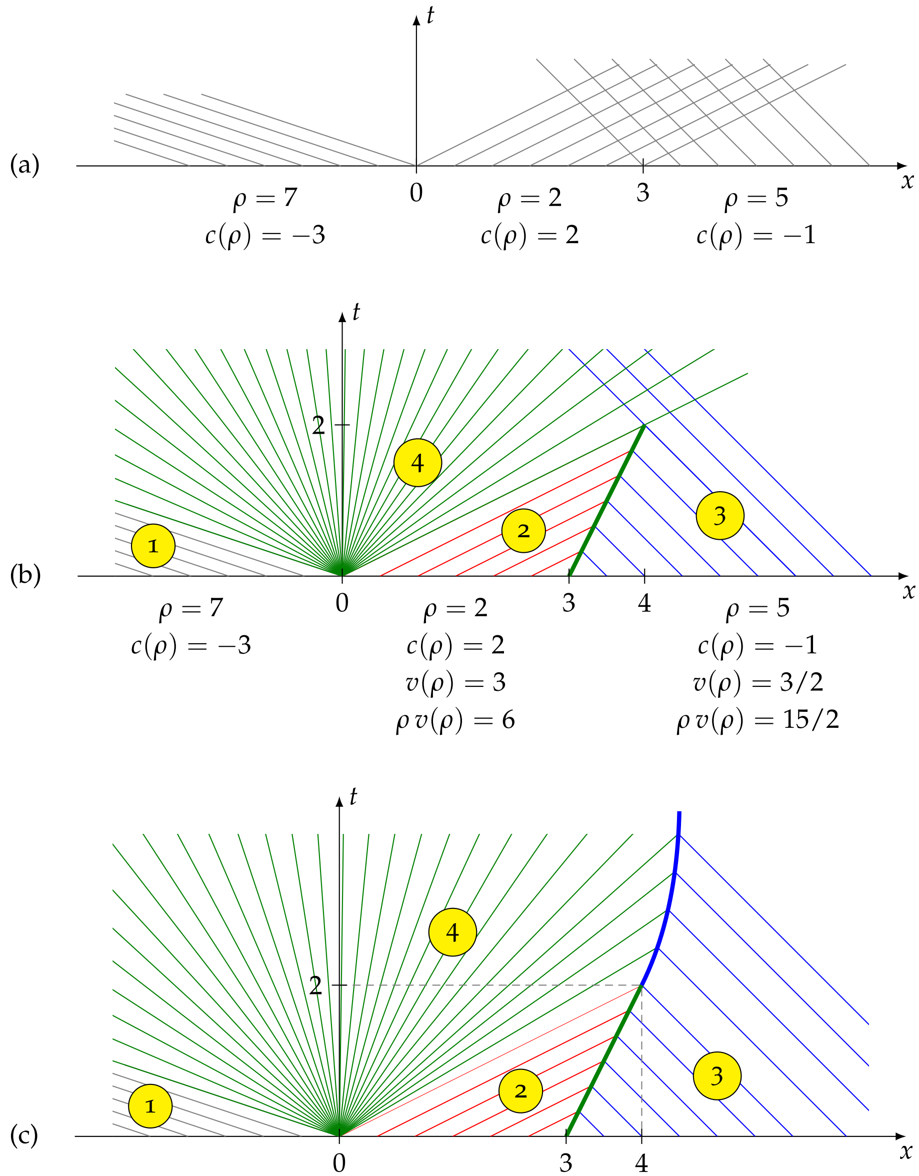


Figure 3.11: These sequence of drawings illustrate the steps involved in the construction of the space-time diagram containing a bowed shock wave.

Then the Rankine-Hugoniot jump condition determines the shock velocity as

$$v_{\text{shock}} = \frac{5\left(\frac{3}{2}\right) - \left(4 - \frac{x}{t}\right)\left(2 + \frac{x}{2t}\right)}{5 - \left(4 - \frac{x}{t}\right)} = \frac{x - t}{2t}.$$

We see that unlike the previous cases, the shock velocity is *not* a constant, and consequently the shock traces a curve in the space-time diagram. Let $x(t)$ be the x coordinate of the point on the shock at time t. Then the shock's velocity would be dx/dt. In view of the expression for the shock velocity calculated above, we arrive at the initial value problem

$$\frac{dx}{dt} = \frac{x - t}{2t}, \quad x(2) = 4.$$

This first order linear ODE may be readily solved via the method of integrating factors. Its solution is the equation of the curved part of the shock:

$$x(t) = 3\sqrt{2t} - t, \quad t \geq 2. \tag{3.18}$$

Figure 3.11(c) shows the curved part of the shock as a thick blue line. Altogether, the complete equation of the shock for all $t \geq 0$ is $x = S(t)$, where

$$S(t) = \begin{cases} 3 + \frac{1}{2}t & \text{if } t < 2, \\ 3\sqrt{2t} - t & \text{if } t \geq 2. \end{cases}$$

That enables us to put the pieces together and arrive at

$$\rho(x,t) = \begin{cases} 7 & \text{if } x < -3t & \text{(region 1)}, \\ 2 & \text{if } x > 2t \text{ and } x < 3 + \dfrac{1}{2}t & \text{(region 2)}, \\ 5 & \text{if } x > S(t) & \text{(region 3)}, \\ 4 - \dfrac{x}{t} & \text{otherwise} & \text{(region 4)}. \end{cases}$$

Remark: The depiction of the bowed shock in Figure 3.11(c) may give the false impression that it tends toward a vertical asymptote. That's certainly not true. The shock's equation in (3.18) indicates that as t grows, $x(t)$ approaches $-t$ since t dominates $\sqrt{t}$ for large t. If there were room within that figure to extend to larger values of t, we would see the shock turn around and ultimately proceed in the northwesterly direction.

Remark: The *strength* of a shock is measured by the jump of the density across it. In Figure 3.11(c) the strength of the segment of the shock bounded by regions 2 and 3 is constant along its length since the densities within the regions 2 and 3 are uniform. The strength of the rest of the shock, bounded by the regions 3 and 4 is *not* constant along its length because the density within region 4 is variable. It can be shown (consider it a challenge!) that the strength of the shock gradually diminishes, and goes to zero, as t gets large. In other words, the shock fades away in the long run. □

3.5 *Exercises*

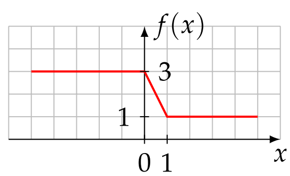
Each of the following exercises provides values of the parameters $v_{\max}$, $\rho_{\max}$, and the initial density $f(x)$. Solve the initial value problem (3.6) for the traffic density $\rho(x,t)$ and sketch a space-time diagram. Additionally, plot one or more graphs of ρ as functions of x, at t values of your choice, in order to illustrate how the solution evolves in time.

3.1. $v_{\max} = 4, \quad \rho_{\max} = 8, \quad f(x) = \begin{cases} 5 & \text{if } x < 0, \\ 3 & \text{if } x > 0. \end{cases}$

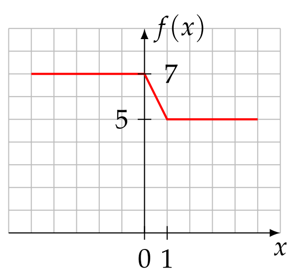
3.2. $v_{\max} = 4, \quad \rho_{\max} = 8, \quad f(x) = \begin{cases} 3 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$

3.3. $v_{\max} = 4, \quad \rho_{\max} = 8, \quad f(x) = \begin{cases} 7 & \text{if } x < 0, \\ 5 & \text{if } x > 0. \end{cases}$

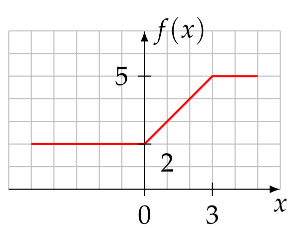
3.4. $v_{\max} = 4, \quad \rho_{\max} = 8,$



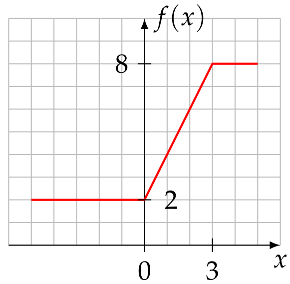
3.5. $v_{\max} = 4, \quad \rho_{\max} = 8,$



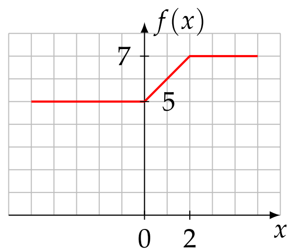
3.6. $v_{\max} = 4, \quad \rho_{\max} = 8,$



3.7. $v_{\max} = 4, \quad \rho_{\max} = 8,$



3.8. $v_{\max} = 2, \quad \rho_{\max} = 8,$



3.9. Show that when the velocity versus density is linear, as in (3.4), then the velocity of a shock is the average of the velocities of the characteristics on its either side.

Hint: Rearrange (3.5) to express ρ in as a function of c. Then show that

$$\rho v(\rho) = \frac{1}{4}\rho_{\max} v_{\max}\left[1 - \left(\frac{c}{v_{\max}}\right)^2\right].$$

Insert these into the Rankine–Hugoniot jump condition and simplify.

3.10. Section (3.1)'s derivation of the traffic model assumes a linear relationship between the traffic's velocity and density; see equation (3.4) and Figure 3.2. Modify that model by making that dependence quadratic:

$$v(\rho) = v_{\max}\left(1 - \left(\frac{\rho}{\rho_{\max}}\right)^2\right).$$

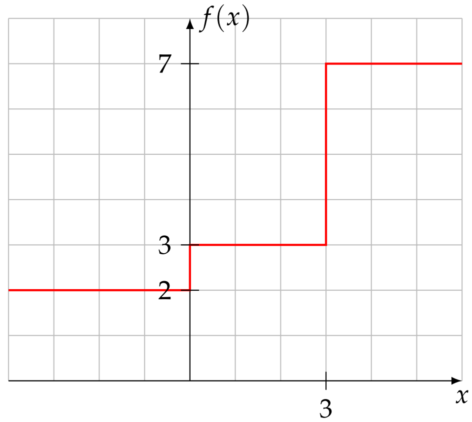
See Figure 3.1 for the graph that function.

1. What is the equivalent of equation (3.5) under this new model?

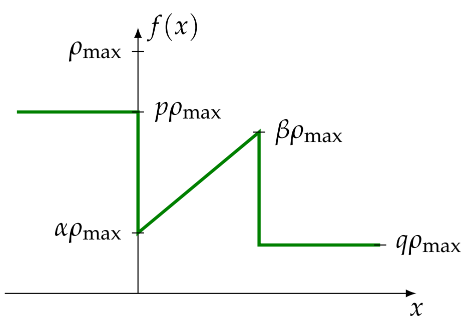
2. Solve Example 3.1's initial value problem under this new model.

3.11. ***Where two shocks merge into one...***

$v_{\max} = 4, \quad \rho_{\max} = 8,$



3.12. ***Where two rarefaction waves merge into a shock...***
In the traffic flow problem, take the initial density $f(x)$ as depicted in the graph



where $a > 0$ and p, q, α, β are constants in $[0, 1]$. Additionally, assume that $\alpha < p$ and $\beta > q$ and $\alpha < \beta$. Show that this gives rise to a pair or rarefaction waves that start out at $x = 0$ and $x = a$ and later merge into a shock wave.
Comment 1: This problem was posed and solved by Owen McMann, an undergraduate student at UMBC, who also made the interesting (and unanticipated) observation that the resulting shock travels at constant speed.
Comment 2: If you feel more comfortable working with numbers rather than symbols, you may at first try solving the problem using the numerical parameters

$$a = 2, \quad v_{\max} = 2, \quad \rho_{\max} = 4, \quad p = \frac{3}{4}, \quad q = \frac{1}{5}, \quad \alpha = \frac{1}{4}, \quad \beta = \frac{2}{3},$$

before tackling the symbolic solution.

3.13. Traffic stopped at a red light at $x = 0$ occupies the interval $[-L, 0]$ on the x axis, and therefore has density $\rho_{\max}$. The light turns green at $t = 0$. Assume that no other vehicles enter the highway.

1. Determine the traffic's density at later times in terms of the (unspecified) symbolic constants $v_{\max}$ and $\rho_{\max}$.

2. Sketch the graphs of $\rho(x,t)$ as a function of x at times $t = k\frac{L}{v_{\max}}$, where $k = 0, \frac{1}{2}, 1, 2, 3, 4, 5$.

3. What is $\lim_{t\to\infty} \rho(x,t)$ at a fixed x?

4. What is the velocity $v(x,t)$ of a vehicle that is passing through the location x at time t?

5. What is $\lim_{t\to\infty} v(x,t)$ at a fixed x?

6. Determine the position $x(t)$ of a vehicle which is located at $x = -a$ $(0 \le a \le L)$ at $t = 0$.

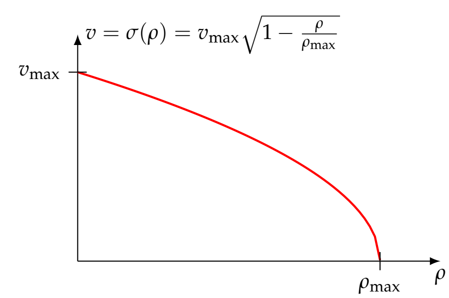
7. Consider two vehicle that are located at $x = -a$ and $x = -b$ $(0 < a < b < L)$ at time $t = 0$. What is the distance between them in the long run?

8. The tail end of the traffic lies on the shock as there are no vehicles to the left of the shock. What is the traffic density at the tail end as a function of time?

3.14 (**A little challenge**). Section (3.1)'s derivation of the traffic model assumes a linear relationship between the traffic's velocity and density; see equation (3.4) and Figure 3.2. Modify that model by changing the velocity profile to

$$v = \sigma(\rho) = v_{\max}\sqrt{1 - \frac{\rho}{\rho_{\max}}},$$

as in the graph shown on the right.



1. What is the equivalent of equation (3.5) under this new model?

2. Solve Example 3.1's initial value problem under this alternative model.

3.15. Consider the linear velocity versus density model, as in (3.4). Show that the Rankine–Hugoniot jump condition may be expressed as

$$v_{\text{shock}} = \frac{v_{\max}}{\rho_{\max}}\Big(\rho_{\max} - \rho_{\text{left}} - \rho_{\text{right}}\Big),$$

where ρ_{left} and ρ_{right} are the densities to the left and to the right of the shock. Conclude that if $\rho_{\text{left}} + \rho_{\text{right}} > \rho_{\max}$, then the shock moves to the left, and if $\rho_{\text{left}} + \rho_{\text{right}} < \rho_{\max}$, then the shock moves to the right.

3.16. Consider the initial value problem (3.6) for traffic flow, where $c(\rho)$ is as in (3.5), and where the initial condition $u(x,0) = f(x)$ is as in

$$f(x) = \begin{cases} \rho_1 & \text{if } x < 0, \\ \rho_1 + (\rho_2 - \rho_1)\dfrac{x}{a} & \text{if } 0 < x < a, \\ \rho_2 & \text{if } x > a, \end{cases}$$

where $0 < \rho_1 < \rho_2 < \rho_{\max}$, and $a > 0$. Show that a shock develops at the point (x_0, t_0) in space-time, where

$$t_0 = \frac{a\rho_{\max}}{2v_{\max}(\rho_2 - \rho_1)}, \quad x_0 = \frac{a(\rho_{\max} - 2\rho_1)}{2(\rho_2 - \rho_1)}.$$

Interesting observation: We see that if $\rho_1 < \frac{1}{2}\rho_{\max}$, then $x_0 > 0$, else $x_0 \leq 0$.

3.17 (**An unexpected shock!**). Solve Example 3.1's initial value problem with the traffic model

$$v = \sigma(\rho) = v_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right)^2. \tag{3.19}$$

3.18 (**Yet another traffic model**). Solve Example 3.1's initial value problem with the traffic model

$$v = \sigma(\rho) = v_{\max}\left(1 - \sqrt{\frac{\rho}{\rho_{\max}}}\right).$$

4
Classification of second order PDEs

In the remainder of this book we study second order PDEs, that is, PDEs that involve second order derivatives of the unknown function. Of particular interest are the classical

Wave equation in the unknown $u(x,t)$:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{4.1}$$

Heat equation in the unknown $u(x,t)$:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \tag{4.2}$$

Laplace's equation in the unknown $u(x,y)$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{4.3}$$

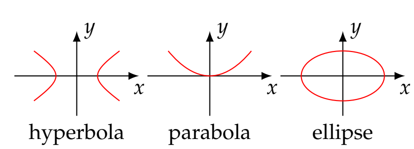
and many of their variations. The reason for interest in these particular equations is twofold. For one thing, these equations occur in a host of mathematical models in engineering, physics, chemistry,[1] and therefore an intimate knowledge of their properties is a prerequisite for any working scientist. For another thing, these equations are prototypes of much wider classes of *hyperbolic*, *parabolic*, and *elliptic* PDEs, respectively. What we learn about the wave, heat, and Laplace's equations extends with some modifications to, and provides intuition and guidance for, the understanding of those more general PDEs.

[1] And to a lesser extent in mathematical biology and finance.

If you pursue your studies of PDEs beyond this introductory textbook, you are likely to encounter the terms hyperbolic, parabolic, and elliptic. We define those terms here in order to place our limited study in the wider context.

You may recall from your studies of calculus and analytic geometry, that the graphs in the Cartesian coordinates of the equation

$$\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1, \quad \frac{x^2}{A^2} - \frac{y}{B} = 1, \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$



are hyperbolas, parabolas, and ellipses, respectively. Furthermore, any quadratic equation of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0,$$

may be reduced, after a change of variables, to one of the preceding three depending on the sign of the *discriminant* $\delta = b^2 - 4ac$. Specifically, if $\delta > 0$, then we have a hyperbola, if $\delta = 0$ then we have a parabola, and if $\delta < 0$, then we have an ellipse.

By analogy, the second order PDE in the unknown $u(x, y)$:

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = q \tag{4.4}$$

is said to be of the hyperbolic, parabolic, or elliptic type depending on whether $\delta = b^2 - 4ac$ is positive, zero, or negative, respectively.[2,3] It turns out, although we won't delve into the subject here, that the behavior of the solutions of (4.4) differ wildly depending on the equation's type, hence the need for classification. Note, in particular, that the wave, heat, and Laplace's equations are of the hyperbolic, parabolic, and elliptic types, in that order.[4]

The left-hand side of (4.4) defines a *partial differential operator* L through

$$L(u) = au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu. \tag{4.5}$$

L is *linear* in the sense that

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v) \tag{4.6}$$

for all sufficiently differentiable functions $u = u(x, y)$ and $v = v(x, y)$, and arbitrary constants α and β.[5] To verify that L indeed is linear, we calculate

$$\begin{aligned}
L(\alpha u + \beta v) &= a(\alpha u + \beta v)_{xx} + b(\alpha u + \beta v)_{xy} + c(\alpha u + \beta v)_{yy} \\
&\qquad + d(\alpha u + \beta v)_x + e(\alpha u + \beta v)_y + f(\alpha u + \beta v) \\
&= a(\alpha u_{xx} + \beta v_{xx}) + b(\alpha u_{xy} + \beta v_{xy}) + c(\alpha u_{yy} + \beta v_{yy}) \\
&\qquad + d(\alpha u_x + \beta v_x) + e(\alpha u_y + \beta v_y) + f(\alpha u + \beta v) \\
&= \alpha(au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu) \\
&\qquad + \beta(av_{xx} + bv_{xy} + cv_{yy} + dv_x + ev_y + fv) \\
&= \alpha L(u) + \beta L(v),
\end{aligned}$$

and thus substantiate the claim.

The PDE

$$L(u) = q,$$

where L is a linear partial differential operator and q is a given function of x and y,[6] is said to be *linear*. If q is zero, then the PDE is said to be *homogeneous*. Otherwise it is *nonhomogeneous*.

[2] The analogy is skin deep. The nomenclature is merely a mnemonic device; there is no ellipse in an elliptic equation.

[3] Here, and what follows, the coefficients a through q may depend on x and y in general. Therefore, the discriminant D, which would depend on x and y, may change sign and consequently (4.4) may change type depending on the region within the xy plane.

[4] The independent variables in (4.4) are x and y. To make the connection to the wave and heat equations, change y to t.

[5] The linearity of L, embodied in the equation (4.6), plays an absolutely crucial role in the rest of this book.

[6] The function q is usually called the PDE's *forcing term*.

For instance, the PDE $u_{xx} + \sqrt{1 + x^2 + y^2}\, u_{yy} = \cos(xy)$ in the unknown $u(x, y)$ is linear because the corresponding partial differential operator

$$L(u) = u_{xx} + \sqrt{1 + x^2 + y^2}\, u_{yy},$$

is linear, as it is a special case of (4.5). The PDE is nonhomogeneous because its forcing term is nonzero.

On the other hand, the PDE $L(u) = 0$ in the unknown function $u(x, t)$, where $L(u) = u_{tt} + uu_{xx}$, is *not* linear. To substantiate that claim, is suffices to demonstrate a single case where the linearity requirement (4.6) fails. Take, for instance, $\alpha = 2$, $\beta = 0$. Then we have

$$\begin{aligned} &2L(u) = 2\big(u_{tt} + uu_{xx}\big), \\ &L(2u) = (2u)_{tt} + (2u)(2u)_{xx} = 2u_{tt} + 2^2 uu_{xx} = 2\big(u_{tt} + 2uu_{xx}\big). \end{aligned}$$

We see that $L(2u) \neq 2L(u)$, so L is not linear.

Here are the linear operators associated with the wave, heat, and Laplace's equations:[7]

$$\begin{aligned} L(u) &= \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} && \text{the wave operator (hyperbolic)} \\ L(u) &= \frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2}, && \text{the heat operator (parabolic)} \\ L(u) &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. && \text{Laplace's operator (elliptic)} \end{aligned}$$

[7] Actually $L(u) = \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}$ expresses *the result of* applying the wave operator to the function u. The wave operator itself is $L = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}$. Same goes for the heat and Laplace's operators.

The wave, heat, and Laplace's equations presented in (4.1), (4.2), and (4.3) are homogeneous. If we add forcing functions q to their right-hand sides, as later we shall, we will obtain the nonhomogeneous versions of those equations.

4.1 *Exercises*

4.1. Determine whether the following PDEs are linear or nonlinear.

1. $tu_t = xu_{xx} + xt^2$
2. $xu_{tt} + tu_{xx} = 0$
3. $u_{xx} + u_{yy} = u^3$

4.2. Determine whether the following PDEs are hyperbolic, parabolic, or elliptic.

1. $u_{xy} = 0$
2. $u_{xx} + u_{xy} + u_{yy} = 1 + u$
3. $u_t + u_{xt} = u_{xx}$

4.3. The **Euler–Tricomi** equation[8] arises in the study of transconic flow:

$$u_{xx} + xu_{yy} = 0, \quad -\infty < x < \infty, \quad -\infty < y < \infty.$$

Is that equation linear or nonlinear? Determine the equation's type.

Hint: The equation's type varies with the location in the (x, y) plane.

4.4. **Chaplygin**'s equation[9] arises in the study of transconic flow:

$$\frac{\partial^2 \phi}{\partial \theta^2} + \frac{v^2}{1 - v^2/c^2} \frac{\partial^2 \phi}{\partial v^2} + v \frac{\partial \phi}{\partial v} = 0,$$

where $\phi(\theta, v)$ is the flow's velocity potential, $\langle v \cos\theta, v \sin\theta \rangle$ is the velocity vector, and c is the speed of sound. Is that equation linear or nonlinear? What is its type?

[8] Named after *Leonhard Euler* (1707–1783) and *Francesco Giacomo Tricomi* (1897–1978).

[9] Named after *Sergei Alekseevich Chaplygin* (1869–1942).

5
The wave equation in 1D

The classical *wave equation* in one space dimension in the unknown $u(x,t)$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

along with its many variants, is the prototype of a very large class of *hyperbolic equations* that arise in a plethora of applications such as

- vibration of solid structures (strings, beams, membranes, plates)
- propagation of sound (acoustics)
- propagation of seismic waves
- geological exploration, oil well detection
- aerodynamics and supersonic flight
- propagation of electromagnetic waves (radiant heat, light, radio waves, microwaves, lasers, fiber optics, antennas)

In this chapter we focus on one elementary and easy-to-grasp application, namely, propagation of waves along a taut string/rope/chain.

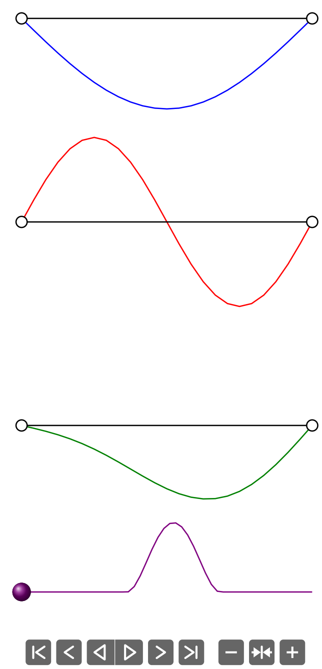


Figure 5.1: The top three animations illustrate the vibration of a taut string segment fixed at its endpoints. The first two correspond to the string's *natural modes* of vibration—their displacements are of the form $u(x,t) = f(x)g(t)$, that is, the string's shape changes in a self-similar manner over time. The third animation is a blend of the first two and it is certainly *not* of the form $f(x)g(t)$. The fourth animation illustrates the propagation of a pulse induced through the motion of the support at one end of a taut semi-infinite string.

5.1 *Waves along a string*

We wish to derive the equation of motion of a taut string, such as that of a guitar, cello, or piano string supported at its ends, or even a conceptually infinitely long string whose endpoints, being infinitely far, are inaccessible to us. See the sample animations on the right.

Every part of the string has some mass. It moves. Naturally Newton's law of motion comes into play. As we shall see, that leads to the wave equation, where $u(x,t)$ is the string's deviation from its equilibrium configuration. In the following subsection we present a "quick and dirty" derivation of the string's equation of motion. The derivation is far from rigorous as it involves quite a few hidden, even unjustified, assumptions as well as sloppy mathematics, but the result is correct within the bounds of the (unstated) assumptions. The purpose of this sketchy derivation is to convey the central idea that leads to the equation of motion of a taut string, The rigorous derivation of that equation is surprisingly nontrivial. See Antman [1] for a careful derivation, Weinberger [25] for a less technical and more accessible treatment, and Keller [14] for a surprising special case.

5.1.1 A quick-and-dirty derivation

We write T for the (constant) magnitude of the tensile force within the string, and $\rho(x)$ for its *linear mass density*[1], that is, its mass per unit length. We set up the x coordinate axis to coincide with the string's undeformed state. We write $u(x,t)$ for the string's *lateral displacement* (also called the *transverse displacement*) away from the equilibrium, and $\theta(x,t)$ for the angle between the string and the equilibrium state at the location x at time t, We assume that the deflection away from equilibrium is small, so we may take the displacement to be perpendicular to the x axis, and we may approximate $\sin\theta \approx \tan\theta \approx \theta$.

[1] This derivation allows for the linear mass density to vary with x but the case where ρ is a constant is just as interesting.

Let us focus on a small (infinitesimal) segment of the string between the locations x and $x + \Delta x$, as shown in Figure 5.2. The segment's mass is $\rho(x)\Delta x$, and its vertical acceleration is $\frac{\partial^2 u}{\partial t^2}$. According to Newton's law of motion, mass times acceleration, $\rho(x)\Delta x \frac{\partial^2 u}{\partial t^2}$, equals the resultant of vertical forces acting on the segment. We see in the figure that the vertical components of tensile forces at the segment's endpoints are $-T\sin\theta(x,t)$ and $T\sin\theta(x+\Delta x,t)$. We conclude that

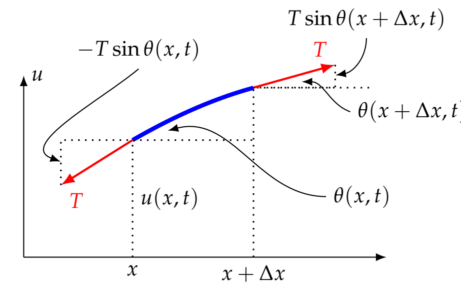


Figure 5.2: We apply Newton's law of motion to an infinitesimal segment of the string between x and $x + \Delta x$. The string's displacement is intentionally exaggerated in order to bring out the details.

$$\rho(x)\Delta x \frac{\partial^2 u}{\partial t^2} = T\sin\theta(x+\Delta x,t) - T\sin\theta(x,t).$$

We divide this through by Δx

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T\frac{\sin\theta(x+\Delta x,t) - \sin\theta(x,t)}{\Delta x}$$

and pass to the limit as $\Delta x \to 0$:

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T\frac{\partial}{\partial x}\big(\sin\theta\big).$$

However, by our smallness assumption of θ we may replace $\sin\theta$ by $\tan\theta$, and noting that $\tan\theta = \frac{\partial u}{\partial x}$, arrive at, $\rho(x)\frac{\partial^2 u}{\partial t^2} = T\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$, that is

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}. \tag{5.1}$$

This is the equation of motion of a taut string. The left-hand side is in effect mass times acceleration, and the right-hand side is the corresponding force.

In the special case of a uniform string where ρ is independent of x, it is customary to let $T/\rho = c^2$, that is,

$$c = \sqrt{\frac{T}{\rho}}, \tag{5.2}$$

and cast (5.1) into the standard form of the one-dimensional *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{5.3}$$

We will soon see that c is the speed of propagation of waves/disturbances/signals along the string.

Remark 5.1. Let us make a note that in the derivation above we have approximated the vertical component of the tensile force within the string as

$$T_{\text{vert}} = T \frac{\partial u}{\partial x}. \tag{5.4}$$

We will have uses for this later on.

5.2 *Reducing the wave equation to a system of 1st order PDEs*

We are going to show how, through a sleight of hand, the second order PDE (5.3) with a constant c may be reduced to an equivalent pair of first order PDEs which then can be solved through Chapter 1's method of characteristics. Toward that end, we introduce the auxiliary variables v and w through

$$v = u_t - cu_x, \quad w = u_t + cu_x, \tag{5.5}$$

and then calculate

$$\begin{aligned}
v_t + cv_x &= \big[u_t - cu_x\big]_t + c\big[u_t - cu_x\big]_x \\
&= \big[u_{tt} - cu_{xt}\big] + c\big[u_{tx} - cu_{xx}\big] = u_{tt} - c^2 u_{xx} \overset{\text{by (5.3)}}{=} 0, \\
w_t - cw_x &= \big[u_t + cu_x\big]_t - c\big[u_t + cu_x\big]_x \\
&= \big[u_{tt} + cu_{xt}\big] - c\big[u_{tx} + cu_{xx}\big] = u_{tt} - c^2 u_{xx} \overset{\text{by (5.3)}}{=} 0.
\end{aligned}$$

In summary:

$$v_t + cv_x = 0, \quad w_t - cw_x = 0. \tag{5.6}$$

This pair of (uncoupled!) first order PDEs (5.6) is equivalent to the second order PDE (5.3), that is, (5.3) $\Leftrightarrow$ (5.6). We have already seen that (5.3) $\Rightarrow$ (5.6). To see the converse, we observe that equations (5.5) imply that

$$u_t = \frac{1}{2}(w + v), \quad u_x = \frac{1}{2c}(w - v). \tag{5.7}$$

Therefore, once we solve (5.6) for v and w, we may construct u through (5.7) as we will see in the following section.

In view of that equivalence, we turn our attention to solving the pair of equations (5.6).

5.3 *Characteristics come to the rescue*

Solving the first order PDEs in (5.6) through the method of characteristics in the usual way, we obtain

$$v(x,t) = \phi(x-ct), \quad w(x,t) = \psi(x+ct),$$

where $\phi(x)$ and $\psi(x)$ are the initial conditions of v and w. Then from (5.7) we get

$$\begin{aligned} u_t(x,t) &= \frac{1}{2}\big[\psi(x+ct) + \phi(x-ct)\big], \\ u_x(x,t) &= \frac{1}{2c}\big[\psi(x+ct) - \phi(x-ct)\big]. \end{aligned}$$

Now that we have succeeded in calculating the derivatives u_t and u_x, we proceed to calculate u itself. For that purpose, it helps to introduce the functions F and G, defined through their derivatives, as

$$F'(x) = -\frac{1}{2c}\phi(x), \quad G'(x) = \frac{1}{2c}\psi(x). \tag{5.8}$$

The we have

$$\begin{aligned} u_t(x,t) &= \frac{1}{2}\big[2c\,G'(x+ct) - 2c\,F'(x-ct)\big] = c\,G'(x+ct) - c\,F'(x-ct), \\ u_x(x,t) &= \frac{1}{2c}\big[2c\,G'(x+ct) + 2c\,F'(x-ct)\big] = G'(x+ct) + F'(x-ct). \end{aligned}$$

Integrating the first equation with respect to t and the second with respect to x, we obtain

$$\begin{aligned} u(x,t) &= G(x+ct) + F(x-ct) + A(x), \\ u(x,t) &= G(x+ct) + F(x-ct) + B(t), \end{aligned}$$

where $A(x)$ and $B(t)$ are the integration "constants". Subtracting the two equations results in $A(x) = B(t)$. This says that $A(x)$ does not depend on x (since it's equal to $B(t)$ for all x). Therefore $A(x)$ is a constant, and therefore $B(t)$ is also a constant. Let's write C for that common constant.

We have thus arrived at $u(x,t) = F(x-ct) + G(x+ct) + C$. But the presence of C is immaterial since F and G are defined through their derivatives only, and therefore they are determined up to additive constants anyway. So we absorb C into F or G and conclude that

$$u(x,t) = F(x-ct) + G(x+ct) \tag{5.9}$$

is *the general solution of the wave equation* (5.3). The functions F and G are defined in (5.8) in terms of the arbitrary functions ϕ and ψ, therefore they may be regarded arbitrary as well. In Section 5.5 we will learn how to express F and G in terms of the wave equation's initial conditions.

Remark 5.2. It's worth noting that the general solution of a second order ODE such as $a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$ comes with two arbitrary *constants*, as in $y(t) = c_1 y_1(t) + c_2 y_2(t)$. In contrast, the general solution of a second order PDE (5.3) comes with two arbitrary *functions*, as the F and G above.

5.4 An alternative approach to solving the wave equation

In this section we present an alternative approach to deriving the general solution (5.9) of the wave equation. We are going to introduce a change of variables, from (x, t) to (ξ, η), defined according to:

$$\xi = x - ct, \quad \eta = x + ct. \tag{5.10}$$

That change of variables is one-to-one as we may verify by solving the equation pair above for x and t:

$$x = \frac{1}{2}(\eta + \xi), \quad t = \frac{1}{2c}(\eta - \xi).$$

Let $u(x, t)$ be a solution of the wave equation (5.3). Define $U(\xi, \eta)$ through

$$U(\xi, \eta) = u(x, t), \tag{5.11}$$

where (x, t) and (ξ, η) are related through (5.10). With the help of the chain rule, and observing that $\xi_x = 1$, $\eta_x = 1$, we calculate

$$u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta,$$

and

$$u_{xx} = (U_\xi + U_\eta)_\xi \xi_x + (U_\xi + U_\eta)_\eta \eta_x = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}. \tag{5.12a}$$

Similarly, we find that

$$u_{tt} = c^2\Big[U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}\Big]. \tag{5.12b}$$

Combining the equations (5.12a) and (5.12b) we see that

$$c^2 u_{xx} - u_{tt} = 4c^2 U_{\xi\eta}.$$

But the left-hand side of that equation is zero since u is a solution of the wave equation. We conclude that

$$U_{\xi\eta} = 0. \tag{5.13}$$

This is an interesting result. It says that after the change of coordinates (5.10), the wave equation (5.3) transforms to the intriguingly simple form of (5.13). That equation is quite straightforward to solve, as it is equivalent to $(U_\eta)_\xi = 0$, which says that U_η is independent of ξ, that is, $U_\eta(\xi, \eta) = \psi(\eta)$ for some arbitrary function ψ. We integrate this with resepct to η and arrive at

$$U(\xi, \eta) = \phi(\xi) + \int \psi(\eta)\, d\eta,$$

where $\phi(\xi)$ plays the role of the arbitrary constant[2] of integration. To align the notation with that of the previous solution (5.9), we write $F(\xi)$ and $G(\eta)$ for the two terms on the right-hand side[3] and thus arrive at

$$U(\xi, \eta) = F(\xi) + G(\eta).$$

[2] The integration is with respect to η, so the "constant" of integration is independent of η, but it can depend on ξ, as it generally does.

[3] Since ϕ and ψ are arbitrary, so are F and G.

Returning to the original (x, t) variables through (5.10) and (5.11), this takes the form

$$u(x, t) = F(x - ct) + G(x + ct),$$

which is identical to (5.9).

5.5 *The initial value problem*

Since the wave equation is of second order in t, the associated initial value problem needs to specify both both u and u_t at $t = 0$:[4]

$$u_{tt} = c^2 u_{xx}, \qquad -\infty < x < \infty, \quad t > 0, \tag{5.14a}$$

$$u(x, 0) = f(x), \qquad -\infty < x < \infty, \tag{5.14b}$$

$$u_t(x, 0) = g(x), \qquad -\infty < x < \infty. \tag{5.14c}$$

[4] This is akin to solving the initial value problem for the ODE

$$m\ddot{u}(t) + ku(t) = 0,$$
$$u(0) = u_0, \quad \dot{u}(0) = u_1$$

that models the oscillations of a simple mass and spring system. Note that we specify both the *initial displacement* and the *initial velocity* to obtain a unique solution.

The general solution of the PDE (5.14a) is available in (5.9). Our task here is to determine F and G in terms of the given initial data f and g. That's a matter of straightforward calculation. We have

$$u(x, t) = F(x - ct) + G(x + ct),$$

and therefore

$$u_t(x, t) = -cF'(x - ct) + cG'(x + ct).$$

Letting $t = 0$ in these, and applying the initial conditions, we see that

$$F(x) + G(x) = f(x), \tag{5.15a}$$

$$-cF'(x) + cG'(x) = g(x). \tag{5.15b}$$

Isolating $G(x)$ in (5.15a) and inserting the result into (5.15b), we obtain

$$-cF'(x) + c\big[f'(x) - F'(x)\big] = g(x),$$

which we solve for $F'(x)$:

$$F'(x) = \frac{1}{2} f'(x) - \frac{1}{2c} g(x),$$

and then integrate:

$$F(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_0^x g(\xi)\, d\xi + K. \tag{5.16a}$$

Having thus determined $F(x)$, we then calculate $G(x)$ from (5.15a):

$$G(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_0^x g(\xi)\, d\xi - K. \tag{5.16b}$$

We conclude that

$$F(x - ct) = \frac{1}{2} f(x - ct) - \frac{1}{2c} \int_0^{x-ct} g(\xi)\, d\xi + K,$$
$$G(x + ct) = \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_0^{x+ct} g(\xi)\, d\xi - K,$$

whence the general solution (5.9) takes the form

$$u(x,t) = \frac{1}{2} \Big[f(x - ct) + f(x + ct) \Big] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(\xi)\, d\xi. \tag{5.17}$$

The representation (5.17) of the solution of the initial value problem (5.14) was discovered by *Jean-Baptiste le Rond d'Alembert* in 1747 [5, 6, 7], and is referred to as *d'Alembert's solution* of the wave equation. See Jouve [13] for historical remarks on controversies that d'Alembert's work engendered at its time.

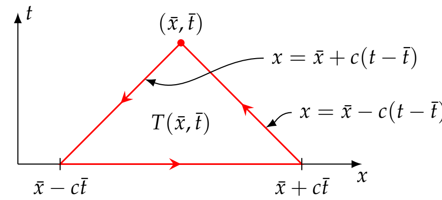
Remark 5.3. It is noteworthy that the general solution (5.9) of the wave equation and the solution (5.17) of the initial value problem impose no explicit restrictions on the continuity or differentiability of the functions F, G, f and g. Thus, for instance, we may plug a step function for f in (5.17) and obtain a discontinuous function for $u(x,t)$. But will that $u(x,t)$ be a solution to the wave equation? Not really, since u needs to be at least twice differentiable in x and in t for terms u_{tt} and u_{xx} in the wave equation to be meaningful. It is possible to introduce alternative formulations of the wave equation that put lesser demands on differentiability [1], but that will take us beyond the scope of this book.

5.6 *Deriving d'Alembert's solution via Green's Theorem*

Here is yet another way of deriving d'Alembert's solution to the 1D wave equation. For the sake of variety, here we add a nonhomogeneous forcing term $q(x,t)$ to the equation:

$$u_{tt} = c^2 u_{xx} + q(x,t) \qquad -\infty < x < \infty,\ t > 0, \tag{5.18a}$$
$$u(x,0) = f(x) \qquad -\infty < x < \infty, \tag{5.18b}$$
$$u_t(x,0) = g(x) \qquad -\infty < x < \infty. \tag{5.18c}$$

Pick any point $(\bar{x}, \bar{t})$ with $\bar{t} > 0$ in the (x, t) plane, and form the triangle $T(\bar{x}, \bar{t})$ with vertices at $(\bar{x}, \bar{t})$, $(\bar{x} - c\bar{t}, \bar{t})$, $(\bar{x} + c\bar{t}, \bar{t})$, as in the adjacent diagram. We are going to derive d'Alembert's solution to (5.18) through an application of Green's Theorem in the triangle $T(\bar{x}, \bar{t})$. Let's note that the equations of the triangle's left and right edges are



$$x = x_{\text{left}}(t) \equiv \bar{x} + c(t - \bar{t}), \qquad x = x_{\text{right}}(t) \equiv \bar{x} - c(t - \bar{t}).$$

According to Green's Theorem, for any pair of smooth function $P = P(x, t)$ and $Q = Q(x, t)$, and any domain T in the (x, t) plane bounded by a simple closed curve, we have

$$\iint_T \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial t}\right] dx\, dt = \int_{\partial T} \big(P\, dx + Q\, dt\big),$$

where the line integral on right-hand side is a taken counter-clockwise around the domain's boundary ∂T. In our case we have

$$\begin{aligned}
\iint_{T(\bar{x},\bar{t})} q(x,t)\, dx\, dt &= \iint_{T(\bar{x},\bar{t})} \big(u_{tt} - c^2 u_{xx}\big)\, dx\, dt \\
&= -\iint_{T(\bar{x},\bar{t})} \big[(c^2 u_x)_x - (u_t)_t\big]\, dx\, dt \\
&= -\int_{\partial T(\bar{x},\bar{t})} \big(u_t\, dx + c^2 u_x\, dt\big) \quad \text{[by Green's Theorem]} \\
&= -\left(\int_{\text{bottom}} + \int_{\text{right}} + \int_{\text{left}}\right) \big(u_t\, dx + c^2 u_x\, dt\big).
\end{aligned} \tag{5.19}$$

We compute each of the three integrals add them up.

- Along the bottom edge we have $dt = 0$ and $u_t(x, 0) = g(x)$. Therefore

$$\int_{\text{bottom}} \big(u_t\, dx + c^2 u_x\, dt\big) = \int_{\bar{x}-c\bar{t}}^{\bar{x}+c\bar{t}} u_t(x, 0)\, dx = \int_{\bar{x}-c\bar{t}}^{\bar{x}+c\bar{t}} g(x)\, dx. \tag{5.20a}$$

- Along the right edge we have $x = x_{\text{right}}(t) = \bar{x} - c(t - \bar{t})$. Therefore, $dx = -c\, dt$ and

$$\begin{aligned}
\int_{\text{right}} \big(u_t\, dx + c^2 u_x\, dt\big) &= \int_0^{\bar{t}} \Big[-c u_t\big(x_{\text{right}}(t), t\big)\, dt + c^2 u_x\big(x_{\text{right}}(t), t\big)\, dt\Big] \\
&= -c \int_0^{\bar{t}} \Big[-c u_x\big(x_{\text{right}}(t), t\big) + u_t\big(x_{\text{right}}(t), t\big)\Big]\, dt.
\end{aligned}$$

We note, however, that

$$-c u_x\big(x_{\text{right}}(t), t\big) + u_t\big(x_{\text{right}}(t), t\big) = \frac{d}{dt} u\big(x_{\text{right}}(t), t\big),$$

and therefore

$$\begin{aligned}\int_{\text{right}} \left(u_t\,dx + c^2 u_x\,dt\right) &= -c\int_0^{\bar t} \frac{d}{dt} u\big(x_{\text{right}}(t), t\big)\,dt \\ &= -c\Big[u\big(x_{\text{right}}(\bar t), \bar t\big) - u\big(x_{\text{right}}(0), 0\big)\Big] \\ &= -c\big[u(\bar x, \bar t) - u(\bar x + c\bar t, 0)\big] \\ &= -c\big[u(\bar x, \bar t) - f(\bar x + c\bar t)\big]. \qquad (5.20\text{b})\end{aligned}$$

- We evaluate the integral along the left edge in a similar way[5] and obtain

$$\int_{\text{left}} \left(u_t\,dx + c^2 u_x\,dt\right) = -c\big[u(\bar x, \bar t) - f(\bar x - c\bar t)\big]. \qquad (5.20\text{c})$$

[5] Note that now the integration limits are $\int_{\bar t}^{0} \cdots dt$.

Substituting the results (5.20a) through (5.20c) into (5.19), we arrive at

$$\begin{aligned}&\iint_{T(\bar x, \bar t)} q(x,t)\,dx\,dt \\ &\quad = -\left(\int_{\bar x - c\bar t}^{\bar x + c\bar t} g(x)\,dx - 2cu(\bar x, \bar t) + c\Big[f(\bar x - c\bar t) + f(\bar x + c\bar t)\Big]\right),\end{aligned}$$

from which we isolate $u(\bar x, \bar t)$:

$$\begin{aligned}u(\bar x, \bar t) = \frac{1}{2}\Big[f(\bar x - c\bar t) + f(\bar x + c\bar t)\Big] \\ + \frac{1}{2c}\int_{\bar x - c\bar t}^{\bar x + c\bar t} g(x)\,dx + \frac{1}{2c}\iint_{T(\bar x, \bar t)} q(x,t)\,dx\,dt.\end{aligned}$$

To simplify the notation, we replace the "dummy" integration variables x and t with ξ and τ. Then we replace all $\bar x$ and $\bar t$ with x and t, and arrive at

$$\begin{aligned}u(x,t) = \frac{1}{2}\Big[f(x - ct) + f(x + ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi \\ + \frac{1}{2c}\iint_{T(x,t)} q(\xi, \tau)\,d\xi\,d\tau. \qquad (5.21)\end{aligned}$$

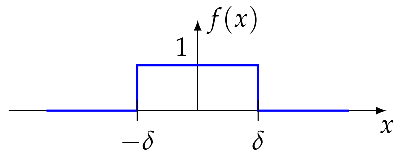
Remark 5.4. The rightmost double integral in (5.21) may be expressed as nested single integrals via Fubini's theorem:

$$\begin{aligned}\iint_{T(x,t)} q(\xi,\tau)\,d\xi\,d\tau &= \int_0^t \int_{x_{\text{left}}(\tau)}^{x_{\text{right}}(\tau)} q(\xi,\tau)\,d\xi\,d\tau \\ &= \int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} q(\xi,\tau)\,d\xi\,d\tau.\end{aligned}$$

5.7 A displacement blip as the initial value

Let us solve the initial value problem (5.14) with the initial velocity $g = 0$ and the initial displacement as a "blip" f given by

$$f(x) = \begin{cases} 1 & \text{if } |x| < \delta, \\ 0 & \text{otherwise.} \end{cases}$$



Since $g = 0$, d'Alembert's solution (5.17) reduces to

$$u(x,t) = \frac{1}{2}\Big[f(x-ct) + f(x+ct)\Big]. \tag{5.22}$$

We know that the graph of $\frac{1}{2}f(x-a)$ is obtained by translating the graph of $f(x)$ by a and reducing the height by a factor of $1/2$. Therefore, in the solution (5.22), the graph of f splits into two half-height blips which travel to the left and to the right at velocities $\pm c$. See the animation in Figure 5.3.

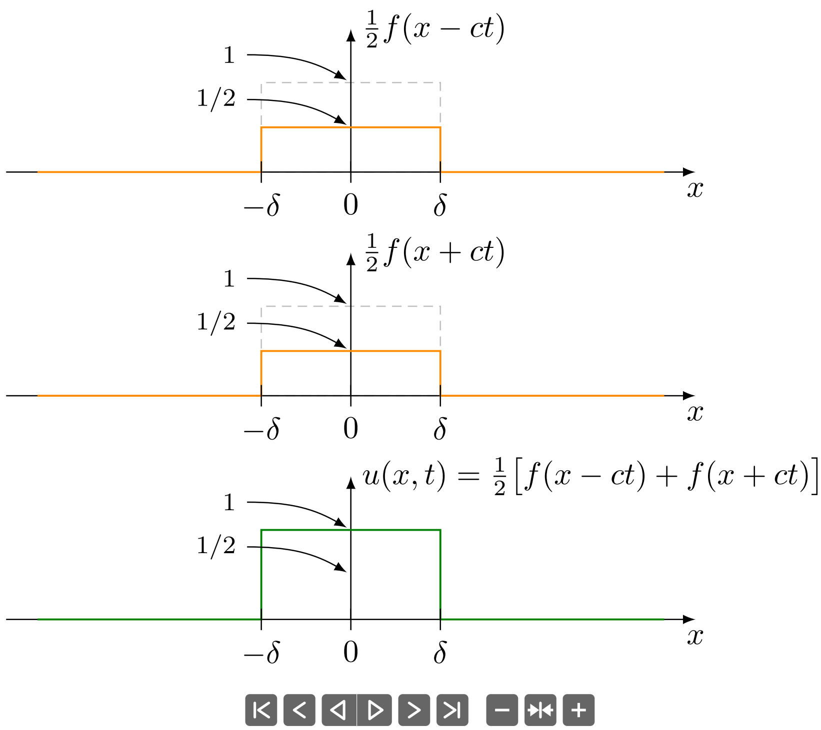


Figure 5.3: The initial displacement blip, drawn in faint dashed lines, splits into a right-moving and left-moving half-blips which travel at speeds $\pm c$, and whose sum, drawn at the bottom, forms the solution of the initial value problem studied in Section 5.7.

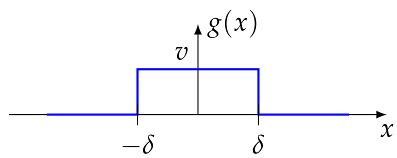
Remark 5.5. With some stretch of the imagination, the IVP just solved may be thought of as a model of the transverse displacement $u(x,t)$ of a long and taut string whose motion starts out with the initial shape as the graph of $f(x)$, and is released with zero initial velocity. That, however, doesn't pass close scrutiny, since in the derivation of the equation of motion of a string in Section 5.1, a key assumption was the smallness of the slope, $u_x(x,t)$. In contrast, the steep rise of the function f is akin to having an infinitely large slope. The actual motion of a string may not be exactly what our solution predicts, but it wouldn't be far. That's because a real string won't sustain the rectangular shape for long due to rapid dissipation of mechanical energy at the steep jump. The rectangular profile will quickly smooth out into a gently curved bleep whose motion will then resemble the motion predicted in (5.22).

Remark 5.6. The IVP may also be viewed as a simplistic model of the evolution of pressure $u(x,t)$ in a gas-filled tube where $f(x)$ represents the initial pressure profile. The previous remark apply to this case as well.

5.8 *A blip as the initial velocity*

Let us solve the initial value problem (5.14) with $f = 0$ and g a "blip" given by

$$g(x) = \begin{cases} v & \text{if } |x| < \delta, \\ 0 & \text{otherwise.} \end{cases}$$



This can be a model of the transverse displacement $u(x,t)$ of piano wire that starts out with zero displacement but with a nonzero velocity v over the interval $-\delta < x < \delta$ due to the impact of the hammer when the player presses a key.

Since $f = 0$, d'Alembert's solution (5.17) reduces to

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi, \tag{5.23}$$

so in a way that's our solution although it is not in a form that can be easily visualized. It's possible to make a better sense of that solution if we define g's antiderivative

$$G(x) = \int_0^x g(\xi)\,d\xi,$$

and then express the solution as

$$u(x,t) = \frac{1}{2c}\big[G(x+ct) - G(x-ct)\big]. \tag{5.24}$$

The graph of G, which is easily determined[6] by inspecting the graph of g, is shown in subfigure (a) in Figure 5.4. The graph of $-G$, which

[6] This marginal note is an overkill since it elaborates the obvious, as you should be able to see its conclusion immediately by staring at the graph of g. Nevertheless, it is here for those who need the extra help to see that.

From $G(x) = \int_0^x g(\xi)\,d\xi$ we have $G'(x) = g(x)$ and $G(0) = 0$. But g takes on the constant values 0, v, and 0 on the intervals $(-\infty,-\delta)$, $(-\delta,\delta)$, and (δ,∞), respectively, and therefore G takes on the values c_1, $vx + c_2$, c_3 on those intervals, where c_1, c_2, c_3 are constants. From $G(0) = 0$ it follows that $c_2 = 0$ and therefore $G(x) = vx$ on the interval $(-\delta,\delta)$. At the endpoints of that interval we have $G(-\delta) = -v\delta$ and $G(\delta = v\delta$. Then the continuity of G implies that $c_1 = -v\delta$ and $c_3 = v\delta$. We conclude that

$$G(x) = \begin{cases} -v\delta & \text{if } x < -\delta, \\ -vx & \text{if } -\delta < x < \delta, \\ v\delta & \text{if } \delta < x, \end{cases}$$

whose graph is shown in Figure 5.4(a).

is obtained by reflecting the graph of G about the horizontal axis, is shown in subfigure (b). The graphs of $G(x+ct)$ and $-G(x-ct)$, which are obtained by shifting the two previous graphs by ct to the left and to the right, are shown in subfigures (c) and (d). Finally, the graph of the solution (5.24), constructed by adding the graphs in (c) and (d) and scaling the result by a factor of $\frac{1}{2c}$, is shown in subfigure (e). Figure 5.5 presents an animation of the solution.

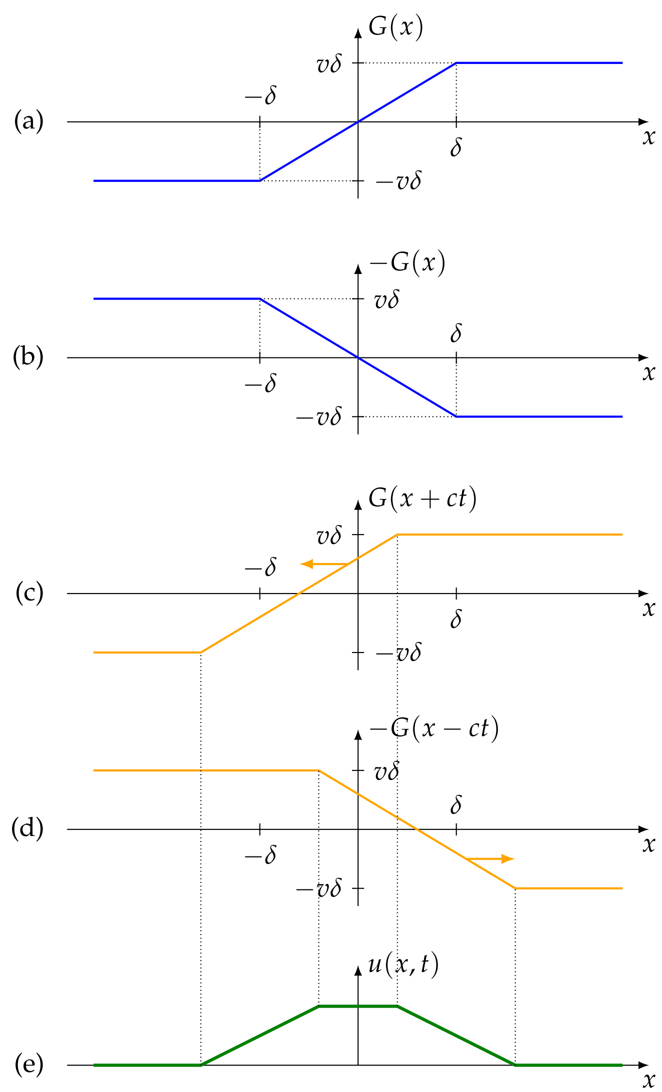


Figure 5.4: Subfigure (a) shows the graph of G which is the antiderivative of g, while subfigure (b) shows the graph of $-G$ which is obtained by reflecting the graph of G about the horizontal axis. Subfigures (c) and (d) depict the graphs of $G(x+ct)$ and $-G(x-ct)$ which are obtained by shifting the graphs in (a) and (b) by ct to the left and to the right. Subfigure (e) depicts the sum the graphs in of (c) and (d), which upon scaling by a factor of $1/(2c)$ yields the solution of the initial value problem studied in Section 5.8.

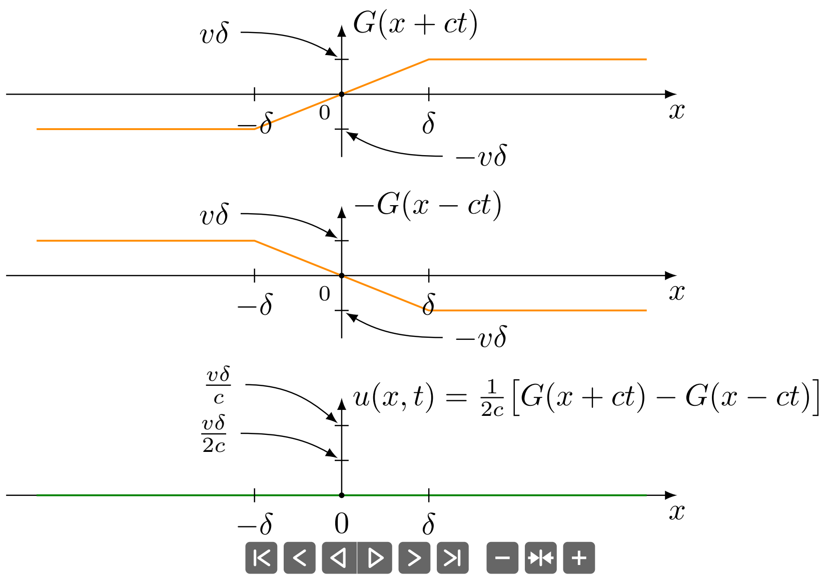


Figure 5.5: Here is an animation constructed through the steps outlined in Figure 5.4.

5.9 *Exercises*

5.1. Sketch the solution $u(x,t)$ of the initial value problem (5.14) with $g = 0$ and

$$f(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geq 0. \end{cases}$$

at $t = 0, 1, 2$.

5.2. Solve the initial value problem for the wave equation with $f(x) = 0$ and

$$g(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0. \end{cases}$$

5.3. Show the details of the calculation that leads to (5.12b).

5.4. We have seen that the initial condition

$$u(x,0) = f(x), \quad u_t(x,0) = 0,$$

gives rise to two waves $\frac{1}{2}f(x-ct)$ and $\frac{1}{2}f(x+ct)$ propagating to the left and to the right. Given the initial displacement $u(x,0) = f(x)$, what should you choose for the initial velocity $u_t(x,0)$ so that to obtain only a right-traveling wave?

5.5. An inextensible chain of uniform linear density ρ and length L is suspended from a fixed point and moves within a vertical plane. Let $u(x,t)$ be the chain's transverse displacement away from the vertical equilibrium, at the location x and time t. Find the PDE that describes the chain's motion.

Hint: You will find it convenient to set up the x coordinate axis pointing up, with the origin at the chain's lowest point.

5.6. Find the solution of the initial value problem

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = 2e^{-x^2}, \quad u_t(x,0) = 0.$$

Sketch the solution $u(x,t)$ at $t = 0, 1, 2, 3$, taking $c = 1$.

5.7. Find the solution of the initial value problem

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = 0, \quad u_t(x,0) = 1/(1+x^2).$$

5.8. Find the solution of the initial value problem

$$u_{tt} = c^2 u_{xx}, \quad u(x,0) = 0, \quad u_t(x,0) = \cos x,$$

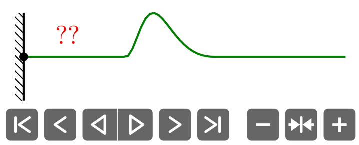
and simplify the answer. What does the solution look like?

6
Reflection and transmission of waves

This chapter introduces various scenarios where waves along a taut string/rope interact with objects placed in their way.

6.1 Semi-infinite rope with fixed end

A taut rope stretches over $0 \leq x < \infty$. The end at $x = 0$ is tied to a fixed (immobile) support. A displacement blip, $u(x,t) = f(ct + x)$, propagates down the x axis from $+\infty$ toward $x = 0$ as in the animation below. How does the blip interact with the support?



Here is an intuitively appealing way of constructing the solution to that problem. Imagine that we extend the semi-infinite rope with an identical twin that covers the negative x axis. Splice the two pieces at the origin and remove the immobility constraint there. Now the blip $f(ct + x)$ will happily sail past $x = 0$, displacing the point at $x = 0$ as it goes through; see the animation in Figure 6.1(a).

What happens if we simultaneously launch a second blip, $-f(ct - x)$, which is the inverted mirror image of the original blip about the origin? As the second blip travels to the right, it will arrive at and continue past the origin toward $+\infty$. As it goes through the origin, it will also displace the point at $x = 0$; see the animation in Figure 6.1(b).

Under the superposition of the two blips, the rope at $x = 0$ will not move at all since the two blips attempt to move it in opposite directions. Another way of seeing this is to observe that the rope's displacement at any x and t is

$$u(x,t) = f(ct + x) - f(ct - x), \tag{6.1}$$

and therefore $u(0,t) = f(ct) - f(ct) = 0$ for all t; see the animation in Figure 6.1(c).

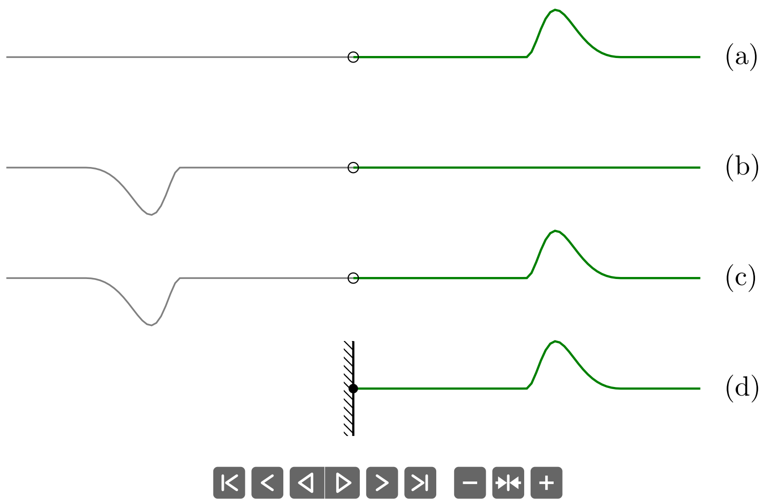


Figure 6.1: (a) The rope is extended to the negative x axis and the immobility constraint at $x = 0$ is removed. The left-moving blip $f(ct + x)$ propagates through $x = 0$ and displaces it in the process.

(b) The right-moving "image" blip $f(ct - x)$ also moves through $x = 0$ unimpeded and displaces it in the process.

(c) The superposition of the right- and left-moving blips cancel each other at $x = 0$, so that point does not move despite there being no constraints on it.

(d) The rope's motion on the $x < 0$ side is the desired solution. The outcome is that the incoming blip is reflected and inverted upon interacting with the boundary.

The function $u(x, t)$ defined above describes the motion of the rope over the entire x axis. If you cover up the negative x axis with your hand in the animation Figure 6.1(c), what remains is the motion of the rope on the positive x axis.[1] That's shown in the animation Figure 6.1(d).

[1] Obviuously your hand does not affect the animation!

We conclude that the motion of the semi-infinite rope on the positive x axis is obtained by limiting the range of x in (6.1) to $x \geq 0$, as in

$$u(x, t) = f(ct + x) - f(ct - x), \quad 0 \leq x < \infty, \quad t \geq 0. \tag{6.2}$$

The trick of adding an imaginary extension to a PDE's domain and adding suitable data there to build the desired boundary condition is known as *the method of images*. We will see other instances of method of images later in this chapter.

6.2 *The vibration of a guitar string*

A taut guitar string extends over the interval $0 < x < a$. When we pluck the string, we give it some initial displacement $f(x)$ and let it go. We wish to determine the string's subsequent motion.

The string's displacement, $u(x, t)$, is the solution to the following initial boundary value problem

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < a, \ t > 0, \tag{6.3a}$$

$$u(x, 0) = f(x) \qquad 0 < x < a, \tag{6.3b}$$

$$u_t(x, 0) = 0 \qquad 0 < x < a, \tag{6.3c}$$

$$u(0, t) = u(a, t) = 0 \qquad t > 0. \tag{6.3d}$$

We may obtain an explicit solution to that problem through a somewhat more sophisticated use of the method of images.

Imagine that we extend the guitar's string to the entire real line $-\infty < x < \infty$, and give it an initial displacement as follows. First extend the initial displacement $f(x)$ to the interval $-a < x < a$ as an odd function:

$$f_{\text{odd}}(x) = \begin{cases} -f(-x) & -a < x < 0, \\ f(x) & 0 < x < a, \end{cases}$$

as in Figure 6.2. Next, we extend f_{odd} as a $(2a)$-periodic function f_{per} to the entire real line as seen in Figure 6.3. It is evident from a quick inspection of that figure that f_{per} is an odd function about $x = 0$ and also an odd function about $x = a$, that is,

$$f_{\text{per}}(-x) = -f_{\text{per}}(x), \qquad f_{\text{per}}(a-x) = -f_{\text{per}}(a+x) \qquad \text{for all } x. \tag{6.4}$$

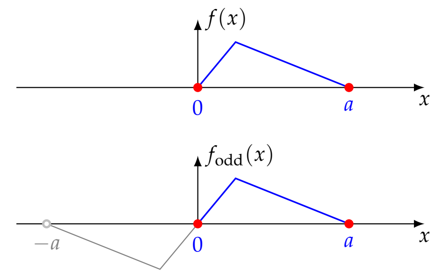


Figure 6.2: The initial displacement f is extended as an odd function from the interval $(0, a)$ to the interval $(-a, a)$. Here we have taken f as a piecewise linear function for the sake of illustration. The argument that leads to the construction of the solution $u(x,t)$ holds for *any choice of* f as long as $f(0) = f(a) = 0$.

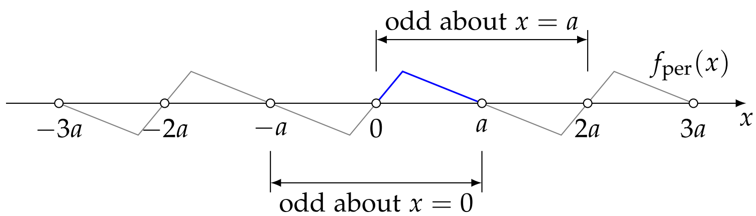


Figure 6.3: The $(2a)$-periodic extension f_{per} of the initial displacement f is an odd function about $x = 0$ as well as about $x = a$.

Now let us set

$$u(x,t) = \frac{1}{2}\Big[f_{\text{per}}(x-ct) + f_{\text{per}}(x+ct)\Big], \qquad -\infty < x < \infty, \quad t > 0. \tag{6.5}$$

This certainly satisfies the wave equation (6.3a) over the entire real line, and in particular in the interval $0 < x < a$. Let us verify that u satisfies the remaining equations in (6.3). We have

$$u(x,0) = \frac{1}{2}\Big[f_{\text{per}}(x) + f_{\text{per}}(x)\Big] = f_{\text{per}}(x).$$

But since f_{per} and f coincide on the interval $0 < x < a$, we see that $u(x,0) = f(x)$ on that interval, and therefore (6.3b) holds. As to (6.3c), we calculate

$$u_t(x,t) = \frac{1}{2}\Big[-cf'_{\text{per}}(x-ct) + cf'_{\text{per}}(x+ct)\Big],$$

where f'_{per} is the derivative of f_{per}. Thus, we have

$$u_t(x,0) = \frac{1}{2}\Big[-cf'_{\text{per}}(x) + cf'_{\text{per}}(x)\Big] = 0 \qquad \text{for all } x,$$

and therefore (6.3c) holds.

Let us also observe that

$$u(0,t) = \frac{1}{2}\Big[f_{\text{per}}(-ct) + f_{\text{per}}(ct)\Big],$$
$$u(a,t) = \frac{1}{2}\Big[f_{\text{per}}(a-ct) + f_{\text{per}}(a+ct)\Big],$$

each of which is zero due to the oddness properties (6.4) of f_{per}, and therefore the equations (6.3d) hold.

We conclude that $u(x,t)$ defined in (6.5) is the solution of the initial boundary value problem (6.3). The animation in Figure 6.4 shows what that solution looks like, and that's what a real guitar string would do if it were not for dissipation of energy due to air resistance and transmission of energy through the soundboard.

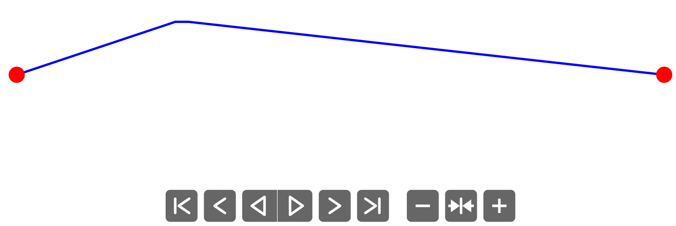


Figure 6.4: A plucked guitar string would move this way forever if it were not for dissipation of energy due to air resistance and transmission of energy through the soundboard.

6.3 *The vibration of a piano string*

A typical piano has 88 sets of wires/strings of varying lengths and weights that generate musical tones when struck by a hammer activated by the player pressing a key. Thus, a distinctive difference between how guitar and piano strings produce sound is that one is plucked while the other is struck.[2] That distinction is reflected in the mathematical models of the strings' motions and the resulting solutions.

[2] For that reason, a piano is commonly classified as a *percussion* instrument.

A good model for the motion $u(x,t)$ of a piano string stretching over the interval $0 < x < a$ is

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < a, \; t > 0, \tag{6.6a}$$

$$u(x,0) = 0 \qquad 0 < x < a, \tag{6.6b}$$

$$u_t(x,0) = g(x) \qquad 0 < x < a, \tag{6.6c}$$

$$u(0,t) = u(a,t) = 0 \qquad t > 0, \tag{6.6d}$$

indicating that the string's motion starts out with zero displacement ($u(x,0) = 0$) but with a velocity $u_t(x,0) = g(x)$ imparted to it by the hammer.

Toward solving this initial boundary value problem, and motivated by d'Alembert's solution of the wave equation, we introduce the antiderivative of g,

$$G(x) = \int_0^x g(\xi)\, d\xi, \quad 0 < x < a,$$

and extend G as an *even function* to the interval $-a < x < a$

$$G_{\text{even}} = \begin{cases} G(-x) & -a < x < 0, \\ G(x) & 0 < x < a. \end{cases}$$

Then, we further extend G_{even} as a $(2a)$-periodic function G_{per} to the entire real line. We leave it to an exercise to show that

$$u(x,t) = \frac{1}{2c}\Big[G_{\text{per}}(x+ct) - G_{\text{per}}(x-ct)\Big], \quad -\infty < x < \infty, \quad t > 0 \tag{6.7}$$

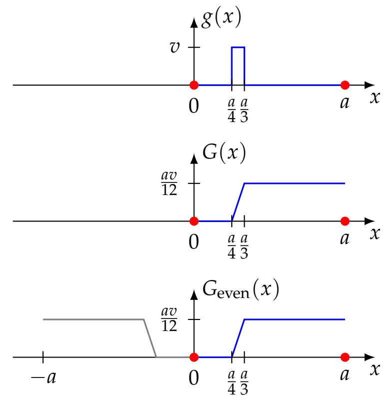


Figure 6.5: The initial velocity $g(x)$ is integrated to produce $G(x)$, and then is extended as an even function $G_{\text{even}}(x)$ to the interval $(-a, a)$.

is the solution of the initial boundary value problem (6.6).

To illustrate, let's take

$$g(x) = \begin{cases} v & \text{if } \frac{a}{4} < x < \frac{a}{3}, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$G(x) = \begin{cases} 0 & \text{if } 0 < x < \frac{a}{4}, \\ v\left(x - \frac{a}{4}\right) & \frac{a}{4} < x < \frac{a}{3}, \\ \frac{av}{12} & \frac{a}{3} < x < a. \end{cases}$$

Figure 6.5 depicts the functions $g(x)$, $G(x)$, and the even extension $G_{\text{even}}(x)$. Figure 6.6 shows the resulting animation. Note how different the motion is from that of a guitar string.

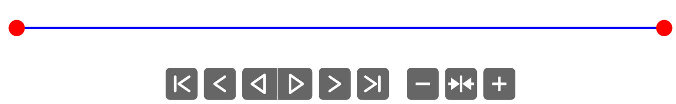


Figure 6.6: A struck piano string would move this way forever if it were not for dissipation of energy due to air resistance and transmission of energy through the soundboard.

6.4 *A two-piece rope*

A taut rope stretching over $-\infty < x < \infty$ is made by splicing two ropes of linear densities ρ_1 in the right half $(x > 0)$ and ρ_2 in the left half $(x < 0)$. A displacement blip originates on the positive x axis, travels toward the origin, reaches the splicing point $x = 0$, and goes beyond. We wish to find the rope's equation of motion.

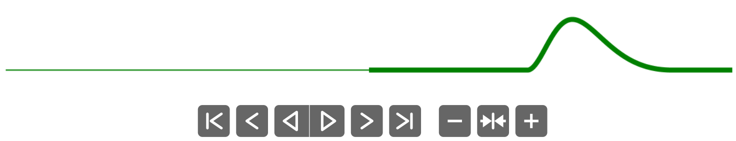


Figure 6.7: The rope's linear density is ρ_1 on $x > 0$ and ρ_2 on $x < 0$ corresponding to the respective wave speeds c_1 and c_2. The incoming blip is partially reflected at $x = 0$. Parameters for this animation are $c_1 = 1$, $c_2 = 2$. The heavier half of the rope is drawn in a thicker line. Note how the wave travels faster in the thinner part.

If T is the magnitude of the tension in the rope, then the wave velocities in the right and left halves are $c_1 = \sqrt{T/\rho_1}$ and $c_2 = \sqrt{T/\rho_2}$, respectively. Observe that signals propagate slower in the heavier part.

Consider the incident blip $f(c_1 t + x)$ where the support of f is contained entirely in the $x > 0$ segment. When the blip reaches the origin, if gives rise to a right-traveling *reflected wave* $g(c_1 t - x)$ and a left-traveling *transmitted wave* $h(c_2 t + x)$. Thus, the motion of the rope is expressed as

$$u(x,t) = \begin{cases} f(c_1 t + x) + g(c_1 t - x) & x \ge 0, \\ h(c_2 t + x) & x < 0, \end{cases} \tag{6.8}$$

where f is given and g and h are to be determined.[3]

To simplify the notation, let us write $u^{\text{right}}(x,t)$ and $u^{\text{left}}(x,t)$ for $u(x,t)$ in the $x > 0$ and $x < 0$ regions, respectively, that is,

$$\begin{aligned} u^{\text{right}}(x,t) &= f(c_1 t + x) + g(c_1 t - x) && x > 0, \\ u^{\text{left}}(x,t) &= h(c_2 t + x) && x < 0. \end{aligned}$$

The continuity of the rope's displacement at $x = 0$ is expressed through $u^{\text{left}}(0,t) = u^{\text{right}}(0,t)$, that is,

$$f(c_1 t) + g(c_1 t) = h(c_2 t). \tag{6.9a}$$

Furthermore, the forces exerted by the rope's two halves on each other must be equal in size and and opposite in direction.[4] But the vertical component of the rope's tensile force is $Tu_x(0,t)$; see (5.4) on page 51. Therefore, $Tu_x^{\text{left}}(0,t) = Tu_x^{\text{right}}(0,t)$, that is

$$f'(c_1 t) - g'(c_1 t) = h'(c_2 t). \tag{6.9b}$$

We introduce $\xi = c_1 t$ and cast the equations (6.9a) and (6.9b) into the forms

$$\begin{aligned} f(\xi) + g(\xi) &= h\Big(\frac{c_2}{c_1}\xi\Big), \\ f'(\xi) - g'(\xi) &= h'\Big(\frac{c_2}{c_1}\xi\Big). \end{aligned}$$

Integrating the second equation with respect to ξ we get $f(\xi) - g(\xi) = \frac{c_1}{c_2} h\Big(\frac{c_2}{c_1}\xi\Big) + C$, where C is the integration constant. To determine C, we observe that $u(0,0) = 0$[5] and therefore from (6.8) we get $f(0) + g(0) = 0$ and $h(0) = 0$. But $f(0) = 0$ since we have assumed that the support of f lies in $x > 0$. In summery, we have $f(0) = g(0) = h(0) = 0$, which then implies that $C = 0$. We conclude that

$$\begin{aligned} f(\xi) + g(\xi) &= h\Big(\frac{c_2}{c_1}\xi\Big), \\ f(\xi) - g(\xi) &= \frac{c_1}{c_2} h\Big(\frac{c_2}{c_1}\xi\Big). \end{aligned}$$

We solve this as a system of two equations in the two unknowns $g(\xi)$ and $h\Big(\frac{c_2}{c_1}\xi\Big)$, and we obtain

$$g(\xi) = \frac{c_2 - c_1}{c_2 + c_1} f(\xi), \quad h\Big(\frac{c_2}{c_1}\xi\Big) = \frac{2c_2}{c_2 + c_1} f(\xi).$$

[3] The incident, reflected, and transmitted waves may be expressed in the equivalent and more familiar forms $f(x+ct)$, $g(x-ct)$ and $h(x+ct$. At the interface at $x = 0$ these evaluate to $f(ct)$, $g(-ct)$, and $h(ct)$. The minus sign in $g(-ct)$ leads to a slightly ugly algebra although that eventually resolves itself. Our alternative formulation, with some help from hindsight, circumvents that ugliness.

[4] Otherwise there will be a nonzero resultant force acting at the massless juncture of the two ropes, which will impart it infinite acceleration by Newton's law.

[5] At $t = 0$ no signal has arrived at the origin yet.

In the second equation we rename $\frac{c_2}{c_1}\xi$ to ξ, and finally arrive at

$$g(\xi) = \frac{c_2 - c_1}{c_2 + c_1} f(\xi), \quad h(\xi) = \frac{2c_2}{c_2 + c_1} f\Big(\frac{c_1}{c_2}\xi\Big).$$

Substituting these into (6.8) we obtain the rope's equation of motion:

$$u(x,t) = \begin{cases} f(c_1 t + x) + \dfrac{c_2 - c_1}{c_2 + c_1} f(c_1 t - x) & x \geq 0, \\ \dfrac{2c_2}{c_2 + c_1} f\Big(\dfrac{c_1}{c_2}(c_2 t + x)\Big) & x < 0. \end{cases} \tag{6.10}$$

The ratio $\frac{c_2 - c_1}{c_2 + c_1}$ of the relative amplitudes of the reflected versus incident waves is called the interface's *reflection coefficient* or *impedance*. The ratio $\frac{2c_2}{c_2 + c_1}$ of the relative amplitudes of the transmitted versus incident waves is called the interface's *transmission coefficient*. Note that when $c_1 = c_2$ the impedance is zero and there is no reflected wave.

6.5 *Bead on a rope*

A bead of mass m is affixed at $x = 0$ to a taut rope that stretches over $-\infty < x < \infty$. A blip in the rope originates in the $x > 0$ region and travels down the x axis, eventually reaching $x = 0$ and going beyond. The animation in Figure 6.8 shows the string's motion, produced based on the equation of motion which we are going to derive.

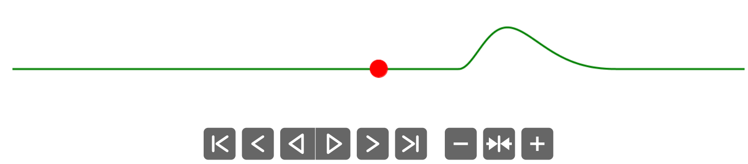


Figure 6.8: The rope stretches over $-\infty < x < \infty$. A bead (a point mass, the red dot) is affixed to the rope and moves up and down with it. The incoming blip is partially reflected when it interacts with the bead. Parameters for this animation are: tension in the rope $T = 1$; wave speed in the rope $c = 1$; mass of the bead $m = 1$.

Let $f(ct + x)$ represent the rope's transverse (that is, up and down) motion due to the incoming blip that originates somewhere in the $x > 0$ region and travels leftward. When the blip reaches $x = 0$, it sets the bead into motion. We write $U(t)$ be the bead's transverse displacement at time t. The interaction of the blip with the bead results in a reflected wave $g(ct - x)$ that propagates to the right (that is, from $x = 0$ toward $x = +\infty$), and a transmitted wave $h(ct + x)$ that propagates to the left (that is, from $x = 0$ toward $x = -\infty$).[6] Then the rope's transverse displacement $u(x,t)$ at any location x and at any time t is

[6] The marginal note 3 on page 68 applies equally well to the current setting.

$$u(x,t) = \begin{cases} f(ct + x) + g(ct - x) & x > 0, \\ U(t) & x = 0, \\ h(ct + x) & x < 0. \end{cases} \tag{6.11}$$

The function f is given. The functions U, g and h are to be determined.

To simplify the notation, let us write $u^{\text{right}}(x,t)$ and $u^{\text{left}}(x,t)$ for $u(x,t)$ in the $x>0$ and $x<0$ regions, respectively, that is,

$$u^{\text{right}}(x,t) = f(ct+x) + g(ct-x) \qquad x>0, \tag{6.12a}$$

$$u^{\text{left}}(x,t) = h(ct+x) \qquad x<0. \tag{6.12b}$$

The continuity of the rope's displacement at $x=0$ is expressed through

$$U(t) = u^{\text{left}}(0,t) = u^{\text{right}}(0,t),$$

that is,

$$U(t) = h(ct) = f(ct) + g(ct). \tag{6.13}$$

For future reference, let's calculate the time derivative of the above

$$U'(t) = ch'(ct) = cf'(ct) + cg'(ct),$$

and then isolate h' and g':

$$h'(ct) = \frac{1}{c}U'(t), \qquad g'(ct) = \frac{1}{c}U'(t) - f'(ct). \tag{6.14}$$

Let T be tension in the rope. The vertical component of the force exerted on the bead by the right-half of the rope is $Tu_x^{\text{right}}(0,t)$; see (5.4) on page 51. Similarly, the vertical component of the force exerted on the bead by the left-half of the rope is $-Tu_x^{\text{left}}(0,t)$. The bead's motion obeys Newton's law:

$$mU''(t) = Tu_x^{\text{right}}(0,t) - Tu_x^{\text{left}}(0,t). \tag{6.15}$$

But from equations (6.12) we have

$$u_x^{\text{right}}(x,t) = f'(ct+x) - g'(ct-x), \qquad u_x^{\text{left}}(x,t) = h'(ct+x),$$

which at $x=0$ yields

$$u_x^{\text{right}}(0,t) = f'(ct) - g'(ct), \qquad u_x^{\text{left}}(0,t) = h'(ct),$$

and thus (6.15) takes the form

$$mU''(t) = T\Big[\Big(f'(ct) - g'(ct)\Big) - h'(ct)\Big]. \tag{6.16}$$

We substitute for $h'(ct)$ and $g'(ct)$ from (6.14), simplify the result, and arrive at

$$U''(t) + \frac{2T}{mc}U'(t) = \frac{2T}{m}f'(ct). \tag{6.17}$$

Integrating on both sides from 0 to t we get

$$\big[U'(t) - U'(0)\big] + \frac{2T}{mc}\big[U(t) - U(0)\big] = \frac{2T}{mc}\big[f(ct) - f(0)\big].$$

Since at time $t = 0$ the incoming blip is away from the origin, we have $U(0) = 0$, $U'(0) = 0$, and $f(0) = 0$, and therefore the equation simplifies to

$$U'(t) + \frac{2T}{mc}U(t) = \frac{2T}{mc}f(ct). \tag{6.18}$$

We multiply through by the integrating factor $e^{\frac{2T}{mc}t}$

$$\frac{d}{dt}\Big(e^{\frac{2T}{mc}t}U(t)\Big) = \frac{2T}{mc}e^{\frac{2T}{mc}t}f(ct)$$

and integrate. Accounting for $U(0) = 0$ we arrive at

$$e^{\frac{2T}{mc}t}U(t) = \frac{2T}{mc}\int_0^t e^{\frac{2T}{mc}\tau}f(c\tau)\,d\tau.$$

The integral may be simplified by changing the variable of integration from τ to $\xi = c\tau$, whereby

$$e^{\frac{2T}{mc}t}U(t) = \frac{2T}{mc^2}\int_0^{ct} e^{\frac{2T}{mc^2}\xi}f(\xi)\,d\xi,$$

and finally[7]

$$U(t) = \frac{2T}{mc^2}\int_0^{ct} e^{-\frac{2T}{mc}((t-\frac{1}{c}\xi)}f(\xi)\,d\xi.$$

[7] Considering that $c = \sqrt{T/\rho}$ (see (5.2) on page 51, the coefficient $\frac{2T}{mc^2}$ may be simplified to $\frac{2\rho}{m}$.

Having thus determined the motion $U(t)$ of the bead, the profiles g and h of the reflected and transmitted waves can be calculated from (6.13) through the change of variable $\xi = ct$:

$$h(\xi) = U\Big(\frac{1}{c}\xi\Big), \quad g(\xi) = U\Big(\frac{1}{c}\xi\Big) - f(\xi). \tag{6.19}$$

Remark 6.1. The term $\frac{2T}{mc}U'(t)$ in (6.17) is akin to the damping term $by'(t)$ in the familiar mass-damper-spring model $my''(t) + by'(t) + ky(t) = 0$. The bead's damped motion is quite evident in the animation in Figure 6.8 as we see that it quickly settles down to its equilibrium position after the blip passes over it. However, unlike damping in the mass-damper-spring model which dissipates mechanical energy, the damping here *does not* dissipate energy—the mechanical energy is conserved in the overall rope+bead system. The bead settles down because its energy is *radiated away* to $x = \pm\infty$.

6.6 *Energy considerations*

Solutions of the wave equation are *conservative*, that is, they don't lose energy and fade out as the travel along. There is no dissipation of energy. We wish to quantify that statement.

For the sake of generality, let's allow for nonuniform string density $\rho(x)$ and therefore look at the (5.1) variant of the wave equation, which we duplicate here for ease of reference:

$$\rho(x)u_{tt} = Tu_{xx}, \tag{6.20}$$

and for the sake of concreteness, let us focus in the case where the spatial domain is the entire real line $-\infty < x < \infty$.[8] Displacements under the wave equation propagate at finite speed. Therefore, if the strings starts out at rest, any disturbance initiated away from infinity will remain bounded away from infinity at all times. Thus, we take $u(x,t)$ to be zero outside of a (generally growing) interval $a(t) < x < b(t)$ at any time t.

[8] You will address solutions over finite intervals in the exercises.

The expression for the string's energy is obtained as follows. We multiply the wave equation (6.20) by u_t

$$\begin{aligned}\rho u_{tt}u_t &= Tu_{xx}u_t\\ &= \big(Tu_xu_t\big)_x - Tu_{xt}u_x,\end{aligned}$$

and then rearrange into

$$\frac{\partial}{\partial t}\Big[\frac{1}{2}\rho u_t^2 + \frac{1}{2}Tu_x^2\Big] = \big(Tu_xu_t\big)_x.$$

Integrating over $-\infty < x < \infty$ we get

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\Big[\frac{1}{2}\rho u_t^2 + \frac{1}{2}Tu_x^2\Big]\,dx = \int_{-\infty}^{\infty}\big(Tu_xu_t\big)_x\,dx &= \lim_{\xi\to\infty}\int_{-\xi}^{\xi}\big(Tu_xu_t\big)_x\,dx\\ &= \lim_{\xi\to\infty}\Big[Tu_xu_t\Big|_{x=\xi} - Tu_xu_t\Big|_{x=-\xi}\Big].\end{aligned}$$

The right-hand side is zero because u is zero near $\pm\infty$. We conclude that

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Big[\frac{1}{2}\rho u_t^2 + \frac{1}{2}Tu_x^2\Big]\,dx = 0,$$

and therefore the quantity E defined as

$$E = \int_{-\infty}^{\infty}\Big[\frac{1}{2}\rho u_t^2 + \frac{1}{2}Tu_x^2\Big]\,dx$$

remains constant at all times. E is called the string's *mechanical energy*.[9]

[9] The $\frac{1}{2}\rho u_t^2$ accounts for the *kinetic energy* while the part $\frac{1}{2}Tu_x^2$ accounts for the *potential energy*.

Referring to the solution (6.10) of the two-piece rope of Section 6.4, you may verify that the energies associated with the incident, reflected, and transmitted waves are

$$E_{\text{incident}} = T\int_0^{\infty} f'(\xi)^2\,d\xi, \tag{6.21a}$$

$$E_{\text{reflected}} = T\Big(\frac{c_2-c_1}{c_2+c_1}\Big)^2\int_0^{\infty} f'(\xi)^2\,d\xi, \tag{6.21b}$$

$$E_{\text{transmitted}} = \frac{4Tc_1c_2}{(c_2+c_1)^2}\int_{-\infty}^{0} f'(\xi)^2\,d\xi. \tag{6.21c}$$

It follows that the fractions of the reflected and transmitted energies are

$$\begin{aligned}\text{fraction reflected} &= \Big(\frac{c_2-c_1}{c_2+c_1}\Big)^2,\\ \text{fraction transmitted} &= \frac{4c_1c_2}{(c_2+c_1)^2}.\end{aligned}$$

Observe that the two fractions add up to 1, confirming the conservation of energy even in this case of non-constant density.

6.7 *Exercises*

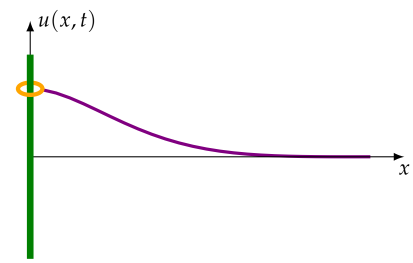
6.1. Show that (6.7) is the solution of the initial boundary value problem (6.6).

6.2. Show that the motion of a plucked guitar string is periodic in time, and find the period.

6.3. Show that the motion of a struck piano string is periodic, and find the period.

6.4. Consider a taut homogeneous rope stretched over $0 < x < \infty$ and initially at rest. Starting at time $t = 0$ we apply the transverse motion $u(0, t) = A \sin \omega t$ to the rope's endpoint at $x = 0$, where A and ω are constants. Find the rope's displacement $u(x, t)$ at all $x > 0$ and $t > 0$.

6.5. Repeat the previous exercise when the motion at the rope's endpoint is a generic $u(0, t) = \phi(t)$.



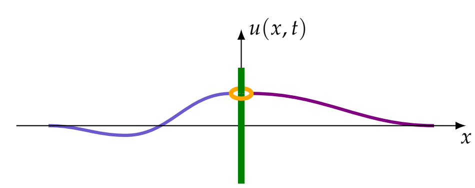
6.6. A taut semi-infinite homogeneous string occupies the interval $0 < x < \infty$ and is terminated at a small ring of negligible mass at $x = 0$. The ring can slide frictionlessly up and down a pole perpendicular to the x axis. A displacement blip, $u(x, t) = f(ct + x)$, propagates down the x axis from $+\infty$ toward $x = 0$. Find the string's motion at all times.

Hint: Applying Newton's law of motion (mass times acceleration = applied force) to the motion of the ring in the pole's direction, we see that the force applied to the ring has no component perpendicular to the x axis since the ring's mass is zero. It follows that the force exerted by the string on the ring is parallel to the x axis. But according to (5.4), the magnitude of that force is $Tu_x(0, t)$. We conclude that $u_x(0, t) = 0$ at all t.

6.7. Find the motion of the system described in the previous exercise if the ring has mass m.

6.8. Referring to the previous exercise, determine the motion $U(t)$ of the ring if the incident wave is $f(x + ct)$, where

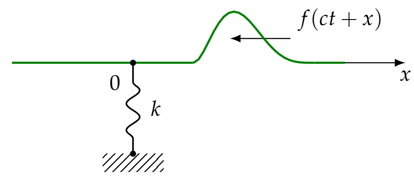
$$f(x) = \begin{cases} 0 & \text{if } 0 < x < a, \\ 1 & \text{if } x > a. \end{cases}$$



6.9. Two semi-infinite strings of equal linear densities ρ occupy the negative and positive x axes. At the origin the strings are hooked to a small massless ring that can slide, without friction, up and down a pole perpendicular to the x axis. Tensions in the right and left strings are T_1 and T_2, respectively. A displacement blip, $u(x,t) = f(ct+x)$, propagates down the x axis from $+\infty$ toward $x=0$. Find the string's motion at all times.

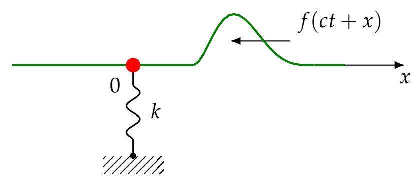
6.10. Repeat the previous exercise but assume that the ring has mass m.

6.11. A taut homogeneous rope rests over the $-\infty < x < \infty$. At $x=0$ it is attached to a (massless) spring of stiffness k, as shown in the figure. The other end of the spring is attached to a fixed (immobile) support. A displacement blip, $u(x,t) = f(ct+x)$, propagates down the x axis from $+\infty$, eventually arriving at $x=0$, and giving rise to reflected and transmitted waves. Find the displacement $u(x,t)$ of the string at all times.



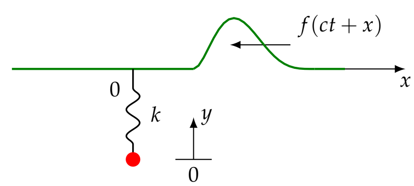
Hint: The steps of the mathematical modeling of this case is very similar to that of Section 6.5's rope. The significant difference occurs at equation (6.16) where previously the balance of forces was equated to mass times acceleration, while here the balance of forces should be equated to the force in the spring, that is, $kU(t)$.

6.12. Repeat Exercise 6.11 but now add a bead of of mass m attached at the juncture of the rope and string.



6.13 (**A mini-project)**. Repeat Exercise 6.11 but now assume that the lower end of the spring is free to bounce up and down and has a bead of mass m attached to it. Ignore the force of gravity, as in an experiment performed in an orbiting space station.

Suggestion: Let $U(t)$ be the displacement of the rope at $x=0$, as in Section 6.5, and let $y(t)$ be the displacement of the bead relative to its equilibrium position, where positive y means upward; see the diagram on the right. Proceed as in Section 6.5 and derive the equivalent of the ODE (6.16). This will involve both $U(t)$ and $y(t)$. Applying Newton's Law of Motion to the bead yields a 2nd order ODE that also involves both $U(t)$ and $y(t)$. Solve the two ODEs as a system for $U(t)$ and $y(t)$.



Although it is possible to carry out the calculations by hand, you will find the algebra to be quite tedious. Here are suggestions for solving the exercise in special cases which are not quite as tedious. Let H be the usual unit step function

$$H(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

and for the incident wave take $f(ct+x) = H(ct+x-6)$. Carry out calculations for:

Case 1: $m=1$, $k=2$, $T=1$, $c=3$;

Case 2: $m=1$, $k=2$, $T=1$, $c=2$.

6.14. Consider the initial boundary value problem for the wave equation over the spatial domain $a < x < b$:

$$\begin{aligned}
&\rho u_{tt} = T u_{xx} && a < x < b, \quad t > 0, \\
&u(a,t) = 0 && t > 0, \\
&u(b,t) = 0 && t > 0, \\
&u(x,0) = f(x) && a < x < b, \\
&u_t(x,0) = g(x) && a < x < b.
\end{aligned}$$

Show that the solution conserves the mechanical energy.

6.15. Verify the energy values given in equations (6.21).

7
The heat equation in 1D

The classical *heat equation* in one space dimension for the unknown $u(x, t)$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

along with its many variants, is the prototype of a very large class of *parabolic equations* that arise in a plethora of applications such as conduction of heat, diffusion of chemicals and solutes, dynamics of populations, analysis of the stock market, and image blurring and deblurring.

In this chapter we focus on one elementary and relatively easy-to-understand application, namely, conduction of heat in solids.

7.1 The physics of heat

The simplest mathematical model of heat conduction involves the concepts of heat, temperature, mass density, specific heat capacity, and thermal conductivity. We introduce these concepts in the following subsections, and then put them together to derive the heat equation. If you are impatient to find out the final result, you may skip to Section 7.2 for now, but it is likely that you will need to return here later in order to understand many of the details.

7.1.1 Heat versus temperature

In common parlance, the words *heat* and *temperature* are used only with fuzzy distinction, if there is any distinction at all. In physics the two words have specific and totally distinct meanings. To understand heat conduction, we need to understand what those words mean.

Temperature is a measure of an object's "hotness". It is a scalar quantity measured in degrees.[1] A hotter object has a higher temperature. The temperature of tap water is less than the temperature of boiling water, and that in turn is less than the temperature of the flame of

[1] The degrees Celsius (°C) and degrees Fahrenheit (°F) are the customary scales for measuring temperature. Their numerical values are related through $F = \frac{5}{9}C + 32$. The absolute temperature scales *Kelvin* (K) and *Rankine* (R) are the prevailing temperature units used in physics and thermodynamics.

a lit match. When a hotter and cooler (that is, less hot) objects are brought together, the hotter object becomes cooler, and the cooler object becomes hotter,[2] and over time, the temperature of both tend to converge to a common value.

[2] But sticking a lit match into a cup of water does not make the flame any cooler; it just extinguishes it.

Temperature is an *intensive quantity*. That's a technical term. It means that if an object is at temperature T, then any subdivision of that object, regardless of its size, is also at temperature T. Intensive quantities are not additive; combining two cups of 50°C water *does not* yield 100°C (boiling) water.

Heat is a measure of the *quantity of thermal energy* stored in an object.[3] Adding heat generally raises an object's temperature. A bathtub of warm water contains a greater amount of heat than what can be produced by lighting a single match. That's despite the fact that the temperature of a match's flame is much greater than that of the bathwater. This should make it clear that heat and temperature are quite distinct concepts.

[3] Thermal energy is measured in an energy unit appropriate to the context. Some such units are the *joule*, *calorie*, and Btu (British thermal unit). For instance, one Btu is the amount of heat required to raise the temperature of one pound of water by one degree Fahrenheit.

Heat is an *extensive quantity*. That's a technical term. It means that the heat contents of two objects add up when they are put together. Thus, half a cup of boiling water contains half the thermal energy of a full cup of boiling water. Similarly, if you double the amount of water in your bathtub, you will double the amount charged to your electric or gas bill for heating it to the same temperature.

7.1.2 *The specific heat capacity*

Despite their differences, heat and temperature are linked through the concept of *specific heat capacity*[4] which is a physical property of a given material—it is *the amount of heat needed to raise the temperature of a unit mass of that material by one degree.*[5]

[4] The word "specific" in this context refers to "that which characterizes the *species*", that is, the *type* of the material under the consideration.

[5] Thus, referring to marginal note 3, we see that the specific heat capacity of the water is 1 Btu/lb/F.

The specific heat capacity of aluminum is 0.897 J/g/K (joules per gram per kelvin) while that of water[6] is 4.1813 J/g/K. This means that it takes about 5 times more thermal energy to raise the temperature of water compared to doing the same with aluminum of the same mass. Think of heating a small piece of aluminum foil by lighting a single match. Think of doing that to comparable mass of water. Isn't it harder to raise the water's temperature?

[6] That is the specific heat capacity of water at at 25°C. The specific heat capacity of a substance generally depends somewhat on the temperature at which it is measured.

Extensive tables of specific heat capacities of a large variety of materials are widely available both in print and online. Such tables generally list the specific heat capacity at constant pressure, (notation c_p), and other measures such as specific heat capacity at constant volume (notation c_v), which we will not be concerned with. To further complicate matters, a material's specific heat capacity is not exactly a constant—it depends on the temperature at which it is measured. To simplify our calculations, we treat c_p as a constant, independent of temperature. A temperature-dependent c_p leads to a nonlinear parabolic PDE which is s beyond the scope of this book.

To conclude this subsection, let us consider the raising of an object's temperature from a reference temperature T_{ref} to some other temperature T. If the mass of the object is m and its specific heat capacity c_p is constant, then the thermal energy $E(T)$ required for the task is

$$E(T) = mc_p(T - T_{\text{ref}}). \tag{7.1}$$

Remark 7.1. We will see that what enters the equation of heat conduction is the *derivative* of $E(T)$ with respect to time t. If m and c_p are independent of time, as they normally are, then

$$\frac{\partial}{\partial t}E(T) = mc_p\frac{\partial T}{\partial t}.$$

As T_{ref} goes away after the differentiation, its value is immaterial to our purposes.

7.1.3 *The mass density*

The mass density of an object is a measure of its mass per unit volume. The density may be constant throughout a material, or it may vary within it. We write $\rho(\boldsymbol{x})$ for the density at the point $\boldsymbol{x}$, or just ρ if it is constant.

The density may also depend on temperature—objects usually expand when heated—but the treatment of mathematical models that account for temperature-dependent density go beyond the scope of this book.

7.1.4 *The thermal conductivity*

When you stir food in a hot sauté pan on the stove with a wooden spoon, the spoon's handle in your hand gets only mildly warm while the spoon's other end is immersed in scorching hot oil. You will have a hard time doing the same with an all-metal spoon; the spoon's handle will become too hot to hold. That's why metal kitchen utensils generally are equipped with a wooden or rubber grip area.

That difference between a wooden and metal spoon is due to the differences in their abilities to conduct heat. heat spreads more easily through metal than wood. They have different *thermal conductivities*.

To get a good idea of the quantitative characterization of conductivity, think of a refrigerator in the kitchen. Let's say the temperature inside the refrigerator is 37°F while that in the kitchen is 75°F. If you unplug the refrigerator, the temperature inside will gradually rise, and over time it will reach the kitchen's temperature, even though you keep the refrigerator's door closed. That shows that there is a constant flow of heat from the kitchen through the refrigerator's walls to the inside of it, and that's why the refrigerator needs a motor (technically, a heat pump) to remove the continuously invading heat in order to keep its insides cool, even when the door is kept shut.

Let ϕ be the amount of heat that invades the refrigerator through a unit area of its wall per unit time. That's the *heat flux* into the refrigerator.[7] Assuming, for simplicity, that the wall is made of a single type of material[8] the temperature inside the wall drops from $T_{\text{kitchen}} = 75°\text{F}$ on the outside face to $T_{\text{fridge}} = 37°\text{F}$ on the inside face. If the thickness of the wall is D, then the slope of the the temperature curve, that is, the *temperature gradient*, is

$$-\frac{T_{\text{kitchen}} - T_{\text{fridge}}}{D}.$$

[7] See Chapter 2 for an introduction to the idea of the flux.

[8] That's almost certainly is not true; a typical refrigerator wall has a metal sheet on its outside face, a plastic sheet in its inside face, and sandwiched between them is Styrofoam or some other insulating material.

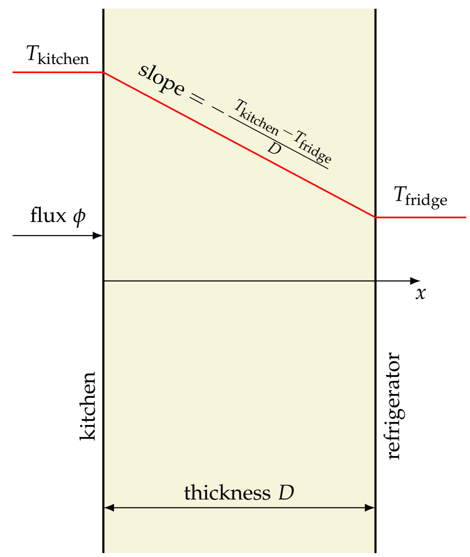


Figure 7.1: The red line graphs the temperature as it drops from T_{kitchen} to T_{fridge} through the thickness D of the refrigerator's wall. The temperature gradient is $-\frac{T_{\text{kitchen}} - T_{\text{fridge}}}{D}$.

A higher temperature difference $T_{\text{kitchen}} - T_{\text{fridge}}$ results in a higher heat flux. After all, if T_{kitchen} and T_{fridge} are almost the same, there is no reason for heat to rush into the refrigerator, while with a large temperature difference, we expect a greater rate of heat transfer. Furthermore, we postulate that a smaller D (a thinner wall) facilitates the transfer of heat, and therefore raises the heat flux. These postulates are formalized in what is known as *Fourier's Law of Heat Conduction*:

> *Heat flux through an object is proportional to the temperature gradient.*

In the refrigerator setting described above, this amount to saying

$$\phi = -K\frac{T_{\text{fridge}} - T_{\text{kitchen}}}{D},$$

where K is the proportionality constant.

The formula above is good if the temperature varies linearly with distance, as that depicted in Figure 7.1. If not, then we express the flux through the *local* change of temperature, that is, the temperature gradient, which in one space dimension is simply the derivative $\partial T/\partial x$:

$$\phi = -K\frac{\partial T}{\partial x}. \tag{7.2}$$

The coefficient K is called the medium's *thermal conductivity*. It is a positive value; the negative sign accounts for the fact that heat flows *against* the direction of rising temperature.

Remark 7.2. The refrigerator wall example above was chosen on purpose to focus our attention to heat conduction in *one spatial dimension*. The temperature in that example depends on one space dimension—the depth x into the refrigerator wall. But in general, heat conduction takes place in three dimensions. Consider, for instance, the dough that's being baked in an oven into a cake. The oven's heat penetrates the cake from *all directions*—there is no single direction x along which the dough is heated. In such a case the temperature T is a function of the location $\boldsymbol{x}$ within the dough, and the temperature gradient is the usual three-dimensional gradient vector ∇T. The flux $\boldsymbol{\phi}$ is also a vector which accounts for the direction in which the heat flows. Fourier's law of heat conduction then is expressed as

$$\boldsymbol{\phi} = -K\nabla T. \tag{7.3}$$

Remark 7.3. Equation (7.3) says that the flux and temperature gradient ∇T are collinear. If the conditions change so that the temperature gradient points in a new direction, the flux will adjust itself to point in that direction. That is a characteristic of *isotropic materials*, that is, materials that "look the same" in every direction. Most ordinary materials such as metals, rubber, Styrofoam, are isotropic for all practical purposes. Laminated wood (plywood), on the other hand, is not quite isotropic. Depending on the construction of the laminae, heat can flow more easily across them than along them. In that case, the vectors ∇T and $\boldsymbol{\phi}$ are no longer collinear. The thermal conductivity is no longer a scalar; it is a *second order tensor*. (Think of it as a matrix if you are not familiar with the tensor terminology.) Then we have

$$\boldsymbol{\phi} = -\mathbf{K}\nabla T.$$

In that way, ∇T and $\boldsymbol{\phi}$ generally point in different directions except when ∇T is an eigenvector of $\mathbf{K}$. We won't deal with anisotropic materials in this book, although most of what we develop here can be extended to the anisotropic case without great effort.

7.2 *The heat equation in 1D*

In the animation in the adjacent figure we have a long and slender metal rod, drawn in light blue, which is initially at zero temperature and thermally insulated all around. At $t = 0$ we expose the rod's left end to a temperature of 100 degrees. The heat propagates through the rod, raising the temperature within it, as depicted by the rising dark green graph. After some time, the rod's temperature stabilizes to a uniform 100 degrees throughout, as you would intuitively expect, and the color changes to red to indicate hot!

The temperature $T(x,t)$ at a point x within the rod and at time t turns out to be a solution of a PDE, the *heat equation*, which we shall now derive by applying the law of conservation of thermal energy. In subsequent chapters we will learn how to solve that PDE in order to get our hands on the function $T(x,t)$.

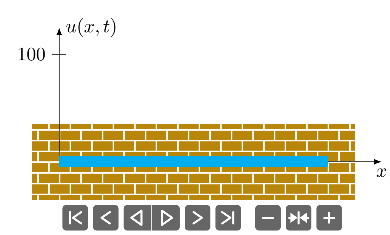


Figure 7.2: The rod, drawn in light blue, is initially insulated all around and is at zero temperature. We then expose the left end to a temperature of 100 degrees and watch the temperature evolve within the rod.

Toward that end we present two different approaches. The first one is quite short but calls for a thorough grasp of the conservation of mass equation of Chapter 2. The second one is more detailed but as a result is longer and repeats much of what was done in Section 2.1.

7.2.1 *The derivation of the heat equation: The quick version*

In Section 2.1 we derived the equation (2.3) that relates density ρ and flux ϕ as a form of enforcing the conservation of mass. Although the density ρ was presented there as the traditional "mass per unit volume", and the flux ϕ as the "mass passing through a unit cross sectional area per unit time", a close examination of the argument reveals that "mass" in those statements is fungible in that it may be substituted by any quantity that can be measured per unit volume. To model heat conduction, for instance, we replace "mass" by "thermal energy" and then (2.3), takes the form

$$\frac{\partial}{\partial t}e(x,t) + \frac{\partial}{\partial x}\phi(x,t) = 0, \tag{7.4}$$

where e is the *thermal energy per unit volume*, and $\phi(x,t)$ is the flux of the thermal energy, that is, thermal energy crossing a unit area per unit time. That is the mathematical statement of the conversation of thermal energy.

Since the thermal energy content of an object of mass m is given by (7.1), the thermal energy content of a unit volume is

$$e(x,t) = \rho(x)c_p(x)\big(T(x,t) - T_{\text{ref}}(x)\big), \tag{7.5}$$

where ρ is the object's mass per unit volume (that is, the traditional mass density). We allow for the possibility of the density ρ and the specific heat capacity c_p, and even the reference temperature T_{ref}, to vary with the position x, but not with time t; see Remark 7.1 in that regard. Thus, our model applies to inhomogeneous materials, that is, materials whose physical properties vary from point to point.

Substituting for e from (7.5) and for ϕ from (7.2) into (7.4), we obtain

$$\frac{\partial}{\partial t}\Big(\rho(x)c_p(x)\big(T(x,t) - T_{\text{ref}}(x)\big)\Big) + \frac{\partial}{\partial x}\Big(-K(x)\frac{\partial}{\partial x}T(x,t)\Big) = 0,$$

which simplifies to

$$\rho(x)c_p(x)\frac{\partial}{\partial t}T(x,t) = \frac{\partial}{\partial x}\Big(K(x)\frac{\partial}{\partial x}T(x,t)\Big). \tag{7.6}$$

This is a general form of the *heat equation* for an inhomogeneous one-dimensional medium. In the case of a *homogeneous medium*, that is, when ρ, c_p, and K, are constants, this simplifies to

$$\frac{\partial}{\partial t}T(x,t) = \frac{K}{\rho c_p}\frac{\partial^2}{\partial x^2}T(x,t).$$

The combination

$$k = \frac{K}{\rho c_p} \tag{7.7}$$

is called the medium's *thermal diffusivity*. In terms of that notation, the heat equation then takes the classic form

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}, \tag{7.8}$$

where $T = T(x, t)$ is the temperature at the point x at time t.

7.2.2 *The derivation of the heat equation: The long version*

In the previous subsection we adapted the conservation of mass equation (2.3) to quickly derive the heat equation.

In the current subsection's approach, instead of appealing to (2.3), we return to the basics and repeat the argument that led to (2.3) but in the context of thermal energy rather than mass. While we are at it, and for the sake of generality, we introduce the additional element of *internal heat generation* within the rod. This may be due to an electrical current flowing through the rod, of perhaps due to exothermic chemical reactions taking place within it.

Thus, as before, consider a thermally insulated rod lying along the x axis. Let A be the rod's (constant) cross-sectional area, and let $T(x, t)$ be the temperature at the point x at time t. The volume of a sliver of infinitesimal thickness dx of the rod at the location x is $A\,dx$, and therefore the mass of that sliver is $\rho(x)A\,dx$, where $\rho(x)$ is the rod's mass density at x. According to (7.1), the thermal energy content of that sliver relative to a reference temperature $T_{\text{ref}}(x)$ is

$$\big(\rho(x)A\,dx\big)c_p(x)\big(T(x,t) - T_{\text{ref}}(x)\big).$$

We conclude that the thermal energy content of an arbitrary segment $a < x < b$ of the rod is

$$A \int_a^b \rho(x)c_p(x)\big(T(x,t) - T_{\text{ref}}(x)\big)\,dx,$$

and therefore the change of that energy content of that segment between two arbitrary times t_1 and t_2 is

$$A \int_a^b \rho(x)c_p(x)\big(T(x,t_2) - T(x,t_1)\big)\,dx. \tag{7.9}$$

The reference temperature $T_{\text{ref}}(x)$ drops out of that calculation as it was noted earlier.

The change of the segment's thermal energy content calculated above is due to two distinct factors:

Heat flowing in and out of the segment's endpoints. Let $\phi(x,t)$ be the heat flux through the rod at the point x at time t. That's the thermal energy that flows per unit cross-sectional area of the rod per unit time. In particular, the thermal energy flowing *into the rod* at the cross-section $x = a$ per unit time is $A\phi(a,t)$, and therefore the energy that flows through that cross-section over the time period $t_1 < t < t_2$ would be

$$A \int_{t_1}^{t_2} \phi(a,t)\,dt. \tag{7.10a}$$

For the same reason, the thermal energy flowing *out of the rod* at $x = b$ is over the time period $t_1 < t < t_2$ is

$$A \int_{t_1}^{t_2} \phi(b,t)\,dt. \tag{7.10b}$$

Heat generated within the rod. Let $q(x,t)$ be the *heat generated per unit mass per unit time* within the rod. Then the heat generated per unit time within the sliver of infinitesimal thickness dx is $(\rho(x)A\,dx)q(x,t)$. Therefore the heat generated per unit time through the entire rod at time t is

$$A \int_a^b \rho(x)q(x,t)\,dx.$$

We conclude that the heat generated throughout the rod over the time period $t_1 < t < t_2$ is

$$A \int_{t_1}^{t_2} \int_a^b \rho(x)q(x,t)\,dx\,dt. \tag{7.11}$$

The principle of conservation of thermal energy implies that the change of internal thermal energy calculated in (7.9) must exactly balance the inflow and outflow in (7.10a) and (7.10b), and the heat generation in (7.11), that is

$$\begin{aligned} &A \int_a^b \rho(x)c_p(x)\big(T(x,t_2) - T(x,t_1)\big)\,dx \\ &\quad = A \int_{t_1}^{t_2} \phi(a,t)\,dt - A \int_{t_1}^{t_2} \phi(b,t)\,dt + A \int_{t_1}^{t_2} \int_a^b \rho(x)q(x,t)\,dx\,dt. \end{aligned} \tag{7.12}$$

That's the end of the physics. The rest is calculus.

On the left-hand side of (7.12) we replace $T(x,t_2) - T(x,t_1)$ via the Fundamental Theorem of Calculus with

$$\int_{t_1}^{t_2} \frac{\partial}{\partial t} T(x,t)\,dt.$$

On the right-hand side of (7.12), we also apply the Fundamental Theorem of Calculus to replace $\phi(b,t) - \phi(a,t)$ with

$$\int_a^b \frac{\partial}{\partial x} \phi(x,t)\,dx.$$

After those replacements, and dividing through by A, (7.12) takes the form

$$\int_a^b \int_{t_1}^{t_2} \rho(x) c_p(x) \frac{\partial}{\partial t} T(x,t)\, dt\, dx \\ = - \int_{t_1}^{t_2} \int_a^b \frac{\partial}{\partial x} \phi(x,t)\, dx\, dt + \int_{t_1}^{t_2} \int_a^b \rho(x) q(x,t)\, dx\, dt.$$

We interchange the order of the integrations on the left-hand side and then combine the three terms into one:

$$\int_{t_1}^{t_2} \int_a^b \left[\rho(x) c_p(x) \frac{\partial}{\partial t} T(x,t) + \frac{\partial}{\partial x} \phi(x,t) - \rho(x) q(x,t) \right] dx\, dt = 0.$$

As argued in going from equation (2.2) to (2.3) in Section 2.1 on page 17, the expression within the square brackets is zero, that is,

$$\rho(x) c_p(x) \frac{\partial}{\partial t} T(x,t) + \frac{\partial}{\partial x} \phi(x,t) - \rho(x) q(x,t) = 0 \quad \text{for all } x \text{ and } t.$$

Substituting for the flux ϕ from the Fourier's Law of heat conduction (7.2), this changes to

$$\rho(x) c_p(x) \frac{\partial}{\partial t} T(x,t) = \frac{\partial}{\partial x} \Big(K(x) \frac{\partial}{\partial x} T(x,t) \Big) + \rho(x) q(x,t), \tag{7.13}$$

which agrees with (7.6) when $q(x,t) = 0$. If, in particular, the rod is homogeneous, that is the coefficients are constants, then this simplifies to

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + \frac{1}{c_p} q, \tag{7.14}$$

where the *thermal diffusivity* k is defined in (7.7).

Remark 7.4. Although we have drived the equations (7.13) and (7.14) in the context heat conduction, those equations arise is numerous context in diverse applications such as diffusion of chemicals, population dynamics, and theoretical probability which have not much to do with conduction of heat. For that reason, in the general study of those equations we forgo the use of the notation $T(x,t)$ (temperature) in favor of a generic $u(x,t)$ which may stand for any type of unknown. You will see that change in notation beginning with the section that follows. Despite that, equations of these type continue to be called "the heat equation" regardless of the domain of application, because historically they first arose in the context of heat conduction.

7.3 *The initial value problem for the heat equation with a step function as the initial condition*

Here we introduce the idea of a similarity solution and derive the fundamental solution of the heat equation.

Consider the initial value problem for the heat equation in the unknown $h(x,t)$:

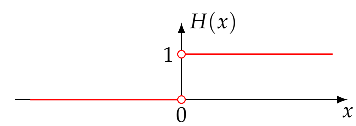
$$h_t = k\,h_{xx} \qquad -\infty < x < \infty, \quad t > 0, \tag{7.15a}$$

$$h(x,0) = H(x) \qquad -\infty < x < \infty, \tag{7.15b}$$

where H is the *Heaviside function*, which is defined as[9]

$$H(x) = \begin{cases} 0 & x < 0, \\ 1 & x > 0, \end{cases} \tag{7.16}$$

[9] The Heaviside function, named after the British mathematician and physicist Oliver Heaviside (1850–1925), is sometimes called the *unit step function* as its graph looks like a step:



The value of the function at $x = 0$ is generally inconsequential and is normally left undefined.

and let

$$w(x,t) = h(\alpha x, \beta t), \tag{7.17}$$

for arbitrary positive constants α and β. Observe that

$$w(x,0) = h(\alpha x, 0) = H(\alpha x) = H(x),$$

and that

$$w_t(x,t) = \beta\,h_t(\alpha x, \beta t), \quad w_{xx}(x,t) = \alpha^2 h_{xx}(\alpha x, \beta t).$$

Since the function h solves the initial value problem (7.15), the function w satisfies the initial value problem

$$w_t(x,t) = \frac{k\beta}{\alpha^2} w_{xx}(x,t), \tag{7.18a}$$

$$w(x,0) = H(x). \tag{7.18b}$$

The initial value problems (7.15) and (7.18) will be identical if $\beta = \alpha^2$, and therefore their solutions will be identical, that is $h(x,t) = w(x,t)$, and thus, in view of (7.17)

$$h(x,t) = h(\alpha x, \alpha^2 t) \qquad \text{for all } \alpha. \tag{7.19}$$

In conclusion, the solution h of the initial value problem (7.15) satisfies the algebraic identity (7.19) for all x, t, α. In particular, taking $\alpha^2 = \frac{1}{4kt}$, this yields[10]

$$h(x,t) = h\Big(\frac{x}{\sqrt{4kt}}, \frac{1}{4k}\Big),$$

[10] At the first attempt it's natural to try taking $\alpha^2 = \frac{1}{t}$. That works—and you are invited to try it too—but then further down you will need to make a change variables in order to connect with the definition of the erf function. The factor of $4k$ in the denominator in inserted here by hindsight in order to avoid the need for a change of variables later on.

which shows that $h(x,t)$ depends on x and t only through the ratio $\eta \equiv \frac{x}{\sqrt{4kt}} = \frac{1}{\sqrt{4k}}\,x t^{-1/2}$. That motivates us to look for a solution of (7.15) in the form

$$h(x,t) = \varphi(\eta).$$

We calculate

$$h_t(x,t) = \varphi'(\eta)\eta_t = -\frac{1}{2\sqrt{4k}} x t^{-3/2} \varphi'(\eta),$$

$$h_x(x,t) = \varphi'(\eta)\eta_x = \frac{1}{\sqrt{4k}} t^{-1/2} \varphi'(\eta),$$

$$h_{xx}(x,t) = \frac{1}{4k} t^{-1} \varphi''(\eta).$$

Substituting these into (7.15a) we arrive at

$$-\frac{1}{2\sqrt{4k}} x t^{-3/2} \varphi'(\eta) = \frac{k}{4k} t^{-1} \varphi''(\eta),$$

which simplifies to

$$\varphi''(\eta) + 2\eta\varphi'(\eta) = 0.$$

Multiplying this through by the integrating factor e^{η^2}, we obtain

$$\left(e^{\eta^2}\varphi'(\eta)\right)' = 0,$$

and therefore

$$\varphi'(\eta) = c_1 e^{-\eta^2},$$

where c_1 is an arbitrary integration constant. Integrating once more, we arrive at

$$\varphi(\eta) = c_1 \int_0^\eta e^{-s^2}\, ds + c_2,$$

where c_2 is another arbitrary constant. Restoring the value of η, we conclude that

$$h(x,t) = c_1 \int_0^{x/\sqrt{4kt}} e^{-s^2}\, ds + c_2.$$

The antiderivative of e^{-s^2} is *not* expressible in terms of elementary functions, but as it occurs quite frequently in various areas of probability and statistics, it is extensively studied under the name of the *Gauss error function* or simply *the error function*, which is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds. \tag{7.20}$$

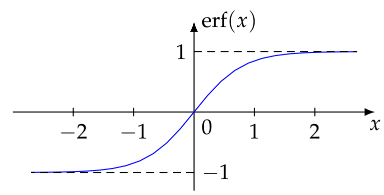


Figure 7.3: The error function, erf, is odd and has asymptotes ±1 at ±∞.

The graph of erf is shown on the right. It is an *odd* function, that is, $\operatorname{erf}(-x) = -\operatorname{erf}(x)$, and it has horizontal asymptotes of ± 1 at $\pm\infty$.

The solution $h(x,t)$ calculated above may be expressed in terms of erf as

$$h(x,t) = c_1 \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big) + c_2$$

after redefining the constant c_1 appropriately. We determine c_1 and c_2 by requiring the solution to satisfy $\lim_{t\to 0^+} h(x,t) = H(x)$. Toward that end, fix x at an arbitrary negative value and let $t \to 0^+$ in the equation above. Then $\frac{x}{\sqrt{4kt}} \to -\infty$, and consequently $\operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big) \to -1$, while the left-hand side of the equation tends to zero. We conclude that

$$-c_1 + c_2 = 0.$$

On the other hand, fixing x at an arbitrary positive value and letting $t \to 0^+$ we get

$$c_1 + c_2 = 1.$$

From the two equations above we see that $c_1 = c_2 = 1/2$, and therefore

$$h(x,t) = \frac{1}{2}\left[1 + \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big)\right]. \tag{7.21}$$

Figure 7.4 shows an animation of this solution.

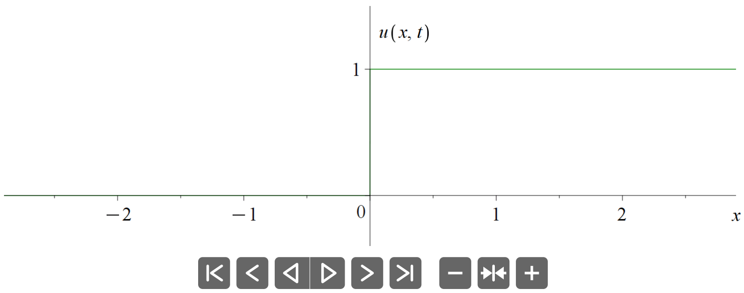


Figure 7.4: The solution of the initial value problem (7.15).

Remark 7.5. The function $h(x,t)$ obtained in (7.21) is undefined at $t = 0$ due to the presence of t in the denominator. The initial condition (7.15b) is satisfied in the sense of the limit $t \to 0^+$ for any $x < 0$ and $x > 0$, as we enforced it in the process of its derivation. We conclude that the proper way of expressing the solution of the initial value problem (7.15) is

$$h(x,t) = \begin{cases} \frac{1}{2}\left[1 + \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big)\right] & t > 0, \\ H(x) & t = 0, \end{cases}$$

where H is the Heaviside function. This, however, is perhaps a bit overly pedantic; one usually refers to (7.21) as the solution of the initial value problem (7.15). The limiting behavior at $t = 0$ is tacitly understood.

Remark 7.6. This section's development will work equally well with the more general initial condition

$$h(x,0) = f(x) = \begin{cases} a & x < 0, \\ b & x > 0. \end{cases}$$

with arbitrary constants a and b. The special property of such initial conditions lies in the observation that $f(\alpha x) = f(x)$ for any positive α, that is, f is *invariant under dilation/compression*[11] *of the x axis*.

[11] But not reflection!

7.4 *The heat kernel*

If u is any solution of the heat equation $u_t = ku_{xx}$, then so is u_x because

$$(u_x)_t = (u_t)_x = (ku_{xx})_x = k(u_x)_{xx}.$$

In particular, since $h(x,t)$ in (7.21) is a solution to the heat equation, so is $G(x,t) \equiv h_x(x,t)$ which can be shown to be

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}. \tag{7.22}$$

The solution $G(x,t)$ to the heat equation is variously called the *fundamental solution of the heat equation*, the *heat kernel*, or the *Green's function* (hence the letter G designation for it.) For any positive t, the graph of G is a bell-shaped curve as in the sample seen on the right. The curve flattens and spreads out as t increases, but the area under the curve remains a constant 1 for all t:

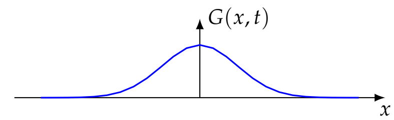


Figure 7.5: The graph of the heat kernel $G(x,t)$ for some $t > 0$.

$$\begin{aligned}\int_{-\infty}^{\infty} G(x,t)\,dx &= \int_{-\infty}^{\infty} h_x(x,t)\,dx \\ &= \lim_{\xi\to\infty} h(\xi,t) - \lim_{\xi\to-\infty} h(\xi,t) = 1 - 0 = 1.\end{aligned} \tag{7.23}$$

It is evident from (7.22) that

$$\lim_{t\to+\infty} G(x,t) = 0 \quad \text{for all } x.$$

The behavior of G as $t \to 0^+$ is more interesting. We leave it as an exercise to show that

$$\lim_{t\to 0^+} G(x,t) = 0 \quad \text{for all } x \neq 0. \tag{7.24}$$

Thus, as we go backward in time to $t = 0+$, the $x < 0$ and $x > 0$ halves of the bell-shaped curve in Figure 7.5 are flattened against the x axis, but since the area under the curve remains 1, the shape becomes increasingly narrow and tall, and the graph climbs up the vertical axis toward infinity. So in the limit a rather strange "function" is obtained which is 0 for all $x \neq 0$, it is ∞ at $x = 0$, and the area under its "graph" is 1 (!). That strange "function" is called the *Dirac delta distribution* and is written as δ:

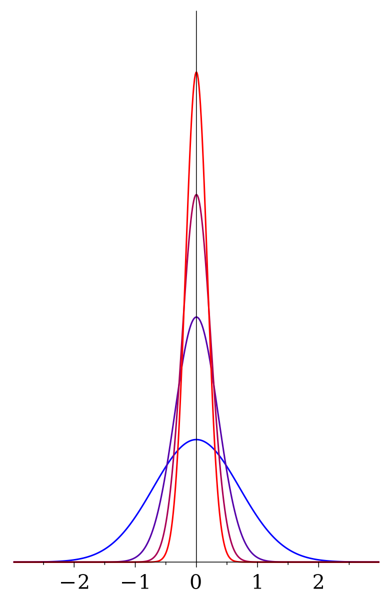


Figure 7.6: The graph of the function $G(x,t)$ becomes narrower and taller as $t \to 0+$. The area under the graph remains constant

$$\lim_{t\to 0^+} G(x,t) = \delta(x). \tag{7.25}$$

I put "function" and "graph" in quotes since what is described here is not a function at all. In fact, the $G(x,t)$ *does not converge* to any function as $t \to 0^+$ in any sense of the word "converge" known in elementary calculus.[12] The following limit, however, does makes an obvious sense:

$$\lim_{t\to 0^+} \int_{-\infty}^{\infty} G(x,t)v(x)\,dx = v(0) \quad \text{for any continuous function } v. \tag{7.26}$$

After all, as t approaches zero, $G(x,t)$ approaches zero everywhere away from $x = 0$ (see equation (7.24) and Figure 7.6), so the values of v at $x \neq 0$ do not matter. Near $x = 0$ the value of v is essentially $v(0)$

[12] It *does converge*, however, in a more esoteric sense of convergence that is studied under the topics of *generalized functions*, *distributions*, *Sobolev spaces*, and *functional analysis*, which are beyond the scope of this book. The limit, in that sense, is usually written as $\delta(x)$.

since v is continuous, and the integral of G is 1, hence it's no surprise that the limit above is $v(0)$.

If we allow ourselves the liberty of interchanging the limit and integration in (7.26)—a step whose justification requires an appeal to the theory of distributions—we get

$$\int_{-\infty}^{\infty} \left[\lim_{t\to 0^+} G(x,t)\right] v(x)\,dx = v(0),$$

or, in view of (7.25)

$$\int_{-\infty}^{\infty} \delta(x)v(x)\,dx = v(0) \quad \text{for any continuous function } v. \tag{7.27}$$

In fact, (7.27) is precisely the technical definition of the Dirac delta distribution—it is a *distribution* (not a function!) with the property that when integrated against any continuous function v, yields $v(0)$. With this notation, the function $G(x,t)$ defined in (7.22) is the solution to the initial value problem

$$G_t = kG_{xx} \qquad -\infty < x < \infty, \quad t > 0, \tag{7.28a}$$

$$G(x,0) = \delta(x) \qquad -\infty < x < \infty. \tag{7.28b}$$

The animation in Figure 7.7 shows the evolution of $G(x,t)$, starting as a Dirac delta distribution at $t = 0$, and very quickly spreading and flattening as we move forward in time.

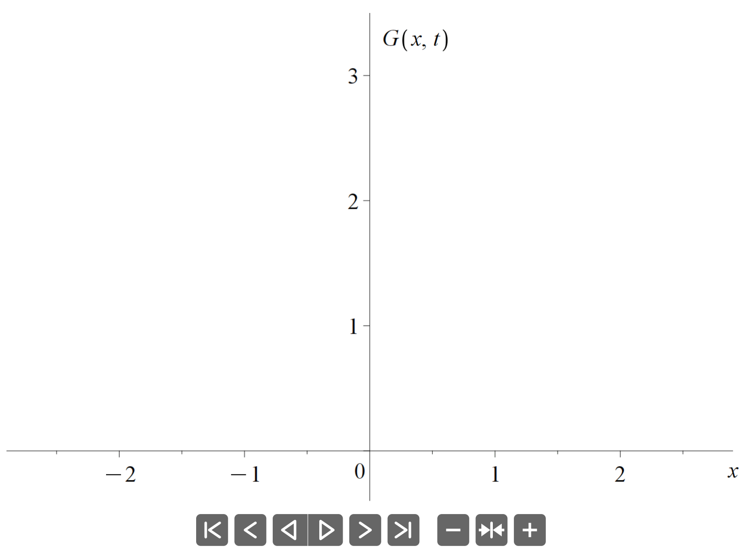


Figure 7.7: The evolution of the heat kernel $G(x,t)$ in time. The initial value, the Dirac delta distribution, quickly changes to a bell-shaped curve and then spreads out and flattens. The area under the graph is 1 at all times.

Remark 7.7. It is interesting (and probably unexpected) that the solution G of the initial value problem (7.28) is nonzero at all x, and all $t > 0$. Consequently, according to our heat conduction model, the effect of injecting heat at $x = 0$ affects the entire x axis *instantly*—heat spreads at infinite speed! There is no such a thing as a "speed of propagation" under the heat equation.

7.5 The initial value problem for the heat equation

In sections 7.3) and 7.4) we obtained explicit expressions for solutions of the heat equations with initial data in the form of the Heaviside step function, $H(x)$, and the Dirac delta distribution, $\delta(x)$. We now turn to the case of general[13] initial data, $f(x)$, as in

$$u_t = ku_{xx} \qquad -\infty < x < \infty, \quad t > 0, \tag{7.29a}$$

$$u(x,0) = f(x) \qquad -\infty < x < \infty, \tag{7.29b}$$

and argue that its solution is[14]

$$u(x,t) = \int_{-\infty}^{\infty} G(x-s,t) f(s)\, ds, \tag{7.30}$$

where G is the heat kernel, defined in (7.22). To verify that assertion, let's observe that

$$u_t(x,t) = \int_{-\infty}^{\infty} G_t(x-s,t) f(s)\, ds, \quad u_{xx}(x,t) = \int_{-\infty}^{\infty} G(x-s,t)_{xx} f(s)\, ds,$$

and therefore

$$u_t(x,t) - ku_{xx}(x,t) = \int_{-\infty}^{\infty} \Big[G_t(x-s,t) - G(x-s,t)_{xx} \Big] f(s)\, ds,$$

which evaluates to zero since G satisfies the equation (7.28a).

That verifies that $u(x,t)$ satisfies the PDE (7.29a). Verifying that it also satisfies the initial condition (7.29b) is a bit tricky. Here is one way of doing it following Strauss [20] (page 48).

Recalling the definition of the heat kernel G in (7.22) as the x derivative of the solution $h(x,t)$ of the IVP (7.15), we may write (7.30) as

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial}{\partial x} \Big[h(x-s,t) \Big] f(s)\, ds \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial s} \Big[h(x-s,t) \Big] f(s)\, ds \\ &= -h(x-s,t) f(s) \Big|_{s=-\infty}^{s=+\infty} + \int_{-\infty}^{\infty} h(x-s,t) f'(s)\, ds. \end{aligned}$$

Integration by parts

Let's note that

$$h(x-s,t) f(s) \Big|_{s=-\infty}^{s=+\infty} = \lim_{s\to-\infty} h(x-s,t) f(s) - \lim_{s\to+\infty} h(x-s,t) f(s),$$

which evaluates to zero since $h(x,t)$ is bounded between 0 and 1, and f tends to zero at $\pm\infty$ by assumption. That leaves us with

$$u(x,t) = \int_{-\infty}^{\infty} h(x-s,t) f'(s)\, ds.$$

[13] Well, not entirely general. We assume that f is differentiable and $f(x)$ tends to zero as $x \to \pm\infty$. See Remark 7.8 regarding less stringent conditions on f.

[14] See (7.31) for an explicit form of equation (7.30).

We evaluate that at $t = 0$, and recalling that $h(x, 0) = H(x)$ (see (7.15b)), we arrive at

$$u(x, 0) = \int_{-\infty}^{\infty} H(x - s) f'(s)\, ds.$$

But $H(x - s)$ is 0 when $s > x$ and 1 when $s < x$. Therefore

$$u(x, 0) = \int_{-\infty}^{x} f'(s)\, ds = f(x) - \lim_{s \to -\infty} f(s) = f(x),$$

since f goes to zero near $-\infty$. This completes the demonstration that u defined in (7.30) satisfies the initial condition (7.29b).

Remark 7.8. The assumption that f goes to zero near $\pm\infty$ is too strong. It's possible to carry out the argument above by assuming that f is *bounded* over the entire real line but not necessarily zero at $\pm\infty$. For the rather technical proof see Evans [9] (pages 47–48).

Remark 7.9. That the function u defined in (7.30) satisfies the initial condition (7.29b) may be surmised intuitively through the following "hand-waving" argument.

Recall that the heat kernel $G(x, t)$ the is the solution of the IVP (7.28), and it approaches[15] Dirac's delta distribution as $t \to 0+$. As $G(x, t)$ is the solution of the IVP (7.28), $G(x - s, t)$ is the solution of the IVP

[15] Play the animation (7.7) backward to visualize that.

$$\begin{aligned} &G_t = kG_{xx} && -\infty < x < \infty, \quad t > 0, \\ &G(x, 0) = \delta(x - s) && -\infty < x < \infty. \end{aligned}$$

for any s. (See Exercise 7.1.) Therefore, as $t \to 0+$, the function $G(x - s, t)$ approaches $\delta(x - s)$. Now we calculate:

$$\begin{aligned} \lim_{t \to 0} u(x, t) &= \lim_{t \to 0+} \int_{-\infty}^{\infty} G(x - s, t) f(s)\, ds \\ &= \int_{-\infty}^{\infty} \lim_{t \to 0+} G(x - s, t) f(s)\, ds \\ &= \int_{-\infty}^{\infty} \delta(x - s) f(s)\, ds \\ &= \int_{-\infty}^{\infty} \delta(\xi) f(x - \xi)\, d\xi && \big[\text{letting } x - s = \xi\big] \\ &= f(x). && \big[\text{by (7.27)}\big] \end{aligned}$$

That "verifies" (7.29b), but the explanation is *far* from rigorous. Interchanging the limit and integration should not be taken casually, especially considering that G blows up at $t = 0$.

Remark 7.10. Inserting the explicit formula for G derived in (7.22), the equation (7.30) takes the form

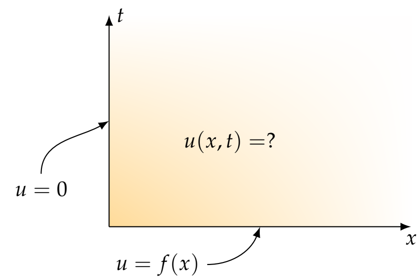
$$u(x, t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\, ds, \tag{7.31}$$

whch is the more commonly seen form of the solution of the initial value problem (7.29).

7.6 *The heat equation on $x > 0$*

We have learned how to solve the initial value problem of the heat equation on the real line $-\infty < x < \infty$. That corresponds to heat conduction in a rod that extends from $-\infty$ to ∞. We now turn to heat conduction on a *semi-infinite rod* that extends over $0 < x < \infty$. This exposes the end of the rod at $x = 0$, therefore it calls for prescribing externally applied conditions there. A a simple case, let's set the temperature at $x = 0$ to zero and examine the following initial boundary value problem (IBVP):

$$u_t = ku_{xx} \qquad 0 < x < \infty, \quad t > 0, \tag{7.32a}$$
$$u(x,0) = f(x) \qquad 0 < x < \infty \tag{7.32b}$$
$$u(0,t) = 0 \qquad t > 0. \tag{7.32c}$$



The *method of images* that was introduced in the context of the wave equation in Section 6.1 applies equally well here. We extend the initial condition $f(x)$ as an odd function f_{ext} to the entire real line,

$$f_{\text{ext}} = \begin{cases} f(x) & \text{if } x > 0, \\ -f(-x) & \text{if } x < 0, \end{cases}$$

and define

$$u(x,t) = \int_{-\infty}^{\infty} G(x-s,t) f_{\text{ext}}(s)\, ds. \tag{7.33}$$

According to (7.30), $u(x,t)$ solves the heat equation with initial data $f_{\text{ext}}(x)$. Therefore $u(x,t)$ satisfies both (7.32a) and (7.32b), since $f_{\text{ext}}(x)$ coincides with $f(x)$ on $x > 0$. It remains to verify that $u(x,t)$ satisfies the boundary condition (7.32c), so let's look at

$$u_{\text{ext}}(0,t) = \int_{-\infty}^{\infty} G(-s,t) f_{\text{ext}}(s)\, ds.$$

But the integrand is an odd function of s since $G(-s,t)$ is even in s and $f_{\text{ext}}(s)$ is odd in s, and therefore the integral evaluates to zero, as asserted.

The solution $u(x,t)$ defined in (7.33) may be expressed solely in terms of the problem's data, f, by splitting the integration into the intervals $-\infty < s < 0$ and $0 < s < \infty$,

$$u(x,t) = \int_{-\infty}^{0} G(x-s,t) f_{\text{ext}}(s)\, ds + \int_{0}^{\infty} G(x-s,t) f_{\text{ext}}(s)\, ds,$$

and noting that

$$\int_{0}^{\infty} G(x-s,t) f_{\text{ext}}(s)\, ds = \int_{0}^{\infty} G(x-s,t) f(s)\, ds,$$

and

$$\int_{-\infty}^{0} G(x-s,t) f_{\text{ext}}(s)\, ds = -\int_{-\infty}^{0} G(x-s,t) f(-s)\, ds$$
$$= -\int_{0}^{\infty} G(x+s,t) f(s)\, ds,$$

Changing the variable of integration from $s \to -s$.

whereby we conclude that the solution of the IBVP (7.32) is

$$u(x,t) = \int_{0}^{\infty} \Big[G(x-s,t) - G(x+s,t) \Big] f(s)\, ds. \tag{7.34}$$

7.7 *The nonhomogeneous heat equation and Duhamel's principle*

Duhamel's principle[16] is more of an idea than a concrete prescription. It amounts to the observation that the solution of a nonhomogeneous differential equation may be expressed in terms of the solution of the corresponding homogeneous equation where the nonhomogeneous terms of the former manifests itself in the initial condition of the latter.

[16] Named after the French mathematician and physicist Jean-Marie Duhamel (1797–1872), pronounced du.a.me'l.

We begin this section with illustrating the idea in the context of an ODE, and later extend our investigations to PDEs.

7.7.1 *Duhamel's principle in ODEs*

Consider the following two initial value problems for the unknowns $w(t)$ and $u(t)$, along with their solutions:

$$w' + aw = 0 \tag{7.35a}$$
$$w(0) = f(\tau) \tag{7.35b}$$
$$\overset{\text{solve}}{\Rightarrow} w(t) = e^{-at} f(\tau) \tag{7.35c}$$

$$u' + au = f(t) \tag{7.36a}$$
$$u(0) = 0 \tag{7.36b}$$
$$\overset{\text{solve}}{\Rightarrow} u(t) = \int_0^t e^{-a(t-\tau)} f(\tau)\, d\tau \tag{7.36c}$$

The symbol τ plays two unrelated roles in those IVPs. In (7.35a)–(7.35b), it's an unspecified parameter that sets the initial value, and is carried over to the solution. In (7.36c), it's the "dummy" variable of integration.[17] A key observation is that the integrand in (7.36c) is exactly $w(t-\tau)$, with w as in (7.35c), and therefore

$$u(t) = \int_0^t w(t-\tau)\, d\tau. \tag{7.37}$$

[17] The solution (7.36c) may be obtained through a straightforward application of the integrating factor technique or the Laplace transform. The solution (7.35c) may be obtained through the same methods, as well as separation of variables.

This equation, which relates the solutions of the two IVPs, is an instance of Duhamel's principle. It's an interesting result as it establishes a connection between the solution w of the *homogeneous* IVP (7.35a)–(7.35b) and the solution u of the *nonhomogeneous* IVP (7.36a)–(7.36b). "Duhamel's principle" is an umbrella terms that coverts such relationships in a variety of contexts.

7.7.2 Duhamel's principle in the heat equation

Consider the following two IVPs for heat equations in the unknowns $w(x,t)$ and $u(x,t)$ in $-\infty < x < \infty,\ t > 0$, and where $f(x,t)$ is given:

$$w_t = w_{xx} \tag{7.38a}$$

$$w(x,0) = f(x,\tau) \tag{7.38b}$$

$$\overset{\text{solve}}{\Rightarrow} w(x,t) = \int_{-\infty}^{\infty} G(x-s,t) f(s,\tau)\, ds \tag{7.38c}$$

$$u_t = ku_{xx} + f(x,t) \tag{7.39a}$$

$$u(x,0) = 0 \tag{7.39b}$$

$$\overset{\text{solve}}{\Rightarrow} u(x,t) = ? \tag{7.39c}$$

The IVP (7.38a)–(7.38b) is identical to (7.29) on page 91. The solution (7.38c) is copied over[18] from (7.30).

[18] Unlike (7.29b), the initial condition (7.38b) depends on a parameter τ, but as far as the IVP is concerned, τ is just a constant, and therefore the solution (7.30) is still applicable.

Can we guess what the solution $u(x,t)$ would be? If Duhamel's principle holds, then $u(x,t)$ would be related to $w(x,t)$ through (7.37), which in this case expands to

$$u(x,t) = \int_0^t \int_{-\infty}^{\infty} G(x-s,t-\tau) f(s,\tau)\, ds\, dt. \tag{7.40}$$

Verifying that (7.40) is indeed the desired solution is a matter of plugging it into (7.39a) and (7.39b). The initial condition (7.39b) is obviously satisfied, so it remains to verify (7.39a), and that turns out to be a non-trivial task. Substituting (7.40) into (7.39b) calls for calculating the derivatives of (7.40) with respect to x and t, and passing those derivatives under the integral signs. The interchange of integration and differentiation operations is a delicate task and it's not always permissible, especially considering that the G in the integrand involves singularities—note the t in the denominator in (7.22). Nevertheless, it is possible to verify that $u(x,t)$ obtained in (7.40) is indeed the expected solution. That takes two pages of calculations in Evans [9] (pages 50–51), so we won't go through that here.

7.8 Uniqueness of the solutions of the heat equation

Let us write D for the upper half-plane in space-time, that is,

$$D = \big\{(x,t) : -\infty < x < \infty, \quad t > 0\big\},$$

and let $\mathscr{C}$ be the set of function defined on D with the property that each such function and its x-derivative are bounded in D, that is for each function there exists a number M such that

$$\big|u(x,t)\big| \le M \quad \text{and} \quad \big|u_x(x,t)\big| \le M \quad \text{for all } (x,t) \in D. \tag{7.41}$$

Then we have:

Theorem 7.1. *The initial value problem*

$$u_t = ku_{xx} + q(x,t) \qquad -\infty < x < \infty, \quad t > 0, \tag{7.42a}$$

$$u(x,0) = f(x) \qquad -\infty < x < \infty, \tag{7.42b}$$

has at most one solution u in $\mathscr{C}$.

Proof. Suppose u_1 and u_2 are solutions of the IVP (7.42). We wish to show that if $u_1 \in \mathscr{C}$ and $u_1 \in \mathscr{C}$, then $u_1 = u_2$.

Let $v = u_1 - u_2$. Since u_1 and u_2 satisfy (7.42), we see that v satisfies the IVP

$$v_t = kv_{xx} \qquad -\infty < x < \infty, \quad t > 0, \tag{7.43a}$$

$$v(x,0) = 0 \qquad -\infty < x < \infty. \tag{7.43b}$$

Showing $u_1 = u_2$ is equivalent to showing that v is identically zero. To show that claim, fix an arbitrary $(x,t) \in D$ and define

$$\psi(\xi,\tau) = G(x-\xi, t-\tau), \tag{7.44}$$

where G is the heat kernel defined in (7.22) on page 89. Let us note that

$$\psi_\tau(\xi,\tau) = -G_t(x-\xi,t-\tau), \quad \psi_{\xi\xi}(\xi,\tau) = G_{xx}(x-\xi,t-\tau),$$

and since G satisfies the heat equation $G_t = kG_{xx}$, we have $\psi_\tau = -k\psi_{\xi\xi}$, that is $\psi_\tau + k\psi_{\xi\xi} = 0.$ [19] At the same time, since v satisfies the equation (7.43a) for all (x,t), we have $v_\tau = kv_{\xi\xi}$, that is $v_\tau - kv_{\xi\xi} = 0$. Now, consider

$$\psi \cdot (v_\tau - kv_{\xi\xi}) + v \cdot (\psi_\tau + k\psi_{\xi\xi}) = \big(v_\tau\psi + v\psi_\tau\big) + k\Big[v\psi_{\xi\xi} - v_{\xi\xi}\psi\Big].$$

The left-hand side is zero because of what was said in the previous paragraph. The first term on the right-hand side is just $(v\psi)_\tau$. The expression in the square brackets may be simplified to

$$v\psi_{\xi\xi} - v_{\xi\xi}\psi = -(v_\xi\psi - v\psi_\xi)_\xi,$$

and thus we conclude that

$$(v\psi)_\tau = k\,(v_\xi\psi - v\psi_\xi)_\xi.$$

We integrate this over the rectangle $-R < \xi < R$, $0 < \tau < t - \epsilon$ for some positive ϵ,

$$\int_{-R}^{R}\int_0^{t-\epsilon}(v\psi)_\tau\,d\tau\,d\xi = k\int_0^{t-\epsilon}\int_{-R}^{R}(v_\xi\psi - v\psi_\xi)_\xi\,d\xi\,d\tau,$$

and evaluate the inner integrals. We get

$$\int_{-R}^{R}(v\psi)\Big|_{\tau=0}^{\tau=t-\epsilon}d\xi = k\int_0^{t-\epsilon}(v_\xi\psi - v\psi_\xi)\Big|_{\xi=-R}^{\xi=R}d\tau.$$

Since $v(\xi,0) = 0$ (see (7.43b)) the left-hand side's integrand simplifies to $v(\xi,t-\epsilon)\psi(\xi,t-\epsilon)$, and thus

$$\int_{-R}^{R}v(\xi,t-\epsilon)\psi(\xi,t-\epsilon)\,d\xi = k\int_0^{t-\epsilon}(v_\xi\psi - v\psi_\xi)\Big|_{\xi=-R}^{\xi=R}d\tau. \tag{7.45}$$

[19] One says that ψ is the solution of the *backward heat equation* since the equation $\psi_\tau(\xi,\tau) = -k\psi_{\xi\xi}(\xi,\tau)$ is obtained by reversing the direction of the flow of time in the heat equation.

What happens as $R \to \infty$? Referring to the definition of ψ in (7.44), and the definition of G in (7.22), we see that ψ goes to zero as $\xi \to \pm\infty$. In view of the boundedness of v and v_ξ, the integrand on the right-hand side of (7.45) goes to zero as $R \to \infty$, and (7.45) reduces to

$$\int_{-\infty}^{\infty} v(\xi, t-\epsilon)\psi(\xi, t-\epsilon)\, d\xi = 0.$$

Substituting for ψ from its definition in (7.44), this takes the form

$$\int_{-\infty}^{\infty} v(\xi, t-\epsilon) G(x-\xi, \epsilon)\, d\xi = 0.$$

As $\epsilon \to 0$, the G factor approaches the Dirac delta distribution $\delta(x-\xi)$, and therefore the integral evaluates to $v(x,t)$, which proves that v is identically zero, as asserted. □

Remark 7.11. According to Theorem 7.1, the IVP (7.42) cannot have more than one solution in the set $\mathscr{C}$. Equivalently, the IVP (7.43), which has $v \equiv 0$ as a solution, cannot have a second nonzero solution in the set $\mathscr{C}$. But can it have a nonzero solution within a larger set of functions? It turns out that the answer is yes! Tychonoff[20] [24] famously constructed a nonzero solution to the IVP (7.43). That doesn't violate Theorem 7.1 since his solution is unbounded in D.

[20] Russian mathematician Andrey Nikolayevich Tikhonov (1906–1993).

Remark 7.12. The class $\mathscr{C}$ of function is far more restrictive that necessary. It's possible to enlarge it by removing the condition on the derivative u_x and allowing u to grow in a controlled way. Specifically, Tychonoff showed that the condition (7.41) may be replaced with:

There exist positive constants M and a such that

$$\big|u(x,t)\big| \le Me^{ax^2}, \quad -\infty < x < \infty.$$

It's possible to further enlarge the class of admissible in various ways and still retain uniqueness; see [4, 3] and references therein.

7.9 *Exercises*

7.1 (**Shift invariance**). Let u be the solution of the IVP

$$\begin{aligned} &u_t = ku_{xx} && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x) && -\infty < x < \infty. \end{aligned}$$

Show that $v(x,t) = u(x-a,t)$ is the solution of the IVP

$$\begin{aligned} &v_t = kv_{xx} && -\infty < x < \infty, \quad t > 0, \\ &v(x,0) = f(x-a) && -\infty < x < \infty. \end{aligned}$$

7.2 (**Additive property**). Consider the solutions u and v of the initial value problems

$$\begin{aligned} &u_t = ku_{xx} && v_t = kv_{xx} \\ &u(x,0) = f(x) && v(x,0) = g(x) \end{aligned}$$

where $-\infty < x < \infty$ and $t > 0$. Show that $w(x,t) = u(x,t) + v(x,t)$ is the solution of the IVP

$$\begin{aligned} &w_t = kw_{xx}, \\ &w(x,0) = f(x) + g(x). \end{aligned}$$

7.3 (**Reflection property**). Let u be the solution of the IVP

$$\begin{aligned} &u_t = ku_{xx} && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x) && -\infty < x < \infty. \end{aligned}$$

Show that $v(x,t) = u(-x,t)$ is the solution of the IVP

$$\begin{aligned} &v_t = kv_{xx} && -\infty < x < \infty, \quad t > 0, \\ &v(x,0) = f(-x) && -\infty < x < \infty. \end{aligned}$$

7.4. The text states that the error function, erf, defined in (7.20) and graphed in Figure (7.3), has horizontal asymptotes ± 1 at $\pm\infty$. Verify that statement by showing that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Hint: Let $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$ and observe that

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\, dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy.$$

The double integral on the right-hand side expresses the volume under the graph of $z = e^{-(x^2+y^2)}$ over the entire xy plane. Evaluate it by changing to polar coordinates.

7.5. Verify that (7.22) is indeed the derivative u_x of the function u defined in (7.21).

7.6. Verify (7.24).

Hint: Let $s = 1/t$ and then look at the limit as $s \to +\infty$.

7.7. Show that the solution (7.31) of the initial value problem of the heat equation may be cast into the equivalent form

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-r^2} f\left(x - r\sqrt{4kt}\right) dr. \tag{7.46}$$

This is known as the *Poisson integral representation* of the solution.

Hint: In (7.31), change the variable of integration from s to r defined through $r = \frac{x-s}{\sqrt{4kt}}$.

7.8. Show that

$$\int e^x \operatorname{erf}(x)\,dx = e^x \operatorname{erf}(x) - e^{1/4} \operatorname{erf}\Big(x - \frac{1}{2}\Big) + C,$$

and generalize it to

$$\int e^{ax} \operatorname{erf}(bx)\,dx = \frac{1}{a} e^{ax} \operatorname{erf}(bx) - \frac{1}{a} e^{a^2/(4b^2)} \operatorname{erf}\Big(bx - \frac{a}{2b}\Big) + C.$$

This is needed for solving Exercise 7.22.

Hint: Integrate by parts.

7.9. Solve the IVP (7.29) with

$$f(x) = \begin{cases} 1 & |x| < 1, \\ 0 & |x| > 1 \end{cases}$$

Hint: Express $f(x)$ as a linear combination of the shifted Heaviside functions $H(x-1)$ and $H(x+1)$, and then appeal to the conclusions of the exercises 7.1 and 7.2 in order to connect this exercise to the solution (7.21).

7.10. Solve the initial value problem (7.29) with $f(x) = \begin{cases} 0 & x < 0 \\ e^{-x} & x > 0 \end{cases}$.

Hint: Apply the formula (7.31) and complete the square (with respect to s) in the exponent.

7.11. Solve the initial value problem (7.29) with $f(x) = e^{-|x|}$.

7.12. Solve the initial value problem for the heat equation with a heat source proportional to temperature:

$$\begin{aligned} &u_t = ku_{xx} + hu && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x). && -\infty < x < \infty. \end{aligned}$$

Hint: Rearrange the PDE into $u_t - hu = ku_{xx}$, multiply through by the "integrating factor" e^{-ht} in order to collapse the two terms on the left-hand side into a single term. Then introduce $v(x,t) = e^{-ht}u(x,t)$ and show that $v(x,t)$ solves the standard heat equation. Apply (7.31) to find $v(x,t)$, and finally find the expression for $u(x,t)$.

7.13. Solve the initial value problem for the *advection-diffusion* problem

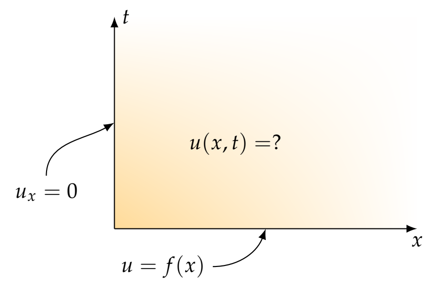
$$\begin{aligned} &u_t + cu_x = ku_{xx} && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x). && -\infty < x < \infty. \end{aligned}$$

Hint: Inspired by the previous exercise, look for $v(x,t) = e^{\alpha x + \beta t}u(x,t)$. Find the PDE satisfied by v, and then pick α and β so that the PDE reduces to the standard heat equation.

7.14. Solve the initial value problem

$$\begin{aligned} &u_t + cu_x = ku_{xx} + hu && -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x) && -\infty < x < \infty. \end{aligned}$$

7.15. Solve the initial boundary value problem of the heat equation on the semi-axis $x > 0$ subject to an insulated boundary condition at $x = 0$:



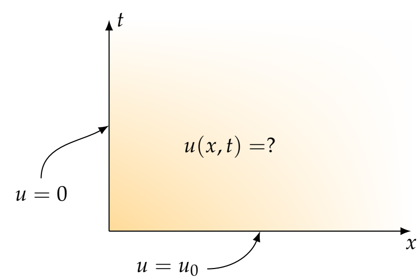
$$\begin{aligned} &u_t = ku_{xx} && 0 < x < \infty, \quad t > 0, \\ &u(x,0) = f(x) && 0 < x < \infty, \\ &u_x(0,t) = 0 && t > 0, \end{aligned}$$

and conclude that

$$u(x,t) = \int_0^\infty \Big[G(x-s,t) + G(x+s,t)\Big] f(s)\, ds.$$

Hint: Method of images.

7.16. Solve the initial boundary value problem

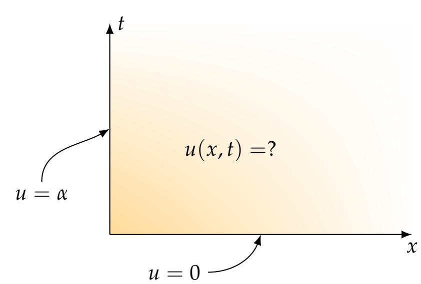


$$\begin{aligned} &u_t = ku_{xx} && 0 < x < \infty, \quad t > 0, \\ &u(x,0) = u_0 && 0 < x < \infty, \\ &u(0,t) = 0 && t > 0, \end{aligned}$$

where u_0 is a constant.

Hint: This is a special case of the IBVP (7.32).

7.17. Show that the solution of the initial boundary value problem



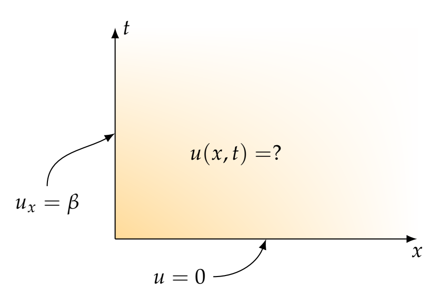
$$\begin{aligned} &u_t = ku_{xx} && 0 < x < \infty, \quad t > 0, \\ &u(x,0) = 0 && 0 < x < \infty, \\ &u(0,t) = \alpha && t > 0, \end{aligned}$$

where α is a given constant, is

$$u(x,t) = \alpha \left[1 - \operatorname{erf}\Big(\frac{x}{\sqrt{4kt}}\Big)\right].$$

Hint: Let $v(x,t) = u(x,t) - \alpha$, find the IBVP satisfied by v, and note that it is a special case of the IBVP (7.32) whose solution was obtained in (7.34).

7.18. Show that the solution of the initial boundary value problem



$$\begin{aligned} &u_t = ku_{xx} && 0 < x < \infty, \quad t > 0, \\ &u(x,0) = 0 && 0 < x < \infty, \\ &u_x(0,t) = -\beta && t > 0, \end{aligned}$$

where β is a given constant, is

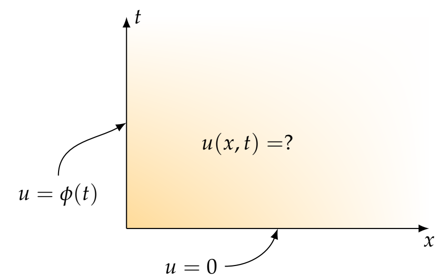
$$u(x,t) = \beta\sqrt{\frac{4kt}{\pi}}e^{-\frac{x^2}{4kt}} - \beta x\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)\right].$$

Hint: Let $v(x,t) = u(x,t) + \beta x$, find the IBVP satisfied by v, and note that it is a special case of the IBVP of exercise 7.15.

Interesting remark: Plugging $x = 0$ in the solution given above, we see that $u(0,t) = \beta\sqrt{\frac{4kt}{\pi}}$ which leads to the (unexpected?) conclusion that the temperature at $x = 0$ grows to infinity due to the supply of a constant heat flux!

7.19. Solve the following generalization of exercise 7.17:

$$u_t = ku_{xx} \qquad 0 < x < \infty, \quad t > 0, \tag{7.47a}$$
$$u(x,0) = 0 \qquad 0 < x < \infty, \tag{7.47b}$$
$$u(0,t) = \phi(t) \qquad t > 0, \tag{7.47c}$$



where ϕ is a prescribed bounded function on $t \geq 0$.

Hint: Let w be the solution of the IBVP

$$w_t = kw_{xx} \qquad 0 < x < \infty, \quad t > 0, \tag{7.48a}$$
$$w(x,0) = 0 \qquad 0 < x < \infty, \tag{7.48b}$$
$$w(0,t) = \phi(\tau) \qquad t > 0, \tag{7.48c}$$

for a fixed parameter τ.[21] Verify that

[21] This is no different from exercise 7.17's IBVP, since $\phi(\tau)$ is just a constant.

$$u(x,t) = \int_0^t w_t(x, t-\tau)\, d\tau \tag{7.49}$$

is the solution of the the IBPV (7.47). Toward that end, you will need to show that (7.49) expands to

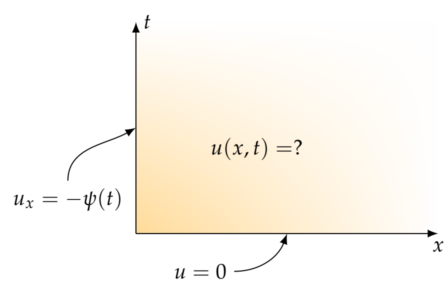
$$u(x,t) = \int_0^t \frac{xe^{-\frac{x^2}{4k(t-\tau)}}}{\sqrt{4\pi k(t-\tau)^3}}\phi(\tau)\, d\tau, \tag{7.50}$$

which, after change the variable of integration from τ to η through $\eta = \frac{x^2}{4k(t-\tau)}$, takes the more useful form

$$u(x,t) = \frac{2}{\sqrt{\pi}}\int_0^\infty \phi\Big(t - \frac{x^2}{4k\eta^2}\Big)\, e^{-\eta^2}\, d\eta - \frac{2}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} \phi\Big(t - \frac{x^2}{4k\eta^2}\Big)\, e^{-\eta^2}\, d\eta.$$

7.20. Solve the following generalization of exercise 7.18:

$$u_t = ku_{xx} \qquad 0 < x < \infty, \quad t > 0, \tag{7.51a}$$
$$u(x,0) = 0 \qquad 0 < x < \infty, \tag{7.51b}$$
$$u_x(0,t) = -\psi(t) \qquad t > 0, \tag{7.51c}$$



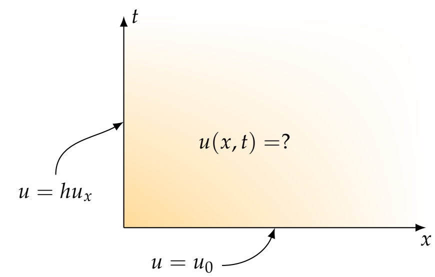
where ψ is a prescribed bounded function on $t \geq 0$.

7.21. Solve the followin IBVP with a *Robin boundary condition*,

$$u_t = ku_{xx} \qquad x > 0, \quad t > 0, \tag{7.52a}$$

$$u(x,0) = u_0 \qquad > 0, \tag{7.52b}$$

$$u(0,t) = hu_x(0,t) \qquad t > 0, \tag{7.52c}$$



where u_0 and h are constants and $h > 0$.
Hint: Let $v(x,t) = u(x,t) - \frac{1}{h}u_x(x,t)$ and show that $v(x,t)$ solves the IBVP

$$v_t = kv_{xx} \qquad x > 0, \quad t > 0, \tag{7.53a}$$

$$v(x,0) = u_0 \qquad x > 0, \tag{7.53b}$$

$$v(0,t) = 0 \qquad t > 0, \tag{7.53c}$$

which can be solved explicitly; see Exercise 7.16. The definition $v = u - \frac{1}{h}u_x$ may be viewed as an ODE, $u_x - hu = -hv$, for $u(x,t)$ as a function of x, for an arbitrary fixed t. Solve the ODE and conclude that

$$u(x,t) = hu_0 \int_0^\infty e^{-hs} \operatorname{erf}\left(\frac{x+s}{\sqrt{4kt}}\right) ds. \tag{7.54}$$

7.22. In the previous exercise show that the temperature at $x = 0$ is

$$u(0,t) = u_0 e^{h^2kt}\left[1 - \operatorname{erf}\left(h\sqrt{kt}\right)\right],$$

and conclude that $u(0,t) \to 0$ as $t \to \infty$.

8
An introduction to the Fourier series

8.1 Preliminaries

You must have learned in elementary calculus courses that some functions admit a power series representation as in

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots,$$

and that the equality holds for all $|x| < R$ for some R (and perhaps also at $x = \pm R$) where R is called the *radius of convergence* of the series.

Expanding a function into a sum of powers of x is not the only possible choice; there are much better choices in applications to PDEs. An expansion of the form

$$\begin{aligned} f(x) = a_0 + a_1 \cos\frac{\pi x}{\ell} + b_1 \sin\frac{\pi x}{\ell} + a_2 \cos\frac{2\pi x}{\ell} + b_2 \sin\frac{2\pi x}{\ell} \\ + a_3 \cos\frac{3\pi x}{\ell} + b_3 \sin\frac{3\pi x}{\ell} + \cdots, \quad -\ell \le x \le \ell, \end{aligned}$$

or written compactly as

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}, \quad -\ell \le x \le \ell, \qquad (8.1)$$

was brought to prominence by Fourier[1] in his 1822 book *Théorie analytique de la chaleur* [10] although there had been hints of such a possibility within mathematical circles earlier. The series (8.1) is now known as the *Fourier trigonometric series*.

Fourier neither proved the convergence of the series, nor explored conditions necessary for its convergence. The first rigorous statement in those regards came in the 1829 Mémoir [8] of his student, Dirichlet[2] That helped to remove some of the early skepticism surrounding the validity and applicability of Fourier's approach, established it as a legitimate and respectable part of mathematics, and spurred active research in the area, leading to the works of Sturm[3] [21, 22] and his

[1] Jean-Baptiste Joseph Fourier (1768–1830), French mathematician and physicist.

[2] Gustav Peter Lejeune Dirichlet (1805–1859), German mathematician.

[3] Jacques Charles François Sturm (1803–1855), French mathematician.

collaboration with LIOUVILLE[4] [23] which lead to a comprehensive generalization of Fourier's ideas into what has now become known as the *Sturm–Liouville theory*, and which will be the subject of Chapter 10.

[4] Joseph Liouville (1809–1882), French mathematician and engineer.

8.2 The vector space $\mathbf{R}^n$

It may come as a surprise that there is a close analogy between the geometric properties of the vector space $\mathbf{R}^n$ and a set of functions such as $\{\sin \frac{n\pi x}{\ell}\}_{n=1}^{\infty}$ that form a part of the Fourier series (8.1). It this section we set out to review some of the basic properties of $\mathbf{R}^n$. In the next section we will show how those properties carry over to sets of functions.

8.2.1 The dot product and the norm in $\mathbf{R}^n$

In elementary linear algebra we learn about the geometry of the space $\mathbf{R}^n$, including the norm $\|\boldsymbol{x}\|$ and the dot product $\boldsymbol{x} \cdot \boldsymbol{y}$ of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbf{R}^n$:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i, \quad \|\boldsymbol{x}\|^2 = \boldsymbol{x} \cdot \boldsymbol{x} = \sum_{i=1}^{n} x_i^2, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n. \tag{8.2}$$

The vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ are said to be *orthogonal* if $\boldsymbol{x} \cdot \boldsymbol{y} = 0$. A set of n nonzero vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is said to be an *orthogonal basis* in $\mathbf{R}^n$ if its vectors are pairwise orthogonal. Any vector $\boldsymbol{v} \in \mathbf{R}^n$ may be expressed as a linear combination of that basis, as in

$$\boldsymbol{v} = \sum_{i=1}^{n} c_i \boldsymbol{e}_i. \tag{8.3}$$

To determine the coefficients c_i of that linear combination, we form the dot product of the expression above with $\boldsymbol{e}_j$ for some j in $1, 2, \dots, n$. We get

$$\boldsymbol{v} \cdot \boldsymbol{e}_j = \sum_{i=1}^{n} c_i \boldsymbol{e}_i \cdot \boldsymbol{e}_j.$$

Due to the orthogonality of the basis, every term in that summation is zero for except for when $i = j$. Thus, as the summation index i runs form 1 to n, only the jth term survives, and leaves us with

$$\boldsymbol{v} \cdot \boldsymbol{e}_j = c_j \boldsymbol{e}_j \cdot \boldsymbol{e}_j = c_j \|\boldsymbol{e}_j\|^2,$$

whence $c_j = \frac{1}{\|\boldsymbol{e}_j\|^2} \boldsymbol{v} \cdot \boldsymbol{e}_j$ for $j = 1, 2, \dots, n$. The name j of the index is immaterial, so we rename it to i and arrive at

$$c_i = \frac{1}{\|\boldsymbol{e}_i\|^2} \boldsymbol{v} \cdot \boldsymbol{e}_i, \quad i = 1, 2, \dots, n. \tag{8.4}$$

In summary, when a vector $\boldsymbol{v}$ is expressed as a linear combination of an orthogonal basis as in (8.3), the coefficients of that linear combination may be calculated from (8.4).

8.2.2 A basic inequality

There exists a very basic, and very important, relationship between the dot product $\boldsymbol{x} \cdot \boldsymbol{y}$ and norms of vectors in $\mathbf{R}^n$, namely

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\| \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n, \tag{8.5}$$

where $|\boldsymbol{x} \cdot \boldsymbol{y}|$ is the absolute value of $\boldsymbol{x} \cdot \boldsymbol{y}$.

To see why that's true, observe that $\|\alpha\boldsymbol{x} + \boldsymbol{y}\|^2 \geq 0$ for all scalar multipliers α. Then expand

$$0 \leq \|\alpha\boldsymbol{x} + \boldsymbol{y}\|^2 = (\alpha\boldsymbol{x} + \boldsymbol{y}) \cdot (\alpha\boldsymbol{x} + \boldsymbol{y}) = \alpha^2\|\boldsymbol{x}\|^2 + 2\alpha\boldsymbol{x} \cdot \boldsymbol{y} + \|\boldsymbol{y}\|^2.$$

For fixed $\boldsymbol{x}$ and $\boldsymbol{y}$, the right-hand side is a quadratic function of α. The left-hand side says that the quadratic function is never negative and therefore it cannot have two real roots. Equivalently, the discriminant of the quadratic cannot be positive, that is

$$\big(2\boldsymbol{x} \cdot \boldsymbol{y}\big)^2 - 4\|\boldsymbol{x}\|^2\|\boldsymbol{y}\|^2 \leq 0,$$

which proves (8.5).

Remark 8.1. The inequality (8.5) is consistent with the customary geometric definition of the dot product

$$\boldsymbol{x} \cdot \boldsymbol{y} = \|\boldsymbol{x}\| \|\boldsymbol{y}\| \cos\theta,$$

where θ is the angle between the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ viewed as oriented line segments, because $|\cos\theta| \leq 1$ for all θ.

8.3 The function space $L^2(a, b)$

The simple geometric structure of $\mathbf{R}^n$ outlined in the previous section carries over essentially with no change to space of function defined over some interval (a, b). It is the goal of this section to flesh out that vague statement.

8.3.1 The inner product and the norm $L^2(a, b)$

With any two functions f and g defined over some interval (a, b), the integral $\int_a^b f(x)g(x)\,dx$ may be approximated as a Riemann sum by splitting the interval (a, b) into n subintervals of lengths $h = (b-a)/n$, and picking x_i, $i = 1, 2, \ldots, n$ in those subintervals:

$$\int_a^b f(x)g(x)\,dx \approx h \sum_{i=1}^{n} f(x_i)g(x_i).$$

The right-hand side has the same algebraic structure as the dot product in $\mathbf{R}^n$, and as a result, algebraic consequences of the dot product, such as the formula for components (8.4) and the inquality (8.5), carry over to the set of functions over $[a, b]$ if that we think of $\int_a^b f(x)g(x)\,dx$ as the "dot product" of the functions f and g. The standard notation for that "dot product" however, is not $f \cdot g$ as you might expect, but but (f, g), and the result is not called "dot product", but *inner product*. Thankfully, the notation for the norm is retained. Thus, we have

$$(f, g) = \int_a^b f(x)g(x)\,dx, \quad \|f\|^2 = (f, f) = \int_a^b |f(x)|^2\,dx. \tag{8.6}$$

Example 8.1. (a) The norm of the function $f(x) = x$ defined over $(0, 1)$ is $1/\sqrt{3}$ because

$$\|f\|^2 = \int_0^1 x^2\,dx = \frac{1}{3}.$$

(b) The norm of the function $f(x) = 1/x^{1/3}$ over the $(0, 1)$ is $\sqrt{3}$:

$$\|f\|^2 = \int_0^1 \frac{1}{x^{2/3}}\,dx = \int_0^1 x^{-2/3}\,dx = 3.$$

(c) The norm of the function $f(x) = 1/x^2$ over the interval $(0, 1)$ is infinity:

$$\|f\|^2 = \int_0^1 \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_0^1 = \infty.$$

Definition 8.1. A function defined over the interval (a, b) which has a finite norm is said to be *square-integrable*. We write $L^2(a, b)$ for the set of all square integrable-functions[5] defined over the interval (a, b).

[5] Technical note: The functions that constitute the space $L^2(a, b)$ consist of *measurable functions* that are square-integrable in the sense of *Lebesgue*. You will learn more on this subject in courses on measure theory and functional analysis, but for the purposes of this textbook you may safely regard all functions and integrals in the same sense that were introduced to you in your calculus courses.

It can be shown (see the Exercises section) that $L^2(a, b)$ is a *linear space* in the sense that if f and g are in $L^2(a, b)$, then so is the linear combination $\alpha f + \beta g$, for all constants α and β. The linear space $L^2(a, b)$ equipped with the inner product (8.6) is called a *Hilbert space*[6]. Due to the parallels with the dot product and norm of $\mathbf{R}^n$, there is a great deal of similarity between the linear spaces $L^2(a, b)$ and $\mathbf{R}^n$. In particular in analogy with $\boldsymbol{x} \cdot \boldsymbol{y} = 0$ indicating the orthogonality of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, the functions f and g are said to be *orthogonal* in $L^2(a, b)$ if their inner product is zero:

[6] Named after the German mathematician David Hilbert (1862–1943).

$$(f, g) = \int_a^b f(x)g(x)\,dx = 0 \quad \Longleftrightarrow \quad f \text{ and } g \text{ are orthogonal.}$$

Example 8.2. (a) The functions $f(x) = x$ and $g(x) = 2x^2 - 1$ are orthogonal in $L^2(0, 1)$:

$$(f, g) = \int_0^1 x(2x^2 - 1)\,dx = \Big(\frac{1}{2}x^4 - \frac{1}{2}x^2\Big)\bigg|_0^1 = 0.$$

(b) The functions $f(x) = \sin x$ and $g(x) = \cos x$ are orthogonal in $L^2[0, \pi]$:

$$(f, g) = \int_0^\pi \sin x \cos x\,dx = \frac{1}{2}\int_0^\pi \sin 2x\,dx = \frac{1}{4}\big(-\cos 2x\big)\bigg|_0^\pi = 0.$$

Example 8.3. Fix an $\ell > 0$ and consider the family of functions $\phi_n(x) = \sin \frac{n\pi x}{\ell}$, $n = 1, 2, \ldots$ in $L^2(0, \ell)$. Let us show that all functions in this family are mutually orthogonal. Specifically, for all integers m and n we have:

$$(\phi_m, \phi_n) = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{1}{2}\ell & \text{if } m = n. \end{cases} \tag{8.7}$$

Solution: Let's consider the $m = n$ case. We have

$$\begin{aligned}(\phi_n, \phi_n) &= \int_0^\ell \sin^2 \frac{n\pi x}{\ell}\, dx \\ &= \frac{1}{2}\int_0^\ell \left[1 - \cos \frac{2n\pi x}{\ell}\right] dx \\ &= \frac{1}{2}\left[x - \frac{\ell}{2n\pi}\sin\frac{2n\pi x}{\ell}\right]\Bigg|_0^\ell = \frac{\ell}{2}.\end{aligned}$$

Now consider the $m \neq n$ case:

$$\begin{aligned}(\phi_m, \phi_n) &= \int_0^\ell \sin\frac{m\pi x}{\ell}\sin\frac{n\pi x}{\ell}\, dx \\ &= \frac{1}{2}\int_0^\ell \left[\cos\frac{(m-n)\pi x}{\ell} - \cos\frac{(m+n)\pi x}{\ell}\right] dx \\ &= \frac{1}{2}\left[\frac{\ell}{(m-n)\pi}\sin\frac{(m-n)\pi x}{\ell} - \frac{\ell}{(m+n)\pi}\sin\frac{(m+n)\pi x}{\ell}\right]\Bigg|_0^\ell \\ &= \frac{1}{2}\left[\frac{\ell}{(m-n)\pi}\sin\big[(m-n)\pi\big] - \frac{\ell}{(m+n)\pi}\sin\big[(m+n)\pi\big]\right] = 0.\end{aligned}$$

That is very interesting (and very important!) result. It says that $L^2(0, \ell)$ can accommodate a set of *infinitely many mutually orthogonal functions*

$$\left\{\sin\frac{\pi x}{\ell}, \sin\frac{2\pi x}{\ell}, \sin\frac{3\pi x}{\ell}, \ldots, \right\}. \tag{8.8}$$

Since a set of mutually orthogonal functions is linearly independent, it follows that $L^2(0, \ell)$ is infinite dimensional! That's in stark contrast with the finite dimensional $\mathbf{R}^n$ that motivated this thread. It is then natural to ask whether the set of functions (8.8) forms a *basis* for $L^2(0, \ell)$. The answer is yes, the set of functions (8.8) is a basis for $L^2(0, \ell)$ in the sense that any function $f \in L^2(0, \ell)$ may be expressed as an (infinite) linear combination of them:

$$f(x) = \sum_{n=1}^\infty b_n \phi_n(x) = \sum_{n=1}^\infty b_n \sin\frac{n\pi x}{\ell}. \tag{8.9}$$

This representation of f is called its *Fourier sine series* of f. We will have much more to say about this in the following sections.

8.4 *Eigenvalue problems for ODEs*

We wish to gain a deeper understanding of the Fourier sine series representation (8.9) that we briefly encountered in the previous section. Specifically, where do the basis functions $\{sin \frac{n\pi x}{\ell}\}_{n=1}^{\infty}$ come from? A good staring point toward that understanding is the analysis of the following boundary value problem (BVP) for an ODE in the unknown $y(x)$ over the interval $0 \le x \le \ell$ for some given ℓ:

$$y''(x) + \lambda y(x) = 0, \qquad 0 < x < \ell, \tag{8.10a}$$

$$y(0) = 0, \tag{8.10b}$$

$$y(\ell) = 0, \tag{8.10c}$$

where the coefficient $\lambda \in \mathbf{R}$ is some constant. It is evident that the identically zero function, $y(x) \equiv 0$, is a solution. But that's neither deep nor interesting, so it is referred to as the *trivial solution* of the boundary value problem (8.10). Are there solutions other than the trivial solution? It turns out that *yes, there do exist nontrivial solutions*, but only for special values of λ. That is an absolutely crucial point, and therefore we set out to demonstrate it immediately.

In search of nontrivial solutions of the BVP (8.10), we fork the analysis into three branches, corresponding to the possibilities of λ being negative, zero, or positive.

Case 1: $\lambda < 0$. To enforce the negativity of λ, take $\lambda = -\gamma^2$, where $\gamma > 0$. Then the differential equation (8.10a) takes the form $y''(x) - \gamma^2 y(x) = 0$ whose general solution is[7]

$$y(x) = a\cosh\gamma x + b\sinh\gamma x.$$

Applying the boundary condition (8.10b), leads to $a = 0$, leaving us with

$$y(x) = b\sinh\gamma x. \tag{8.11}$$

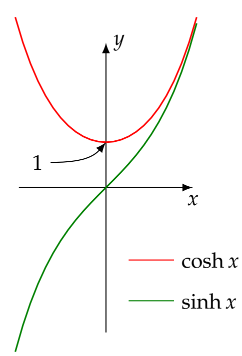
Then the boundary condition (8.10c) implies that $b\sinh\gamma\ell = 0$. We don't want to take $b = 0$ because then (8.11) would reduce to the trivial solution, so we take $\sinh\gamma\ell = 0$ which implies that $\gamma\ell = 0$ (see the comment at the bottom of the marginal note). But that's impossible since both γ and ℓ are positive. Having thus reached a dead end, we abandon the $\lambda < 0$ case.

Case 2: $\lambda = 0$. The differential equation (8.10a) takes the form $y''(x) = 0$ whose general solution is $y(x) = ax + b$. Applying the boundary condition (8.10b) implies that $b = 0$, and therefore $y(x) = ax$. Then the boundary condition (8.10c) implies that $a\ell = 0$. We are forced to take $a = 0$ and thus arrive at the trivial solution again. So we abandon the $\lambda = 0$ case.

[7] In your ordinary differential equations course you must have learned that the general solution of the ODE $y'' - \gamma^2 y = 0$ is $y(x) = ae^{\gamma x} + be^{-\gamma x}$, which is also correct. But since a and b are arbitrary, we may replace them with $(a+b)/2$ and $(a-b)/2$, and obtain

$$\begin{aligned} y(x) &= \frac{a+b}{2}e^{\gamma x} + \frac{a-b}{2}e^{-\gamma x} \\ &= a\frac{e^{\gamma x}+e^{-\gamma x}}{2} + b\frac{e^{\gamma x}-e^{-\gamma x}}{2} \\ &= a\cosh\gamma x + b\sin\gamma x. \end{aligned}$$

This alternative form is more suitable in applications to BVPs. Here is what the graphs of cosh and sinh look like.



Important observations: (a) The cosh function is nowhere zero; (b) $\cosh(0) = 1$; and (c) if $\sinh x = 0$, then $x = 0$. Be sure to make a mental note of these facts. You will need them!

Case 3: $\lambda > 0$. To enforce the positivity of λ, take $\lambda = \gamma^2$, where $\gamma > 0$. Then the differential equation (8.10a) takes the form $y''(x) + \gamma^2 y(x) = 0$ whose general solution is

$$y(x) = a\cos\gamma x + b\sin\gamma x.$$

Applying the boundary condition (8.10b), leads to $a = 0$, leaving us with

$$y(x) = b\sin\gamma x. \tag{8.12}$$

Then the boundary condition (8.10c) implies that $b\sin\gamma\ell = 0$. Since b cannot be zero—otherwise we will have a trivial solution—we are left with $\sin\gamma\ell = 0$ which is possible if $\gamma\ell$ is any (nonzero) integer multiple of π. Thus we arrive at infinitely many choices for γ:

$$\gamma_n = \frac{n\pi}{\ell}, \quad n = 1, 2, \ldots, \tag{8.13}$$

and since $\lambda = \gamma^2$, we have infinitely many choices for λ:

$$\lambda_n = \Big(\frac{n\pi}{\ell}\Big)^2, \quad n = 1, 2, \ldots. \tag{8.14}$$

Then (8.12) yields the solution $y_n(x) = b\sin\gamma_n x = b\sin\frac{n\pi x}{\ell}$ for each n. The factor b is not essential for our purposes. After all, the equations in (8.10) are homogeneous, and therefore *any nonzero multiple of a nontrivial solution is also a nontrivial solution*. We set $b = 1$ to identify one member of the infinite family of solutions for each n:

$$y_n(x) = \sin\gamma_n x = \sin\frac{n\pi x}{\ell}, \quad n = 1, 2, \ldots. \tag{8.15}$$

Definition 8.2. Each of the numbers λ_n, $n = 1, 2, \ldots$, obtained in (8.14) is called an *eigenvalue* of the BVP (8.10). The function $y_n(x)$ in (8.15) is called the *eigenfunction* corresponding to the eigenvalue λ_n.

Remark 8.2. The eigenfunctions y_n are identical to the functions ϕ_n that were introduced in Example 8.3 on page 107. It was shown there that the functions ϕ_n are mutually orthogonal in the sense of $L^2(0, \ell)$. It follows that the eigenfunctions y_n are mutually orthogonal in the same sense, that is, for all positive integers m and n, we have

$$(y_m, y_n) = \int_0^\ell y_m(x) y_n(x)\, dx = \begin{cases} \frac{\ell}{2} & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases} \tag{8.16}$$

8.5 *A second look at positiveness and orthogonality*

The previous section's calculations led to at least two noteworthy results: First, the eigenvalues are the BVP (8.10a) are *positive*; and second, its eigenfunctions are *mutually orthogonal*. We arrived at those results through somewhat tedious calculations involving the explicit solutions of the ODE (8.10a). In more complicated cases, for instance, when the ODE (8.10a) is replaced with one with non-constant coefficients, or when the boundary conditions (8.10b) and (8.10c) are replaced with more complicated ones,[8] explicit solutions in terms of elementary function may no longer be available, but the positivity of the the eigenvalues and orthogonality of the eigenfunctions persist. Therefore, there ought to be a way of verifying those properties *without appealing to explicit solutions*. Indeed there is a way. In this section we do that in the context of the very elementary eigenvalue problem (8.10a). In later chapters we will see how it is done in more complicated problems.

[8] These are not complications merely for the sake of complications; these arise naturally in many interesting applications of PDEs, as we will see in Chapters **??** and **??**.

The main trick behind the following calculations is what is called *integration by parts* in calculus courses, and which is nothing but an application of the product rule of differentiation. Specifically, consider any two continuously differentiable functions u and v. According to the product rule of differentiation, we have $\big[u(x)v(x)\big]' = u'(x)v(x) + u(x)v'(x)$. We rearrange that into $u'(x)v(x) = \big[u(x)v(x)\big]' - u(x)v'(x)$, and integrate:

$$\int u'(x)v(x)\,dx = \int \big[u(x)v(x)\big]'\,dx - \int u(x)v'(x)\,dx.$$

But the integral of $\big[u(x)v(x)\big]'$ is just $u(x)v(x)$. We conclude that

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx. \tag{8.17}$$

That's the integration by parts formula! We see that on transforming the integral on the left to the integral on the right, the prime shifts from $u(x)$ to $v(x)$, and the sign of the integral gets reversed. The $u(x)v(x)$ is the "penalty" we pay for making that transformation.

A similar calculation may be made in the case of a definite integral over an interval $[a, b]$:

$$\begin{aligned}\int_a^b u'(x)v(x)\,dx &= \int_a^b \big(u(x)v(x)\big)'\,dx - \int_a^b u(x)v'(x)\,dx \\ &= u(x)v(x)\Big|_a^b\,dx - \int_a^b u(x)v'(x)\,dx.\end{aligned}$$

That evaluates to

$$\int_a^b u'(x)v(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u(x)v'(x)\,dx. \tag{8.18}$$

We leave the verification of a closely related variant to an exercise:

$$\int_a^b u''(x)v(x)\,dx = u'(b)v(b) - u'(a)v(a) - \int_a^b u'(x)v'(x)\,dx. \tag{8.19}$$

8.5.1 *The positivity of the eigenvalues*

Equipped with these tools, let us show that the eigenvalues λ of the BVP (8.10a) are positive. Toward that end, multiply (8.10a) by $y(x)$, integrate over $[0, \ell]$,

$$\int_0^\ell y''(x)y(x)\,dx + \lambda \int_0^\ell y(x)^2\,dx = 0,$$

and then apply the integration by parts formula (8.19) to the first term on the left-hand side to get

$$y'(\ell)y(\ell) - y'(0)y(0) - \int_0^\ell y'(x)^2\,dx + \lambda \int_0^\ell y(x)^2\,dx = 0.$$

We have $y(0) = 0$ and $y(\ell) = 0$ due to the boundary conditions (8.10b) and (8.10c). That leaves us with

$$\lambda \int_0^\ell y(x)^2\,dx = \int_0^\ell y'(x)^2\,dx.$$

It follows that λ cannot be negative since the integrands are nonnegative. But can it be zero? No, it cannot be. To see that, we proceed with an *argument by contradiction*. Thus, suppose for the sake of contradiction, that λ is zero. Then the equation above says that $\int_0^\ell y'(x)^2\,dx = 0$ which implies that $y'(x)$ is identically zero,[9] which implies that $y'(x)$ is a constant, and that constant is zero since $y(0) = 0$. It follows that $y(x)$ is identically zero on $[0, \ell]$. That contradicts the fact that y is an eigenfunction—an eigenfunction cannot be identically zero by definition. We conclude that $\lambda > 0$. That agrees with the outcome of the analysis of the three cases in Section 8.4.

[9] If $y'(x)$ is nonzero at some x, then $y'(x)^2$ would be positive at that x, and therefore the integral $\int_0^\ell y'(x)^2\,dx$ would be positive.

Important! The argument that was just presented is central to the analysis of ODEs and PDEs and should be taken to heart. This approach, in contrast to that of Section 8.4, does not appeal to the explicit solution of the differential equation, and therefore it is applicable in more complex cases where no explicit solution is available.

8.5.2 *The orthogonality of the eigenfunctions*

In Remark 8.2 we noted that the eigenfunctions y_n of the BVP (8.10c) are mutually orthogonal in $L^2(0, \ell)$; see equation (8.16). That was accomplished through a tedious calculation by appealing to the explicit form $\sin \frac{n\pi x}{x}$ of the eigenfunctions—see the calculations leading to the displayed expression (8.8) on page 107. Our purpose in this subsection is to show that the same conclusions may be reached more quickly and without a knowledge of explicit form of the eigenfunctions. Here is how.

Consider eigenfunctions y_m and y_n, and the corresponding eigenvalues λ_m and λ_n of the BVP (8.10), that is,

$$\begin{aligned} y_m'' + \lambda_m y_m &= 0 & \qquad y_n'' + \lambda_n y_n &= 0, \\ y_m(0) &= 0 & y_n(0) &= 0, \\ y_m(\ell) &= 0 & y_n(\ell) &= 0. \end{aligned}$$

Multiplying the first ODE by y_n and the second by y_m, and subtracting, we get

$$y_m'' y_n - y_n'' y_m + (\lambda_m - \lambda_n) y_m y_n = 0,$$

which simplifies to[10]

$$\big(y_m' y_n - y_n' y_m\big)' + (\lambda_m - \lambda_n) y_m y_n = 0.$$

[10] Expand $(y_m' y_n - y_n' y_m)'$ and verify that it indeed evaluates to $y_m'' y_n - y_n'' y_m$.

Integrating the result over $[0, \ell]$, we obtain

$$\big(y_m' y_n - y_n' y_m\big)\Big|_0^\ell + (\lambda_m - \lambda_n) \int_0^\ell y_m y_n \, dx = 0.$$

But y_m and y_n are zero at $x = 0$ and $x = \ell$, so the first terms drops out and we are left with

$$(\lambda_m - \lambda_n) \int_0^\ell y_m(x) y_n(x) \, dx = 0.$$

It follows that if $\lambda_m - \lambda_n \neq 0$, then

$$(y_m, y_n) = \int_0^\ell y_m(x) y_n(x) \, dx = 0,$$

which confirms that the eigenfunctions corresponding to distinct eigenvalues are mutually orthogonal. It is worth stressing again that we did not appeal to the explicit form $\sin \frac{n\pi x}{\ell}$ of the eigenfunctions to reach that conclusion. We do need that explicit form, however, in order to evaluate the *norm* of the eigenfunctions

$$\|y_n\|^2 = \frac{\ell}{2},$$

as was done in Example 8.3.

8.6 *The significance of orthogonality*

Let us return to the Fourier's idea of expanding an arbitrary function f into a sine series (8.9). In light of what we learned in the preceding two sections, that series is actually the expansion of the function f into a series of the eigenfunctions y_n of the BVP (8.10), that is

$$f(x) = \sum_{n=1}^\infty b_n \sin \frac{n\pi x}{\ell} = \sum_{n=1}^\infty b_n y_n(x), \tag{8.20}$$

where y_n is defined in (8.15). The algebra that led from the representation (8.3) to the the coefficients in (8.4) in $\mathbf{R}^n$ carries over to this infinite-dimensional setting without change. To find the coefficients b_n, we form the inner product of (8.20) with y_m for some integer m:

$$(f, y_m) = \sum_{n=1}^{\infty} b_n (y_n, y_m),$$

and observe that according to (8.16), the inner product (y_n, y_m) is zero for all n except for when $n = m$. Thus, as the summation index n runs form 1 to infinity, only the mth term survives and that leaves us with

$$(f, y_m) = b_m (y_m, y_m).$$

Referring to (8.16) again, we see that $(y_m, y_m) = \ell/2$. We conclude, after renaming m to n, that

$$b_n = \frac{2}{\ell}(f, y_n) = \frac{2}{\ell}\int_0^\ell f(x)\sin\gamma_n x\,dx = \frac{2}{\ell}\int_0^\ell f(x)\sin\frac{n\pi x}{\ell}\,dx. \quad (8.21)$$

In summary, the coefficients b_n of the eigenfunction expansion (8.20) are given by (8.21).

If you see clearly the logic that takes us from (8.20) to (8.21), then you have mastered the central idea of the Fourier series theory.

8.7 Examples

Example 8.4. Find the Fourier sine series (8.20) of $f(x) = x(1-x)$ over the interval $0 \le x \le 1$.

Solution. We have $\ell = 1$ and therefore $\gamma_n = n\pi$. According to (8.21), calculating the coefficients b_n calls for evaluating the integral[11]

$$\begin{aligned}\int x(1-x)\sin\gamma_n x\,dx = \big(x(1-x)\big)\Big(-\frac{1}{\gamma_n}\cos\gamma_n x\Big) \\ -(1-2x)\Big(-\frac{1}{\gamma_n^2}\sin\gamma_n x\Big) \\ +(-2)\Big(\frac{1}{\gamma_n^3}\cos\gamma_n x\Big).\end{aligned}$$

[11] Here we apply Kronecker's method (see Appendix 8.9) to evaluate the integral but you may do it with any other integration method that you are comfortable with.

Evaluating the resulting expression at $x = 0$ yields $-\frac{2}{\gamma_n^3}$. Evaluating it at $x = 1$ yields $-\frac{\sin\gamma_n}{\gamma_n^2} - \frac{2\cos\gamma_n}{\gamma_n^3}$. Therefore

$$\begin{aligned}b_n = 2\int_0^1 x(1-x)\sin nx\,dx &= 2\Big[-\frac{\sin\gamma_n}{\gamma_n^2} - \frac{2\cos\gamma_n}{\gamma_n^3} + \frac{2}{\gamma_n^3}\Big] \\ &= \frac{2}{\gamma_n^3}\Big[2 - 2\cos\gamma_n - \gamma_n\sin\gamma_n\Big].\end{aligned}$$

Since $\gamma_n = n\pi$, we have $\sin\gamma_n = 0$ and $\cos\gamma_n = (-1)^n$. Thus, we arrive at

$$b_n = \frac{4}{\pi^3}\frac{1-(-1)^n}{n^3},$$

and conclude that

$$x(1-x) = \frac{4}{\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\sin n\pi x.$$

The result simplifies somewhat by noting that $1-(-1)^n$ is zero when n is even and is 2 when n is odd:

$$x(1-x) = \frac{8}{\pi^3}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty}\frac{1}{n^3}\sin n\pi x.$$

Since every odd integer is of the form $2n-1$, then we may replace n by $2n-1$ and drop the "odd" qualifier:

$$\begin{aligned} x(1-x) &= \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\big[(2n-1)\pi x\big] \\ &= \frac{8}{\pi^3}\left[\sin\pi x + \frac{1}{3^3}\sin 3\pi x + \frac{1}{5^3}\sin 5\pi x + \cdots\right]. \end{aligned}$$

□

Example 8.5. Calculate the Fourier sine series of $f(x) = x$ over the interval $0 \le x \le \pi$.

Solution. We have $\ell = \pi$ and therefore $\gamma_n = n$. To calculate the coefficients of the series according to (8.21) calls for evaluating the integral[12]

$$\begin{aligned} \int x\sin nx\,dx &= (x)\Big(-\frac{1}{n}\cos nx\Big) - (1)\Big(-\frac{1}{n^2}\sin nx\Big) \\ &= -\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx. \end{aligned}$$

[12] Here we apply Kronecker's method (see Appendix 8.9) to evaluate the integral but you may do it with any other integration method that you are comfortable with.

Therefore

$$\begin{aligned} b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx &= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]\Bigg|_0^{\pi} \\ &= -2\frac{\cos n\pi}{n} = -2\frac{(-1)^n}{n}. \end{aligned}$$

We conclude that

$$\begin{aligned} x &= -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx \\ &= -2\left[-\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \cdots\right]. \end{aligned}$$

□

8.8 Convergence

The convergence of a series is defined through the convergence the sequence of its partial sums. Specifically, referring to (8.9), let's write $f_N(x)$ for its Nth partial sum, that is

$$f_N(x) = \sum_{n=1}^{N} b_n \sin \frac{n\pi x}{\ell},$$

By saying that the series converges to $f(x)$ we really mean to say that

$$f_N(x) \to f(x) \quad \text{as } N \to \infty. \tag{8.22}$$

That statement, however, is ambiguous. Its intention is to say that the function $f_N(x)$ is "getting close" to the function $f(x)$ as $N \to \infty$. We know what "getting close" means when talking about numbers—their difference gets smaller—but what does "getting close" mean when talking about functions?

The norm $\|f\|$ of a function $f \in L^2(a,b)$ is an indication of the "size" of the function. If the the function is identically zero, then the norm, defined in (**??**), is exactly zero. If the function is close to zero throughout, then $\|f\|$ is small. Thus, one way of making sense of (8.22) is to say

$$\|f_N - f\| \to 0 \quad \text{as } N \to \infty. \tag{8.23}$$

Yet another way of interpreting (8.22) is as a *pointwise convergence* which says that the value of $f_N(x)$ approaches to the value of $f(x)$ at every x in the interval (a,b), as N gets large. There is no norm associated with pointwise convergence of functions.

The two definitions of convergence above are not quite compatible with each other. The sequence of functions f_N may converge to f is one sense but not in another. That's why we referred to (8.22) as "ambiguous"—it says that the left-hand side approaches the right-hand side but does not say in which sense. In what follows, we state, without proofs, two rigorous convergence results for the Fourier sine series.

Theorem 8.1. *Suppose $f \in L^2(0,\ell)$. Then the Fourier sine series* (8.9) *converges to f in the $L^2(0,\ell)$, that is,*

$$\|f_N - f\| \to 0 \quad \textit{as } N \to \infty.$$

You will find the proof in books dedicated to Fourier series, such as [2], or Hilbert space theory, such as [16].

The pointwise convergence is more delicate and more complex to state, although historically it the concept was developed decades before that of the L^2 convergence.

Definition 8.3. A function is said to have a *jump discontinuity* at a point $x = x_0$ in its domain if its left and right limits exist at x_0 but are unequal. A function is said to be *piecewise smooth* if it is continuous and has a continuous derivative everywhere with the exception of a finite number of points where the function or its derivative may undergo jump discontinuities.

Figure 8.1 shows the graphs of a few piecewise smooth functions.

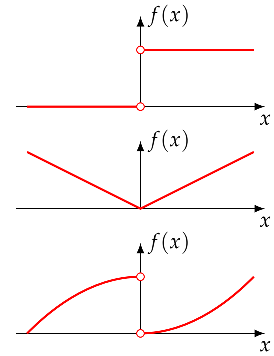


Figure 8.1: Samples of piecewise smooth functions.

Theorem 8.2. *Suppose the function* $f : [0, \ell] \to \mathbf{R}$ *is* piecewise smooth *Then the Fourier series* (8.9) *converges pointwise to* $f(x)$ *at every* $x \in (0, \ell)$ *where* f *is continuous, while at a point of* f*'s jump discontinuity, it converges to the average* $\frac{1}{2}\big[f(x^+) + f(x^-)\big]$ *of the left and right limits of* f *at the jump.*

Remark 8.3. Note that Theorem 8.2 applies to the *interior points* of the interval $[0, \ell]$. At the endpoints the series adds up to zero because the the functions ϕ_n are zero at the endpoints. Consequently, the equality in (8.9) holds at an endpoint if and only if f is zero there.

8.9 Appendix: Integration by parts à la Kronecker

The following approach to integrating by parts is attributed to Kronecker in [2].

Lemma 8.1. *Let* $p(x)$ *be a polynomial of degree* m*, and suppose that* $f(x)$ *is continuous. Then*

$$\int p(x)f(x)\,dx = p(x)F_1(x) - p'(x)F_2(x) + p''(x)F_3(x) - \cdots (-1)^m p^{(m)}(x)F_{m+1}(x), \quad (8.24)$$

where $p^{(m)}$ *is the mth derivative of* p*,* F_1 *is the antiderivative of* f*, and each subsequent* F_j *is the antiderivative of* F_{j-1}*. The signs of the terms on the right-hand side alternate between plus and minus.*

The successive terms in the formula are obtained by repeatedly differentiating p and integrating f, and adding them with alternating signs. Here is an illustration:

$$\begin{aligned}\int & x^2 \cos bx\,dx \\ &= (x^2)\Big(\frac{1}{b}\sin bx\Big) - (2x)\Big(-\frac{1}{b^2}\cos bx\Big) + (2)\Big(-\frac{1}{b^3}\sin bx\Big) + C \\ &= \frac{x^2}{b}\sin bx + \frac{2x}{b^2}\cos bx - \frac{2}{b^3}\sin bx + C.\end{aligned}$$

8.10 Appendix: Trigonometric identities

Calculating with Fourier series often involves the application of one or another trigonometric identity. A host of identities relate the sine

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{8.25a}$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b \tag{8.25b}$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b \tag{8.25c}$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b \tag{8.25d}$$
$$\cos a \cos b = \frac{1}{2}\Big[\cos(a-b) + \cos(a+b)\Big] \tag{8.25e}$$
$$\sin a \sin b = \frac{1}{2}\Big[\cos(a-b) - \cos(a+b)\Big] \tag{8.25f}$$
$$\sin a \cos b = \frac{1}{2}\Big[\sin(a+b) + \sin(a-b)\Big] \tag{8.25g}$$
$$\cos^2 a + \sin^2 a = 1 \tag{8.25h}$$
$$cos2a = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a \tag{8.25i}$$
$$\cos^2 a = \frac{1}{2}(1 + \cos 2a) \tag{8.25j}$$
$$\sin^2 a = \frac{1}{2}(1 - \cos 2a) \tag{8.25k}$$

Table 8.1: A table of basic trigonometric identities. The entire table may be derived from (8.25a) through simple algebra with the knowledge that cosine is an even function, sine is an odd function, $\cos(\frac{\pi}{2} - a) = \sin a$, and $\sin(\frac{\pi}{2} - a) = \cos a$.

and cosine functions, some of which are listed in Table 8.1. The table's entries look quite varied but it turns out—as you should be able to verify—that the entire table may be derived through simple algebra from the first identity, (8.25a). Thus, if there is anything to memorize there, it is (8.25a).

8.11 Exercises

8.1. Show that a set of nonzero orthogonal vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ in $\mathbf{R}^n$ is linearly independent.

Hint: Show that $\sum_{i=1}^{n} c_i \boldsymbol{e}_i = \mathbf{0}$ implies that $c_i = 0$, $i = 1, 2, \dots, n$.

8.2. Show that $\|\boldsymbol{x} \pm \boldsymbol{y}\| \le \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n$. (Do you see why this is called the *triangle inequality*?)

8.3. Show that

$$\big|(f,g)\big| \le \|f\|\|g\| \quad \text{for all } f, g \in L^2(a,b).$$

This is known as the *Cauchy–Schwarz inequality*.

Hint: Adapt the proof of the inequality (8.5).

8.4. Adapt the solution of Exercise 8.2 to show that $\|f \pm g\| \le \|f\| + \|g\|$ for all f and g in $L^2(a,b)$. This is called the *triangle inequality* in $L^2(a,b)$.

Remark: This says that if f and g are square-integrable, then so is $f+g$. That is the main step toward showing that $L^2(a,b)$ is a linear space.

8.5. Consider the functions $f(x) = x + c$ and $g(x) = \sin x$. Determine the constant c so that f and g are orthogonal in $L^2(0,\pi)$.

8.6. Consider the functions $f(x) = 1$, $g(x) = x + a$, and $h(x) = x^2 + bx + c$. Find the constants a, b, and c so that the three functions are mutually orthogonal in $L^2(0,1)$, that is, $(f,g) = 0$, $(g,h) = 0$, $(h,f) = 0$.

8.7. Evaluate $\Phi_{m,n}$ for integers $m, n \in \{0,1,2,\dots\}$, where

$$\Phi_{m,n} = \int_0^\pi \cos mx \cos nx \, dx$$

8.8. Find the eigenvalues λ_n and eigenfunctions y_n of the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < \ell, \tag{8.26}$$
$$y(0) = 0, \tag{8.27}$$
$$y'(\ell) = 0, \tag{8.28}$$

and then follow the method of Section 8.5 to show that

$$(y_m, y_n) = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{1}{2}\ell & \text{if } m = n. \end{cases}$$

8.9. Find the eigenvalues λ_n and eigenfunctions y_n of the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < \ell, \tag{8.29}$$
$$y'(0) = 0, \tag{8.30}$$
$$y'(\ell) = 0, \tag{8.31}$$

and then follow the method of Section 8.5 to show that

$$(y_m, y_n) = \begin{cases} \ell & \text{if } m = n = 0, \\ \frac{1}{2}\ell & \text{if } m = n \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

9
Separation of variables

In this chapter we apply what we have learned about the Fourier series to solve the basic heat and wave equations on bounded spatial domains. The central idea is a technique called *separation of variables*. We illustrate the technique through examples.

9.1 Separation of variables in the heat equation

Consider the initial boundary value problem (IBVP) for heat equation for the unknown $u(x,t)$ on the interval $[0,\ell]$:

$$u_t = ku_{xx} \qquad 0 < x < \ell, \quad t > 0, \tag{9.1a}$$

$$u(0,t) = 0 \qquad t > 0, \tag{9.1b}$$

$$u(\ell,t) = 0 \qquad t > 0, \tag{9.1c}$$

$$u(x,0) = f(x) \qquad 0 < x < \ell. \tag{9.1d}$$

The crucial feature of problem (9.1) is the *homogeneity* of its first three equations, which implies that if u_1 and u_2 satisfy those three equations, then so does the linear combination $c_1u_1 + c_2u_2$ for arbitrary constants c_1 and c_2. That "superposition principle" may be extended by a straightforward induction argument to linear combinations of any number of functions. That is, if $u_1, u_2, \ldots, u_n$ satisfy the first three equations of (9.1), then so does $u = \sum_{j=1}^{n} c_j u_j$ with arbitrary constants c_j. We will see that there lies the crux of the application of the Fourier series to solving the initial boundary value problem (9.1).

The classical *separation of variables* method for solving the initial boundary value problem (9.1) begins with looking for a function of the form

$$u(x,t) = X(x)T(t) \tag{9.2}$$

that satisfies *the first three equations* of (9.1). We have

$$u_t(x,t) = X(x)T'(t), \quad u_{xx}(x,t) = X''(x)T(t),$$

where primes indicate derivatives. Therefore

$$X(x)T'(t) = kX''(x)T(t) \qquad 0 < x < \ell, \quad t > 0, \tag{9.3a}$$
$$X(0)T(t) = 0 \qquad t > 0, \tag{9.3b}$$
$$X(\ell)T(t) = 0 \qquad t > 0. \tag{9.3c}$$

We divide the equation (9.3a) through by $kX(x)T(t)$ [1] and arrive at

$$\frac{1}{k}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}. \tag{9.4}$$

[1] Including k in that divisor is optional but omitting it leads to a messier algebra.

Thus, we have *separated the variables* by getting the t-dependent quantities to one side and the x-dependent quantities to the other side of the equation. Next, we tiptoe through a delicate logical argument: The left-hand side of (9.4) has no x in it, so it cannot equal $X''(x)/X(x)$ unless $X''(x)/X(x)$ itself does not depend on x, that is, it is a constant, let's say $-\lambda$. It that case, the left-hand side also equals $-\lambda$, and thus we are led to the pair of equation

$$\frac{X''(x)}{X(x)} = -\lambda, \quad \frac{1}{k}\frac{T'(t)}{T(t)} = -\lambda,$$

or equivalently

$$X''(x) + \lambda X(x) = 0, \tag{9.5}$$
$$T'(t) + \lambda k T(t) = 0. \tag{9.6}$$

The constant λ is called the *separation constant* in this context.

The separation of variables trick reduces the analysis of the heat equation (9.1a), a PDE, to that of the pair of ODEs (9.5) and (9.6). To satisfy the boundary conditions (9.3b) and (9.3c), it suffices to take $X(0) = 0$ and $X(\ell) = 0$. This, along with (9.5) leads to the eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < \ell \tag{9.7a}$$
$$X(0) = 0, \tag{9.7b}$$
$$X(\ell) = 0, \tag{9.7c}$$

which is identical to the eigenvalue problem that we analyzed in Section 8.4, and where we found out that the problem has eigenvalues λ_n and eigenfunction X_n given by

$$\gamma_n = \frac{n\pi}{\ell}, \quad \lambda_n = \gamma_n^2 = \Big(\frac{n\pi}{\ell}\Big)^2, \quad X_n(x) = \sin\gamma_n x = \sin\frac{n\pi x}{\ell}, \qquad n = 1, 2, \dots. \tag{9.8}$$

Now that the possible values of the separation constant λ are determined, we solve the ODE (9.6) with $\lambda = \lambda_n$ and obtain

$$T_n(t) = e^{-k\lambda_n t}, \quad n = 1, 2, \dots,$$

At this point we have have constructed infinitely many functions $u_n(x,t) = X_n(x)T_n(t) = e^{-k\lambda_n t}\sin\gamma_n x$, $n = 1, 2, \ldots$, that satisfy the first three equations of (9.1). Due to the homogeneity of those equations and the superposition principle, any *finite* linear combination of the functions $u_n(x,t)$ also satisfy those three equations. In a leap of faith, we extend that statement to a linear combination of *infinitely many* such functions[2] and arrive at

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-k\lambda_n t}\sin\gamma_n x, \tag{9.9}$$

[2] The proof of the validity of the extension of the finite sum to infinitely many terms is a subject for a more advanced course on PDEs.

where γ_n and λ_n are given in (9.8).

The function u constructed in (9.9) satisfies the first three equations of (9.1). It remains to account for the remaining equation, that is, the initial condition $u(x,0) = f(x)$. So we evaluate (9.9) at $t = 0$ and equate the result to $f(x)$:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\gamma_n x.$$

This is exactly the Fourier sine series of the function f over $(0,\ell)$; see (8.20) on page 112. The coefficients b_n are given in (8.21) as

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\gamma_n x\,dx. \tag{9.10}$$

We conclude that the solution $u(x,t)$ of the IBVP (9.1) is that given in (9.9), with b_n given above, and where γ_n and λ_n are given in (9.8).

Example 9.1. Solve the IBVP

$$\begin{aligned}
&u_t = u_{xx} && 0 < x < 1, \quad t > 0,\\
&u(0,t) = 0, && t > 0,\\
&u(1,t) = 0, && t > 0,\\
&u(x,0) = 1 && 0 < x < 1.
\end{aligned}$$

Solution. This is a special case of problem (9.1) with $k = 1$, $\ell = 1$, and $f(x) \equiv 1$ for which we have obtained the solution (9.9). All there remains is to calculate the coefficients b_n. We have

$$b_n = 2\int_0^1 \sin\gamma_n x\,dx = -\frac{2}{\gamma_n}\Big[\cos\gamma_n x\Big]\Big|_0^1 = -\frac{2}{\gamma_n}\Big[\cos\gamma_n - 1\Big].$$

But $\gamma_n = n\pi$ in this case, and therefore $\cos\gamma_n = \cos n\pi = (-1)^n$. We conclude that

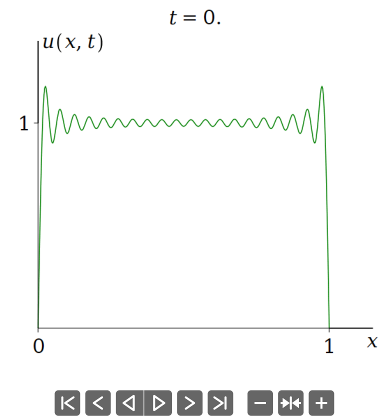
$$b_n = -\frac{2}{n\pi}\Big[(-1)^n - 1\Big].$$

The expression in the square brackets is zero when n is even and -2 when n is odd. Therefore all even-indexed terms in the sum (9.9) are absent, and the solution reduces to

$$u(x,t) = \frac{4}{\pi} \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n} e^{-n^2\pi^2 t} \sin n\pi x.$$

We remove the "odd" qualifier by replacing n by $2n-1$, and arrive at the final form of the solution:

$$u(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} e^{-(2n-1)^2\pi^2 t} \sin\big((2n-1)\pi x\big).$$



Here are the first few terms of that series:

$$u(x,t) = \frac{4}{\pi}\Big[e^{-\pi^2 t} \sin \pi x + \frac{1}{3} e^{-9\pi^2 t} \sin 3\pi x + \frac{1}{5} e^{-25\pi^2 t} \sin 5\pi x + \cdots\Big].$$

The animation on the right displays snapshots of the solution at a sequence of times over the range $0 \le t \le 1/2$. For the purpose of plotting, we have truncated the infinite sum of the solution to 20 terms. [This comments is out of sequence. Fix!] Note the prominent Gibbs phenomena at the interval's edges at $t = 0$ where the odd periodic extension of f is discontinuous. □

9.2 *Separation of variables in the wave equation*

Consider the initial boundary value problem (IBVP) for wave equation for the unknown $u(x,t)$ on the interval $[0,\ell]$:

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < \ell, \quad t > 0, \tag{9.11a}$$

$$u(0,t) = 0 \qquad t > 0, \tag{9.11b}$$

$$u(\ell,t) = 0 \qquad t > 0, \tag{9.11c}$$

$$u(x,0) = f(x) \qquad 0 < x < \ell, \tag{9.11d}$$

$$u_t(x,0) = g(x) \qquad 0 < x < \ell. \tag{9.11e}$$

Following the separation of variables idea introduced in Section 9.1, we look for a function of the form

$$u(x,t) = X(x)T(t) \tag{9.12}$$

that satisfies *the first three equations* of (9.11). We have

$$u_{tt}(x,t) = X(x)T''(x), \quad u_{xx}(x,t) = X''(x)T(t),$$

and therefore

$$X(x)T''(x) = c^2 X''(x)T(t) \qquad 0 < x < \ell, \quad t > 0, \tag{9.13a}$$

$$X(0)T(t) = 0 \qquad t > 0, \tag{9.13b}$$

$$X(\ell)T(t) = 0 \qquad t > 0. \tag{9.13c}$$

We divide the equation (9.13a) through by $c^2 X(x)T(t)$ and arrive at

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \tag{9.14}$$

where λ is the separation constant, and therefore

$$X''(x) + \lambda X(x) = 0, \tag{9.15}$$

$$T''(t) + \lambda c^2 T(t) = 0. \tag{9.16}$$

Thus, the separation of variables trick has reduced the analysis of the wave equation, a PDE, to that of a pair of ODEs. To satisfy the boundary conditions (9.13b) and (9.13c), it suffices to take $X(0) = 0$ and $X(\ell) = 0$. This, along with (9.15) leads to the eigenvalue problem

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \qquad 0 < x < \ell \\ X(0) &= 0, \\ X(\ell) &= 0, \end{aligned}$$

which is identical to the eigenvalue problem that we analyzed in Section 8.4—and which we also encountered in Section 9.1—and where we found out that the problem has eigenvalues λ_n and eigenfunction X_n given by

$$\gamma_n = \frac{n\pi}{\ell}, \quad \lambda_n = \gamma_n^2 = \Big(\frac{n\pi}{\ell}\Big)^2, \quad X_n(x) = \sin\gamma_n x = \sin\frac{n\pi x}{\ell}, \qquad n = 1, 2, \ldots. \tag{9.17}$$

Then we solve the ODE (9.16) with $\lambda = \lambda_n = \gamma_n^2$ and obtain

$$T_n(t) = \alpha_n \cos\gamma_n ct + \beta_n \sin\gamma_n ct \quad n = 1, 2, \ldots.$$

Thus, we have have constructed infinitely many functions $u_n(x,t) = X_n(x)T_n(t)$, $n = 1, 2, \ldots$, that satisfy the first three equations of (9.11). As in the previous section, we invoke the principle of superposition and arrive at the following candidate for the solution of our IBVP:

$$u(x,t) = \sum_{n=1}^{\infty} \big(\alpha_n \cos\gamma_n ct + \beta_n \sin\gamma_n ct\big)\sin\gamma_n x, \tag{9.18}$$

where γ_n and λ_n are given in (9.17).

To satisfy the initial condition (9.11e), we calculate u_t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \big(-\alpha_n\gamma_n c \sin\gamma_n ct + \beta_n\gamma_n c\cos\gamma_n ct\big)\sin\gamma_n x. \tag{9.19}$$

We evaluate (9.18) and (9.19) at $t = 0$ equate to f and g:

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\gamma_n x,$$

$$g(x) = \sum_{n=1}^{\infty} \beta_n\gamma_n c\sin\gamma_n x.$$

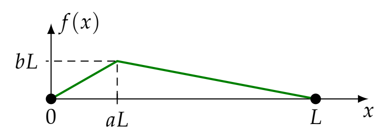
These are the Fourier sine series expansions of the function f and g over $(0, \ell)$; see (8.20) on page 112. The coefficients of those expansions are obtained from (8.21):

$$\alpha_n = \frac{2}{\ell} \int_0^\ell f(x) \sin \gamma_n x \, dx, \tag{9.20a}$$

$$\beta_n = \frac{2}{\ell c \gamma_n} \int_0^\ell g(x) \sin \gamma_n x \, dx. \tag{9.20b}$$

Example 9.2. Consider the transverse displacement $u(x,t)$ of a taut guitar string stretched between the points $x = 0$ and $x = \ell$ along the x axis. At time $t = 0$ the string is deformed (plucked) into some shape $u(x,0) = f(x)$ and released. Determine the string's motion for all $t > 0$ if

$$f(x) = \begin{cases} \frac{b}{a} x & 0 < x < a\ell, \\ \frac{b}{1-a}(\ell - x) & a\ell < x < \ell, \end{cases} \tag{9.21}$$



for some $0 < a < 1$.

Solution. The initial boundary value problem corresponding to the string's motion is

$$\begin{aligned} &u_{tt} = c^2 u_{xx} && 0 < x < \ell, \quad t > 0, \\ &u(0,t) = 0 && t > 0, \\ &u(\ell,t) = 0 && t > 0, \\ &u(x,0) = f(x) && 0 < x < \ell, \\ &u_t(x,0) = 0 && 0 < x < \ell. \end{aligned}$$

This a special case of (9.11) with $g = 0$ and a prescribed f. The solution is given in (9.18) and the coefficients $\alpha_n$and β_n are given in (9.20). We see that all β_n are zero since $g = 0$. All there remains to do is to evaluate the coefficients α_n. We have:

$$\begin{aligned} \alpha_n &= \frac{2}{\ell} \int_0^{a\ell} \left[\frac{b}{a} x \right] \sin \gamma_n x \, dx + \frac{2}{\ell} \int_{a\ell}^{\ell} \left[\frac{b}{1-a} (\ell - x) \right] \sin \gamma_n x \, dx \\ &= \frac{2b}{a\ell} \int_0^{a\ell} x \sin \gamma_n x \, dx + \frac{2b}{(1-a)\ell} \int_{a\ell}^{\ell} (\ell - x) \sin \gamma_n x \, dx. \end{aligned}$$

We calculate the two integrals separately. Let's call them A and B:[3]

[3] Kronecker's algorithm (Section 8.9 on page 116) comes quite handy for this calculation.

$$\begin{aligned} A &= \int_0^{a\ell} x \sin(\gamma_n x) \, dx = \frac{1}{\gamma_n^2} \Big[\sin(\gamma_n a\ell) - \gamma_n a\ell \cos(\gamma_n a\ell) \Big], \\ B &= \int_{a\ell}^{\ell} (\ell - x) \sin(\gamma_n x) \, dx = \frac{1}{\gamma_n^2} \Big[\sin(\gamma_n a\ell) + (1-a)\gamma_n \ell \cos(\gamma_n a\ell) - \sin(\gamma_n \ell) \Big]. \end{aligned}$$

But $\sin(\gamma_n \ell) = 0$ (do you see why?), and therefore we are left with

$$\alpha_n = \frac{2}{\ell} \left[\frac{b}{a} A + \frac{b}{1-a} B \right] = \frac{2b}{a(1-a)\gamma_n^2 \ell} \sin(\gamma_n a\ell).$$

Substituting this in (9.18) we arrive at

$$u(x,t) = \frac{2b}{a(1-a)\ell} \sum_{n=1}^{\infty} \frac{\sin(\gamma_n a \ell)}{\gamma_n^2} \cos(c\gamma_n t) \sin(\gamma_n x).$$

If you wish, you may substitute for γ_n from (9.17) and rewrite that result as

$$u(x,t) = \frac{2b\ell}{a(1-a)\pi^2} \sum_{n=1}^{\infty} \frac{\sin(a\pi n)}{n^2} \cos\frac{n\pi ct}{\ell} \sin\frac{n\pi x}{\ell}.$$

On the right, we have an animation of the solution where for the parameters we have taken

$$c = 1, \quad \ell = 1, \quad a = 1/4, \quad b = 1/5,$$

and have truncated the series solution to the first six nonzero terms. In the first frame you see that the apex of the plucked string is rounded while according to (9.21), it should be a sharp corner. That's due to the *very coarse approximation* resulting from truncating the infinite series to six terms.

Compare this with the very simple *exact solution* (6.5) of the same problem obtained in Section 6.2 (page 64) and the accompanying animation in Figure 6.4. The lesson learned? There may be more than one way of solving a problem and some solutions are better than others! □

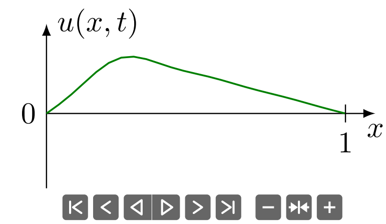


Figure 9.1: An animation of the solution $u(x,t)$ of Exercise 9.2.

9.3 Exercises

Solve the following IBVPs. You may freely refer to the conclusions of the analysis of the eigenvalue problem (8.10) (page 108) of Chapter 8, and the analysis of the IBVPs in the current chapter, where applicable. No need to re-derive those results.

9.1.
$$\begin{array}{ll} u_t = u_{xx} & 0 < x < 1, \quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u(1,t) = 0 & t > 0 \\ u(x,0) = \sin \pi x & 0 < x < 1 \end{array}$$

9.2.
$$\begin{array}{ll} u_t = u_{xx} & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin \dfrac{x}{2} & 0 < x < \pi \end{array}$$

9.3.
$$\begin{array}{ll} u_t = u_{xx} & 0 < x < \pi, \quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u(\pi,t) = 0 & t > 0 \\ u(x,0) = \cos \dfrac{x}{2} & 0 < x < \pi \end{array}$$

9.4.
$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(1,t) = 0 && t > 0 \\ &u(x,0) = \sin \pi x && 0 < x < 1 \\ &u_t(x,0) = 0 && 0 < x < 1 \end{aligned}$$

9.5.
$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(1,t) = 0 && t > 0 \\ &u(x,0) = 0 && 0 < x < 1 \\ &u_t(x,0) = \sin \pi x && 0 < x < 1 \end{aligned}$$

9.6.
$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(\pi,t) = 0 && t > 0 \\ &u(x,0) = \sin \frac{x}{2} && 0 < x < \pi \\ &u_t(x,0) = 0 && 0 < x < \pi \end{aligned}$$

9.7.
$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(\pi,t) = 0 && t > 0 \\ &u(x,0) = \cos \frac{x}{2} && 0 < x < \pi \\ &u_t(x,0) = 0 && 0 < x < \pi \end{aligned}$$

9.8.
$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(\pi,t) = 0 && t > 0 \\ &u(x,0) = 0 && 0 < x < \pi \\ &u_t(x,0) = \cos \frac{x}{2} && 0 < x < \pi \end{aligned}$$

9.9.
$$\begin{aligned} &u_{tt} = u_{xx} && 0 < x < \pi, \quad t > 0 \\ &u(0,t) = 0 && t > 0 \\ &u(\pi,t) = 0 && t > 0 \\ &u(x,0) = 0 && 0 < x < \pi \\ &u_t(x,0) = 1 - \frac{2}{\pi}\left|x - \frac{\pi}{2}\right| && 0 < x < \pi \end{aligned}$$

10

The Sturm–Liouville theory

In Chapter 8 we analyzed the eigenvalue problem (8.10), and in Chapter 9 we saw its applications to solving the heat and wave equations in homogeneous media. Those applications were limited to PDEs with boundary conditions of the type $u(0,t) = 0$, $u(\ell,t) = 0$ because our eigenvalue problem imposed the boundary conditions $y(0) = 0$, $y(\ell) = 0$. The Sturm–Liouville theory, due to Sturm [21, 22], and his collaboration with Liouville [23] introduces a less restrictive eigenvalue problem, enabling us to handle a greater variety of boundary conditions, and PDEs with variable coefficients.[1]

[1] Sturm's and Liouville's efforts may not have been directed toward supplementing and generalizing Fourier's theory, as the only mention of Fourier and the Fourier series occurs only as a passing remark in [22].

10.1 The Sturm–Liouville eigenvalue problem

The Sturm–Liouville theory is a study of the following boundary value problem of a second order ODE over an interval (a,b)

$$\frac{d}{dx}\Big(p(x)\frac{dy}{dx}\Big) + q(x)y = -\lambda w(x)y, \qquad a < x < b \tag{10.1a}$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0, \qquad \big[(\alpha_1,\alpha_2) \neq (0,0)\big] \tag{10.1b}$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0, \qquad \big[(\beta_1,\beta_2) \neq (0,0)\big] \tag{10.1c}$$

where the functions $p(x)$, $q(x)$, $w(x)$, and the coefficients α_1, α_2, β_1, β_2, are given, and the unknown function $y(x)$ and the constant λ are to be determined. The condition $(\alpha_1,\alpha_2) \neq (0,0)$ says that α_1 and α_2 cannot be both zero,[2] although one or the other may possibly be zero. The same goes for the condition $(\beta_1,\beta_2) \neq (0,0)$.

[2] If α_1 and α_2 are both zero, then (10.1b) reduces to $0 = 0$ and the boundary condition at $x = a$ is lost.

The identically zero function $y(x) \equiv 0$ certainly satisfies all three equations in (10.1) and is referred to as the *trivial solution*. Are there other, nontrivial, solutions? Yes, but only for certain special choices of the coefficient λ in (10.1a) and some restrictions on the functions $p(x)$, $q(x)$, and $w(x)$.

The statement, presented below, of the Sturm–Liouville's central theorem regarding the eigenvalue problem (10.1) is not difficult to comprehend since it exhibits close parallels to what we have leaned about the eigenvalue problem (8.10) in Chapter 8. We state it here as a formal theorem but we omit the rather technical proof. See Zettl's monograph [26] for a comprehensive survey of the subject. The following special case of that monograph's Theorem 4.3.1 amply suffices for our purposes.[3]

[3] Actually parts 5 and (6) are not present in Theorem 4.3.1, as they are implied by the general theory of self-adjoint operators in Hilbert space.

Theorem 10.1 (Sturm–Liouville). *Suppose that*

$$\begin{gathered} p(x) > 0, \quad w(x) > 0, \quad x \in [a,b], \\ \int \frac{1}{p(x)}\,dx < \infty, \quad \int_a^b |q(x)|\,dx < \infty, \quad \int_a^b w(x)\,dx < \infty. \end{gathered} \tag{10.2}$$

Then

1. *There exists an infinite sequence of special values*[4]

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \lambda_n < \cdots \to \infty \tag{10.3a}$$

of λ so that the BVP (10.1) *has a nontrivial solution, $y_n(x)$, whenever $\lambda = \lambda_n$.*

[4] The special values λ_n are called (10.1)'s *eigenvalues*, and the corresponding nontrivial solutions $y_n(x)$ are called the *eigenfunctions*. The strict inequalities in (10.3a) indicate that the eigenvalue problem (10.1) has no repeated eigenvalues.

2. *The eigenvalues λ_n grow proportional to n^2. Specifically,*

$$\lim_{n\to\infty} \frac{1}{n^2}\lambda_n = K^2, \quad \text{where } K = \frac{\pi}{\int_a^b \sqrt{\frac{w(x)}{p(x)}}\,dx}. \tag{10.3b}$$

3. *Each eigenfunction y_n has exactly n zeros in the open interval (a,b).*[5]

4. *The eigenfunctions are mutually orthogonal in the sense that*[6]

$$\int_a^b y_m(x) y_n(x) w(x)\,dx = 0 \quad \text{for all } m \neq n. \tag{10.3c}$$

[5] Thus, y_0 has *no zeros* in the open interval (a,b), y_1 has one zero, y_2 has two zeros, and so on.

[6] Note the modified inner product with $w(x)$ as a factor in the integrand. One says that the inner product is *weighted* by w. The weight in the eigenvalue problem studied in Chapter 8 was $w(x) \equiv 1$, and therefore it did not show up in the inner product.

5. *Any (not necessarily continuous) function $f(x)$ which is integrable in the sense that*

$$\int_a^b f(x)^2 w(x)\,dx < \infty, \tag{10.3d}$$

may be expressed as an infinite linear combination of the eigenfunctions:

$$f(x) = \sum_{n=0}^{\infty} c_n y_n(x). \tag{10.3e}$$

6. *The coefficients c_n in* (10.3e) *may be computed through*

$$c_n = \frac{1}{\|y_n\|^2} \int_a^b f(x) y_n(x) w(x)\,dx, \tag{10.3f}$$

where

$$\|y_n\|^2 = \int_a^b y_n(x)^2 w(x)\,dx. \tag{10.3g}$$

Remark 10.1. The meaning of the series representation (10.3e) is understood as follows. Let $f_N(x)$ be the Nth partial sum of the series, that is,

$$f_N(x) = \sum_{n=0}^{N} c_n y_n(x).$$

Then $f_N(x)$ converges to $f(x)$ as $N \to \infty$ in the sense that[7]

$$\lim_{N\to\infty} \int_a^b \big[f(x) - f_N(x)\big]^2 w(x)\, dx = 0.$$

[7] This statement can be formulated and stated much more transparently and succinctly in the setting of Hilbert space. Specifically, let $L^2([a,b],w)$ be the Hilbert space of measurable functions on $[a,b]$ equipped with the weighted inner product

$$(f,g) = \int_a^b f(x)g(x)w(x)\, dx.$$

Then the infinite series in (10.3e) converges to f in the Hilbert space $L^2([a,b],w)$.

Don't worry about this marginal note if you are unfamiliar with the Hilbert space terminology.

Remark 10.2. The set of equations (10.1) is homogeneous in y, therefore if y is any nontrivial solution, then so is cy for any nonzero constant multiplier c. This indicates that the eigenfunctions y_n of the Sturm–Liouville problem (10.1) are determined up to an arbitrary (nonzero) multiplicative constant. In particular, the multiplier may be selected so that $\|y_n\|^2 = 1$, whereby c_n in (10.3f) would take the simple form

$$c_n = \int_a^b f(x) y_n(x) w(x)\, dx.$$

That simplification, however, turns out to be illusory—it tends to lead to unnecessary algebraic manipulations—and therefore we won't pursue it in what follows.

Example 10.1. Find the eigenvalues and eigenfunctions of the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad 0 < x < \ell, \tag{10.4a}$$

$$y(0) = 0, \tag{10.4b}$$

$$y'(\ell) = 0. \tag{10.4c}$$

Note that this is a special case of the eigenvalue problem (10.1) with

$$a = 0, \quad b = \ell, \quad p(x) \equiv 1, \quad q(x) \equiv 0, \quad w(x) \equiv 1,$$
$$\alpha_1 = 1, \quad \alpha_2 = 0, \quad \beta_1 = 0, \quad \beta_2 = 1.$$

Solution. Following a procedure similar to that we employed in Section 8.4, we fork the analysis of the eigenvalue problem into three branches.

Case 1: $\lambda < 0$. To enforce the negativity of λ, take $\lambda = -\gamma^2$, where $\gamma > 0$. Then the differential equation (10.4a) takes the form $y''(x) - \gamma^2 y(x) = 0$ whose general solution is

$$y(x) = a \cosh \gamma x + b \sinh \gamma x.$$

Applying the boundary condition (10.4b), leads to $a = 0$, leaving us with

$$y(x) = b \sinh \gamma x, \tag{10.5}$$

and therefore $y'(x) = b\gamma \cosh \gamma x$. The boundary condition (10.4c) implies that $b\gamma \cosh \gamma \ell = 0$. We don't want to take $b = 0$ because then (10.5) would reduce to the trivial solution, and since $\gamma > 0$ by assumption and the cosh function is never zero (see the marginal note on page 108), we reach a dead end and abandon the $\lambda < 0$ case.

Case 2: $\lambda = 0$. The differential equation (10.4a) takes the form $y''(x) = 0$ whose general solution is $y(x) = ax + b$. Applying the boundary condition (10.4b) implies that $b = 0$, and therefore $y(x) = ax$. Then the boundary condition (10.4c) that $a\ell = 0$. We are forced to take $a = 0$ and thus arrive at the trivial solution again. So we abandon the $\lambda = 0$ case.

Case 3: $\lambda > 0$. To enforce the positivity of λ, take $\lambda = \gamma^2$, where $\gamma > 0$. Then the differential equation (10.4a) takes the form $y''(x) + \gamma^2 y(x) = 0$ whose general solution is

$$y(x) = a \cos \gamma x + b \sin \gamma x.$$

Applying the boundary condition (10.4b), leads to $a = 0$, leaving us with

$$y(x) = b \sin \gamma x, \tag{10.6}$$

and therefore $y'(x) = b\gamma \cos \gamma x$. The boundary condition (10.4c) implies that $b\gamma \cos \gamma \ell = 0$. Since b cannot be zero—otherwise we will have a trivial solution—and since $\gamma > 0$, we are left with $\cos \gamma \ell = 0$ which is possible only if $\gamma \ell$ is an odd multiple of $\pi/2$, as in $(2n-1)\frac{\pi}{2}$. Thus, we arrive at infinitely many choices for γ:

$$\gamma_n = \frac{(2n-1)\pi}{2\ell}, \quad n = 1, 2 \ldots, \tag{10.7}$$

and since $\lambda = \gamma^2$, we have infinitely many choices for λ:

$$\lambda_n = \left(\frac{(2n-1)\pi}{2\ell} \right)^2, \quad n = 1, 2 \ldots. \tag{10.8}$$

Finally, (10.6) yields an eigenfunction $y_n(x)$ corresponding to each eigenvalue λ_n:

$$y_n(x) = \sin \gamma_n x = \sin \frac{(2n-1)\pi x}{2\ell}, \quad n = 1, 2 \ldots. \tag{10.9}$$

For future reference, let us note that

$$\|y_n\|^2 = \int_0^1 y_n(x)^2 \, dx = \int_0^1 \sin^2 \frac{(2n-1)\pi x}{2\ell} = \frac{1}{2}\ell. \tag{10.10}$$

□

Example 10.2. Find the expansion of the function $f(x) = x, \;\; 0 < x < \ell$ into the series of eigenfunctions y_n calculated in (10.9).

Solution. According to (10.3e), the function f may be expressed as

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \gamma_n x = \sum_{n=1}^{\infty} c_n \sin \frac{(2n-1)\pi x}{2\ell}$$

where the coefficients c_n may be calculated according to (10.3f) and (10.10):

$$\begin{aligned} c_n &= \frac{1}{\|y_n\|^2} \int_0^\ell f(x) \sin \frac{(2n-1)\pi x}{2\ell}\, dx \\ &= \frac{2}{\ell} \int_0^\ell x \sin \frac{(2n-1)\pi x}{2}\, dx = -\frac{8\ell}{\pi^2} \cdot \frac{(-1)^n}{(2n-1)^2}, \end{aligned} \tag{10.11a}$$

and therefore

$$\begin{aligned} x &= -\frac{8\ell}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2\ell} \qquad (10.12) \\ &= \frac{8\ell}{\pi^2} \left[\sin \frac{\pi x}{2\ell} - \frac{1}{3^2} \sin \frac{3\pi x}{2\ell} + \frac{1}{5^2} \sin \frac{5\pi x}{2\ell} - \frac{1}{7^2} \sin \frac{7\pi x}{2\ell} + \cdots \right]. \square \end{aligned}$$

10.2 The signs of the eigenvalues

In solving Example 10.1 we followed the procedure established in Chapter 8 where we examined the three cases $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$, and concluded that $\lambda > 0$ was the only possibility. If you found that procedure tedious, you will be pleased to learn that there is a clever way of concluding $\lambda > 0$ with hardly any work at all. Here is how.

Let λ be an eigenvalue and $y(x)$ be the corresponding eigenfunction of the eigenvalue problem (10.4). Multiply the equation (10.4a) by $y(x)$ and integrate the result over the interval $[0, \ell]$:

$$\int_0^\ell y''(x)y(x)\, dx + \lambda \int_0^\ell y(x)^2\, dx = 0. \tag{10.13}$$

However, $y''(x)y(x) = \big(y'(x)y(x)\big)' - y'(x)^2$ (do you see why?) and therefore

$$\begin{aligned} \int_0^\ell y''(x)y(x)\, dx &= \int_0^\ell \big(y'(x)y(x)\big)'\, dx - \int_0^\ell y'(x)^2 \\ &= y'(x)y(x)\Big|_0^\ell - \int_0^\ell y'(x)^2 \\ &= y'(\ell)y(\ell) - y'(0)y(0) - \int_0^\ell y'(x)^2, \end{aligned}$$

which, in view of the boundary conditions (10.4b) and (10.4c), that is, $y(0) = 0$ and $y'(\ell) = 0$, reduces to

$$\int_0^\ell y''(x)y(x)\, dx = -\int_0^\ell y'(x)^2.$$

Substracting this from (10.13) we arrive at $\lambda \int_0^\ell y(x)^2\,dx = \int_0^\ell y'(x)^2$, that is[8]

$$\lambda = \frac{\int_0^\ell y'(x)^2}{\int_0^\ell y(x)^2\,dx}.$$

We see right away that λ cannot be negative since neither of the integrands is negative. But can λ be zero? No, because if it were so, then the numerator would be zero, which would imply that $y'(x)$ would be zero throughout the interval $(0, \ell)$ and consequently $y(x)$ would be a constant, and that constant would be zero since $y(0) = 0$. That is impossible since an eigenfunction cannot be identically zero by defintion. We conclude that $\lambda > 0$.

[8] An eigenfunction is cannot be identically zero by definition. Therefore dividing by $\int_0^\ell y(x)^2\,dx$ is harmless.

Important! The argument that was just presented is central to the analysis of ODEs and PDEs and should be taken to heart. This method, in contrast to what was done in the solution of Example (10.1), does not appeal to the explicit solution of the differential equation, and therefore it is applicable in more complex cases where no explicit solution is available.

Example 10.3. Show that the eigenvalues of the following problem are positive:

$$\Big(xy'(x)\Big)' + \lambda x y(x) = 0, \qquad 1 < x < 2, \tag{10.14a}$$

$$y'(1) = y(1), \tag{10.14b}$$

$$y(2) = 0 \tag{10.14c}$$

Solution. Multiply (10.14a) by $y(x)$ and integrate:

$$\int_1^2 \Big(xy'(x)\Big)' y(x)\,dx + \lambda \int_1^2 x y(x)^2\,dx = 0. \tag{10.15}$$

Observe that

$$\Big(xy'(x)\Big)' y(x) = \Big(xy'(x)y(x)\Big)' - xy'(x)^2,$$

and therefore

$$\begin{aligned}
\int_1^2 \Big(xy'(x)\Big)' y(x)\,dx &= \int_1^2 \Big(xy'(x)y(x)\Big)'\,dx - \int_1^2 xy'(x)^2 \\
&= xy'(x)y(x)\Big|_1^2 - \int_1^2 xy'(x)^2 \\
&= 2y'(2)y(2) - y'(1)y(1) - \int_1^2 xy'(x)^2 \\
&= -y(1)^2 - \int_1^2 xy'(x)^2,
\end{aligned}$$

where in the last step we have applied the boundary conditions (10.14b) and (10.14c). Subtracting this from (10.15) we arrive at

$$\lambda \int_1^2 x y(x)^2\,dx = y(1)^2 + \int_1^2 xy'(x)^2.$$

Then an argument along the lines presented earlier leads to the conclusion that $\lambda > 0$. □

10.3 *Eigenfunction expansion of solutions of PDEs*

According to part 5 of Theorem 10.1, any function f that satisfies the integrability condition (10.3d) may be expanded into a series of Sturm–Liouville eigenfunctions (10.3e), and the coefficients of the expansion may be calculated from (10.3f). That is a powerful result and it enables us to express the solutions of a variety of boundary value problems as series of eigenfunctions. A crucial property of that construction is that the sum of the series satisfies the same boundary conditions as those that are built into the Sturm–Liouville eigenvalue problem since each term of the series does that.

To illustrate, consider the heat equation with a heat source $q(x,t)$ and mixed boundary conditions

$$u_t = ku_{xx} + q(x,t) \qquad 0 < \ell < x, \quad t > 0 \tag{10.16a}$$

$$\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = 0 \qquad t > 0, \tag{10.16b}$$

$$\beta_1 u(\ell,t) + \beta_2 u_x(\ell,t) = 0 \qquad t > 0, \tag{10.16c}$$

$$u(x,0) = f(x) \qquad 0 < \ell < x. \tag{10.16d}$$

First, we look at the *homogeneous equations* consisting of the first three of (10.16) after setting $q = 0$:[9]

$$u_t = ku_{xx} \qquad 0 < \ell < x, \quad t > 0 \tag{10.17a}$$

$$\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = 0 \qquad t > 0, \tag{10.17b}$$

$$\beta_1 u(\ell,t) + \beta_2 u_x(\ell,t) = 0 \qquad t > 0. \tag{10.17c}$$

[9] This is reminiscent of the common technique of solving the second order ODE $ay'' + by' + cy = f(x)$, where we find the solution $y_h(x)$ of the homogeneous equation $ay'' + by' + cy = 0$ first, and then we adjust the result as $y(x) = y_h(x) + y_p(x)$, where $y_p(x)$ is a particular solution of the ODE.

Then, following the procedure that we learned in Chapter 9, we look for separable solutions $u(x,t) = X(x)T(t)$ of (10.17). Substituting $u(x,t) = X(x)T(t)$ into (10.17), we obtain

$$X(x)T'(t) = kX''(x)T(t),$$

$$\alpha_1 X(0)T(t) + \alpha_2 X'(0)T(t) = 0,$$

$$\beta_1 X(\ell)T(t) + \beta_2 X'(\ell)T(t) = 0,$$

From the first of these equations it follows that

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)} = -\lambda,$$

where λ is a constant, and thus we arrive at[10]

$$X''(x) + \lambda X(x) = 0, \tag{10.18a}$$

$$\alpha_1 X(0) + \alpha_2 X'(0) = 0 \tag{10.18b}$$

$$\beta_1 X(\ell) + \beta_2 X'(\ell) = 0, \tag{10.18c}$$

[10] The separation of variables also yields the ODE $T'(t) = kT(t)$, but we have no use for it now.

which is a special case of the Sturm–Liouville eigenvalue problem (10.1), and therefore, according to Theorem 10.1, there are infinitely many eigenvalues λ_n and eigenfunctions $X_n(x)$, $n = 0, 1, 2, \ldots$.[11]

We are now ready to tackle the original IBVP (10.16). We expand the solution $u(x,t)$, and the functions $q(x,t)$ and $f(x)$, into series of eigenfunctions, as in

$$u(x,t) = \sum_{n=0}^{\infty} \eta_n(t) X_n(x), \tag{10.19a}$$

$$q(x,t) = \sum_{n=0}^{\infty} \zeta_n(t) X_n(x), \tag{10.19b}$$

$$f(x) = \sum_{n=0}^{\infty} \phi_n X_n(x), \tag{10.19c}$$

where η_n and ζ_n depend on t because $u(x,t)$ and $q(x,t)$ depends on t.

The coefficients ζ_n and ϕ_n may be readily calculated according to (10.3f):[12]

$$\zeta_n(t) = \frac{1}{\|X_n\|^2} \int_0^{\ell} q(x,t) X_n(x)\, dx, \tag{10.20a}$$

$$\phi_n = \frac{1}{\|X_n\|^2} \int_0^{\ell} f(x) X_n(x)\, dx, \tag{10.20b}$$

where the norm $\|X_n\|^2$ is

$$\|X_n\|^2 = \int_0^{\ell} X_n(x)^2\, dx.$$

Observe that due to (10.18b) and (10.18c), the function $u(x,t)$ defined in (10.19a) satisfies the boundary conditions (10.16b) and (10.16c), so it remains to satisfy the PDE (10.16a) and the initial condition (10.16d).

To enforce the PDE (10.16a), we substitute (10.19a) into it and obtain:

$$\sum_{n=0}^{\infty} \eta_n'(t) X_n(x) = k \sum_{n=0}^{\infty} \eta_n(t) X_n''(x) + q(x,t).$$

Recalling (10.18a), we replace $X_n''(x)$ by $-\lambda_n X_n(x)$ and then rearrange the equation into

$$\sum_{n=0}^{\infty} \left[\eta_n'(t) + k\lambda_n \eta_n(t)\right] X_n(x) = q(x,t).$$

Comparing this with (10.19b) we conclude that

$$\eta_n'(t) + k\lambda_n \eta_n(t) = \zeta_n(t), \quad n = 0, 1, 2, \ldots, \tag{10.21a}$$

where $\zeta_n(t)$ is given in (10.20a).

To enforce the initial condition (10.16d), we substitute $t = 0$ in (10.19a) and equate the result to $f(x)$:

$$\sum_{n=0}^{\infty} \eta_n(0) X_n(x) = f(x).$$

[11] In this general treatment, the index n starts at zero for the sake of consistency with the statement of Theorem 10.1. In specific problems that we will solve later, the index will start at zero or one, as convenient.

[12] Note that $w(x) \equiv 1$ in this case.

Comparing this with (10.19c), we see that

$$\eta_n(0) = \phi_n, \quad n = 0, 1, 2, \ldots, \tag{10.21b}$$

where ϕ_n is given in (10.20b).

The first order linear ODE (10.21a) along with the initial condition (10.21b) may be solved to determine $\eta_n(t)$ for each n. Then (10.19a) completely determines the function $u(x,t)$ which in turn solves the IBVP (10.16).

Remark 10.3. The procedure outlined above for solving the heat equation applies equally well to the IBVP of the wave equation

$$u_{tt} = c^2 u_{xx} + q(x,t) \qquad 0 < x < \ell, \quad t > 0 \tag{10.22a}$$
$$\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = 0 \qquad t > 0, \tag{10.22b}$$
$$\beta_1 u(\ell,t) + \beta_2 u_x(\ell,t) = 0 \qquad t > 0, \tag{10.22c}$$
$$u(x,0) = f(x) \qquad 0 < x < \ell, \tag{10.22d}$$
$$u_t(x,0) = g(x) \qquad 0 < x < \ell. \tag{10.22e}$$

Separation of variables applied to the first three equations in (10.22) leads to eigenvalue problem (10.18)—same as that what we obtained for the heat equation. We expand u, q, f, and g into eigenfunctions series as in (10.19):

$$u(x,t) = \sum_{n=0}^{\infty} \eta_n(t) X_n(x), \tag{10.23a}$$
$$q(x,t) = \sum_{n=0}^{\infty} \zeta_n(t) X_n(x), \tag{10.23b}$$
$$f(x) = \sum_{n=0}^{\infty} \phi_n X_n(x), \tag{10.23c}$$
$$g(x) = \sum_{n=0}^{\infty} \psi_n X_n(x). \tag{10.23d}$$

Mimicking steps that lead to the equations (10.21a), leads us to the initial value problems

$$\eta_n''(t) + c^2 \gamma_n^2 \eta_n(t) = \zeta_n(t), \tag{10.24a}$$
$$\eta_n(0) = \phi_n, \tag{10.24b}$$
$$\eta_n'(0) = \psi_n, \tag{10.24c}$$

for $n = 1, 2, \ldots$. Upon solving these ODEs, we obtain the solution of our IBVP from (10.23a).

10.4 Examples

Let us apply the techniques described in the previous section to solving a few IBVPs.

Example 10.4. Consider heat conduction in a laterally insulated rod whose end at $x = \ell$ is also insulated while the end at $x = 0$ is maintained at zero temperature.[13]

$$u_t = ku_{xx} \qquad 0 < x < \ell, \quad t > 0, \tag{10.25a}$$
$$u(0,t) = 0 \qquad t > 0, \tag{10.25b}$$
$$u_x(\ell,t) = 0 \qquad t > 0, \tag{10.25c}$$
$$u(x,0) = f(x) \qquad 0 < x < \ell, \tag{10.25d}$$

[13] According to Fourier's law of heat conduction, the flux in the bar $-Ku_x$ (see (7.2) on page 80). Insulation prevents flux, and therefore $u_x = 0$ at an insulated end.

The eigenvalue problem (10.18) in this case takes the form

$$X''(x) + \lambda X(x) = 0 \qquad 0 < x < \ell, \tag{10.26a}$$
$$X(0) = 0, \tag{10.26b}$$
$$X'(\ell) = 0. \tag{10.26c}$$

This is identical to the eigenvalue problem analyzed in Example 10.1, and therefore we have eigenfunctions X_n given by

$$\gamma_n = \frac{(2n-1)\pi}{2\ell}, \tag{10.27a}$$
$$\lambda_n = \left(\frac{(2n-1)\pi}{2\ell}\right)^2, \tag{10.27b}$$
$$X_n(x) = \sin\gamma_n x = \sin\frac{(2n-1)\pi x}{2\ell}, \quad n = 1, 2 \ldots. \tag{10.27c}$$

We expand the solution u and the initial condition f into series of eigenfunctions as in (10.19a) and (10.19c). The coefficients ϕ_n may be immediately calculated:

$$\phi_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\gamma_n x\,dx.$$

Since the heat source term, $q(x,t)$, is absent in (10.25a), the coefficients $\zeta_n(t)$ calculated in (10.19b) are zero and therefore the initial value problem (10.21) reduces to

$$\eta_n'(t) + k\lambda_n\eta_n(t) = 0,$$
$$\eta_n(0) = \phi_n,$$

whose solution is[14]

$$\eta_n(t) = \phi_n e^{-k\lambda_n t}.$$

[14] The solution may be obtained through separation of variables or the integrating factor methods.

Then, according to (10.19b), the solution of our IBVP is

$$u(x,t) = \sum_{n=1}^{\infty}\phi_n e^{-k\lambda_n t}\sin\gamma_n x = \sum_{n=1}^{\infty}\phi_n e^{-k\lambda_n t}\sin\frac{(2n-1)\pi x}{2\ell}.$$

Example 10.5. We wish to model heat conduction in a laterally insulated rod which has an internal heat source (think electric current) that varies periodically in time, and where the rod's ends are maintained at zero temperature. A plausible IBVP would be

$$u_t = ku_{xx} + \sigma \sin \omega t \qquad 0 < x < \ell, \quad t > 0, \tag{10.28a}$$

$$u(0,t) = 0 \qquad t > 0, \tag{10.28b}$$

$$u(\ell,t) = 0 \qquad t > 0, \tag{10.28c}$$

$$u(x,0) = 0 \qquad 0 < x < \ell, \tag{10.28d}$$

where σ is a constant.

The eigenvalue problem (10.18) in this case takes the form

$$X''(x) + \lambda X(x) = 0 \qquad 0 < x < \ell, \tag{10.29a}$$

$$X(0) = 0, \tag{10.29b}$$

$$X(\ell) = 0. \tag{10.29c}$$

This is identical to problem (8.10) (page (8)) of Chapter 8 where we saw that the eigenvalues λ_n and the eigenfunctions X_n are given by

$$\gamma_n = \frac{n\pi}{\ell}, \tag{10.30a}$$

$$\lambda_n = \gamma_n^2 = \left(\frac{n\pi}{\ell}\right)^2, \tag{10.30b}$$

$$X_n(x) = \sin \gamma_n x = \sin \frac{n\pi x}{\ell}, \quad n = 1, 2 \ldots. \tag{10.30c}$$

We expand the solution u, the source term q, and the initial condition f into series of eigenfunction as in (10.19). Since the initial condition is zero in this case, the coefficients ϕ_n are all zero. We calculate the coefficients $\zeta_n(t)$:

$$\begin{aligned}
\zeta_n(t) &= \frac{2}{\ell} \int_0^\ell \sigma \sin \omega t \sin \gamma_n x \, dx = \frac{2\sigma \sin \omega t}{\ell} \int_0^\ell \sin \gamma_n x \, dx \\
&= \frac{2\sigma \sin \omega t}{\gamma_n \ell} \left(-\cos \gamma_n x\right)\Big|_0^\ell = \frac{2\sigma \sin \omega t}{\gamma_n \ell} \left[1 - \cos \gamma_n \ell\right] \\
&= Q_n \sin \omega t,
\end{aligned}$$

where, after substituting for γ_n from (10.30a), we have let

$$Q_n = \frac{2\sigma}{\pi} \cdot \frac{1 - (-1)^n}{n}. \tag{10.31}$$

Then, the initial value problem (10.21) takes the form

$$\eta_n'(t) + k\lambda_n \eta_n(t) = Q_n \sin \omega t, \tag{10.32a}$$

$$\eta_n(0) = 0, \tag{10.32b}$$

where $n = 1, 2, \ldots$. This may be solved through the Laplace transform, or by splitting the solution into the sum of homogeneous and particular solutions, or via the integrating factor method. Here is how it's done by the first two techniques. The third is left for an exercise.

Solution through the Laplace transform. Applying the Laplace transform to (10.32) and taking the initial condition into account, we obtain

$$\begin{aligned}\mathscr{L}\{\eta_n(t)\} &= \frac{\omega Q_n}{(s+k\lambda_n)(s^2+\omega^2)} \\ &= \frac{\omega Q_n}{\omega^2+k^2\lambda_n^2}\left[\frac{1}{s+k\lambda_n} - \frac{s-k\lambda_n}{s^2+\omega^2}\right], \quad \text{(partial fractions)}\end{aligned}$$

whence

$$\eta_n(t) = \frac{\omega Q_n}{\omega^2+k^2\lambda_n^2}\left[e^{-k\lambda_n t} - \cos\omega t + \frac{k\lambda_n}{\omega}\sin\omega t\right].$$

Solution through splitting into a sum of homogeneous and particular solutions. The homogenous equation corresponding to (10.32a) is $\eta_n'(t) + k\lambda_n\eta_n(t) = 0$. Its general solution is $\eta_n^{(h)}(t) = Ce^{-k\lambda t}$, where C is an arbitrary constant.[15] As to the particular solution of (10.32a), we appeal to the so-called *method of judicious guessing* to surmise that a particular solution should be something of the form $\eta_n^{(p)}(t) = A\cos\omega t + B\sin\omega t$. To determine A and B, we substitute this into (10.32a):

[15] The superscript "(h)" in $\eta_n^{(h)}(t)$ indicates that this is the solution of the *homogeneous* equation. Further down, the "(p)" in $\eta_n^{(p)}(t)$ indicates the *particular solution* of the nonhomogeneous ODE (10.32a).

$$\big[-A\omega\sin\omega t + B\omega\cos\omega t\big] + k\lambda_n\big[A\cos\omega t + B\sin\omega t\big] = Q_n\sin\omega t,$$

and collect the sine and cosine terms:

$$\big[k\lambda_n A + B\omega\big]\cos\omega t + \big[-A\omega + k\lambda_n B\big] = Q_n\sin\omega t,$$

which implies that

$$\begin{aligned} k\lambda_n A + B\omega &= 0 \\ -A\omega + k\lambda_n B &= Q_n. \end{aligned}$$

We solve this linear system for A and B and obtain

$$A = -\frac{\omega}{\omega^2+k^2\lambda_n^2}Q_n, \quad B = \frac{k\lambda_n}{\omega^2+k^2\lambda_n^2}Q_n,$$

and therefore

$$\eta_n^{(p)}(t) = -\frac{\omega}{\omega^2+k^2\lambda_n^2}Q_n\cos\omega t + \frac{k\lambda_n}{\omega^2+k^2\lambda_n^2}Q_n\sin\omega t.$$

We conclude that the general solution of the ODE (10.32a) is

$$\begin{aligned}\eta_n(t) &= \eta_n^{(h)}(t) + \eta_n^{(p)}(t) \\ &= Ce^{-k\lambda t} - \frac{\omega}{\omega^2+k^2\lambda_n^2}Q_n\cos\omega t + \frac{k\lambda_n}{\omega^2+k^2\lambda_n^2}Q_n\sin\omega t.\end{aligned}$$

We determine the constant C by setting $t = 0$ and applying the initial condition (10.32b):

$$0 = C - \frac{\omega}{\omega^2+k^2\lambda_n^2}Q_n,$$

whence $C = \frac{\omega}{\omega^2+k^2\lambda_n^2} Q_n$. Substituting this into the expression for $\eta_n^{(p)}(t)$ we arrive at

$$\eta_n(t) = \frac{\omega Q_n}{\omega^2 + k^2\lambda_n^2}\left[e^{-k\lambda_n t} - \cos\omega t + \frac{k\lambda_n}{\omega}\sin\omega t\right],$$

which is identical to what we obtained earlier through the Laplace transform.

Having thus obtaintaed $\eta_n(t)$, we are now in a position to write down the solution $u(x,t)$ of our IBVP. A minor simplification is achieved by noting that in view of (10.31), Q_n is zero for even n and $4\sigma/(n\pi)$ for odd n, and therefore (10.19a) yields

$$u(x,t) = \frac{4\sigma\omega}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n(\omega^2+k^2\lambda_n^2)}\left[e^{-k\lambda_n t} - \cos\omega t + \frac{k\lambda_n}{\omega}\sin\omega t\right]\sin\gamma_n x. \quad (10.33)$$

This is animated in Figure 10.1 with the parameter values of

$$\ell = 1, \quad \sigma = 1, \quad \omega = 1, \quad k = 1/40$$

and by summing the first 8 (odd-indexed) terms of the series.

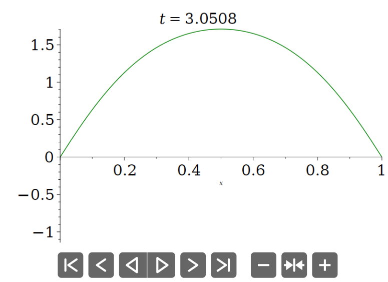


Figure 10.1: An animation of the solution (10.33) over the time range $0 < t < 36$.

Remark 10.4. The solution (10.33) may be rearranged into a more informative form by splitting it into three parts:

$$\begin{aligned} u(x,t) &= \frac{4\sigma\omega}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n(\omega^2+k^2\lambda_n^2)} e^{-k\lambda_n t}\sin\gamma_n x \\ &- \left[\frac{4\sigma\omega}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{1}{n(\omega^2+k^2\lambda_n^2)}\sin\gamma_n x\right]\cos\omega t \\ &+ \left[\frac{4\sigma k}{\pi}\sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} \frac{\lambda_n}{n(\omega^2+k^2\lambda_n^2)}\sin\gamma_n x\right]\sin\omega t. \end{aligned}$$

The first term on the right-hand side represents the system's *transient behavior* as the exponential factor dies out as t increases. The other two terms represent the system's *steady state oscillations* at the temporal angular frequency ω. Figure 10.2 depicts the the graph of $u(\frac{1}{2}, t)$, that is, the temperature at $x = 1/2$, corresponding to the animation in Figure 10.1. The transient is clearly visible. See if you can spot the transient in the animation.

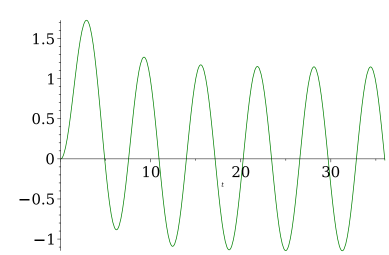


Figure 10.2: The graph of $u(1/2, t)$ of the solution (10.33).

Example 10.6. A taut homogeneous string occupies the horizontal interval $[0, \ell]$ in its rest state. The end at $x = 0$ is fixed (immobile) while the end at $x = \ell$ is attached to a small ring of negligible mass that can slide frictionlessly up and down a vertical pole. The string is given an initial transverse displacement $f(x)$ and released. Find the displacement $u(x,t)$ of the string at all $t > 0$.

Solution. The equation of motion[16]

[16] See the hint to Exercise 6.6s on page 73

$$\begin{aligned}
&u_{tt} = c^2 u_{xx} && 0 < x < \ell, \quad t > 0,\\
&u(0,t) = 0 && t > 0,\\
&u_x(\ell,t) = 0 && t > 0,\\
&u(x,0) = f(x), && 0 < x < \ell,\\
&u_t(x,0) = 0, && 0 < x < \ell,
\end{aligned}$$

is a special case of (10.22). The associated eigenvalue problem

$$\begin{aligned}
&X''(x) + \lambda X(x) = 0 && 0 < x < \ell,\\
&X(0) = 0,\\
&X'(\ell) = 0
\end{aligned}$$

is identical to that treated in Examples 10.1 and 10.4, whose eigenvalues and eigenfunctions are given in equations (10.7) through (10.10). In the eigenfunction expansions (10.23) the coefficients ζ_n and ψ_n are zero since q and g are zero, while the expansions (10.23a) and (10.23c) take the forms

$$u(x,t) = \sum_{n=1}^{\infty} \eta_n(t) \sin \gamma_n x, \tag{10.34a}$$

$$f(x) = \sum_{n=1}^{\infty} \phi_n \sin \gamma_n x. \tag{10.34b}$$

The coefficients ϕ_n may be immediately calculated:

$$\phi_n = \frac{2}{\ell} \int_0^{\ell} f(x) \sin \gamma_n x \, dx. \tag{10.35}$$

Then the initial value problems (10.24) take the form

$$\eta_n''(t) + c^2 \gamma_n^2 \eta_n(t) = 0, \tag{10.36a}$$

$$\eta_n(0) = \phi_n, \tag{10.36b}$$

$$\eta_n'(0) = 0, \tag{10.36c}$$

$n = 1, 2, \ldots$, which may be readily solved for $\eta_n(t)$:

$$\eta_n(t) = \phi_n \cos \gamma_n ct.$$

We conclude that

$$u(x,t) = \sum_{n=1}^{\infty} \phi_n \sin \gamma_n x \cos \gamma_n ct,$$

where ϕ_n are given in (10.35), and $\gamma_n = (2n-1)\frac{\pi}{2\ell}$.

□

10.5 *Exercises*

10.1. Verify that the eigenfunctions y_n of the Sturm–Liouville eigenvalue problem (10.1) are indeed orthogonal, as claimed in (10.3c).

Hint: Mimic the calculation of Section 8.5.

10.2. Verify that the eigenfunctions y_n calculated in (10.9) are indeed orthogonal, that is, they do satisfy (10.3c).

10.3. Verify that the eigenvalues λ_n of Example (10.1) satisfy the assertion (10.3b).

10.4. Show that the eigenvalues of the following problem are positive:

$$\begin{aligned} &y'' + \lambda y = 0, && 0 < x < \ell, \\ &y'(0) = y(0), \\ &y(\ell) = 0. \end{aligned}$$

10.5. Show that the eigenvalues of the following problem are positive:

$$\begin{aligned} &y'' + \lambda y = 0, && 0 < x < \ell, \\ &y'(0) = y(0), \\ &y'(\ell) = -y(\ell). \end{aligned}$$

10.6. In Chapter 7 we derived the equation of conduction of heat in an *nonhomogeneous* bar as equation (7.6) on page 82. Consider the following IBVP for such a bar:

$$\rho(x)c_p(x)\frac{\partial}{\partial t}u(x,t) = \frac{\partial}{\partial x}\Big(K(x)\frac{\partial}{\partial x}u(x,t)\Big) \qquad 0 < x < \ell, \quad t > 0, \tag{10.37a}$$

$$u(0,t) = 0 \qquad t > 0, \tag{10.37b}$$

$$u(\ell,t) = 0 \qquad t > 0, \tag{10.37c}$$

$$u(x,0) = f(x) \qquad 0 < x < \ell. \tag{10.37d}$$

Show that looking for a separable solution $u(x,t) = X(x)T(t)$ that satisfies the first three of those four equations leads to a Sturm–Liouville eigenvalue problem for X.

10.7. Referring to Remark 10.3, the initial value problem (10.24) is stated without much explanation. Provide the details.

10.8. Find the eigenvalues and eigenfunctions of the boundary value problem

$$y''(x) + \lambda y(x) = 0, \quad -\ell < x < \ell, \tag{10.38a}$$

$$y(-\ell) = 0, \tag{10.38b}$$

$$y'(\ell) = 0. \tag{10.38c}$$

10.9. Solve the heat conduction problem in a homogeneous rod which is insulated all around, including its ends, and whose initial temperature is $f(x)$, $0 \leq x \leq \ell$.

Hint: See Exercise 8.9.

10.10. Re-do Example 10.6 by taking the initial displacement f as zero but imposing an initial velocity $u_t(x, 0) = g(x)$.

10.11. A taut homogeneous steel wire occupies the horizontal interval $0 \leq x \leq \ell$ in its equilibrium state and is fixed at its ends. We subject the wire to a variable magnetic field that exerts a force of $q(x, t) = \sigma \sin \omega t$ per unit mass in the vertical direction. Find the wire's motion if it starts from rest at time $t = 0$. Is resonance possible?

10.12. In Section 10.4 the initial value problem (10.32) is solved in two different ways. Do it a third way through the method of integrating factors. Which of the three approaches do you prefer?

11

Nonhomogeneous boundary data

The eigenfunction expansion method depends crucially on the *homogeneity* of the boundary conditions (10.16b) and (10.16c), that is, the fact that their right-hand sides are zero. After all, each term $c_n(t)X_n(x)$ in the eigenfunction expansion (10.19a) satisfies those boundary conditions, and consequently their sum, that is $u(x,t)$, also satisfies them. That would not be the case if the boundary conditions were nonhomogeneous.

It turns out, as we are going to see now, that nonhomogeneous boundary conditions may be reduced to homogeneous ones through a change of variable. We begin with the simple case where the values of the solution are specified at the boundaries, and later we treat a more general case.

11.1 A simple case

To illustrate the idea, consider the IBVP for the heat equation in a rod where the (possibly time-dependent) temperature is specified at its ends:

$$u_t = ku_{xx} + q(x,t) \qquad 0 < x < \ell, \quad t > 0, \tag{11.1a}$$

$$u(0,t) = \alpha(t) \qquad t > 0, \tag{11.1b}$$

$$u(\ell,t) = \beta(t) \qquad t > 0, \tag{11.1c}$$

$$u(x,0) = f(x) \qquad 0 < x < \ell. \tag{11.1d}$$

Split the solution $u(x,t)$ of (11.1) into a sum

$$u(x,t) = v(x,t) + \eta(x,t) \tag{11.2}$$

where η is any (twice differentiable) function that satisfies the given boundary conditions

$$\eta(0,t) = \alpha(t), \quad \eta(\ell,t) = \beta(t), \quad t > 0. \tag{11.3}$$

There are infinitely many possible choices here; any η with those properties will do. For any such η, since u and η share identical boundary conditions, their difference, v, satisfies zero boundary conditions, that is,

$$v(0,t) = 0, \quad v(\ell,t) = 0, \quad t > 0.$$

Moreover, since u satisfies the PDE (11.1a) and the initial condition (11.1d), we have

$$\begin{aligned} &v_t + \eta_t = k(v_{xx} + \eta_{xx}) + q, \\ &v(x,0) + \eta(x,0) = f(x). \end{aligned}$$

In summary, v satisfies the following IBVP:

$$\begin{aligned} &v_t = kv_{xx} + q + k\eta_{xx} - \eta_t && 0 < x < \ell, \quad t > 0 \\ &v(0,t) = 0 && t > 0, \\ &v(\ell,t) = 0 && t > 0, \\ &v(x,0) = f(x) - \eta(x,0) && 0 < x < \ell. \end{aligned}$$

Letting

$$\tilde{q}(x,t) = q(x,t) + k\eta_{xx}(x,t) - \eta_t(x,t), \quad \tilde{f}(x) = f(x) - \eta(x,0), \tag{11.4}$$

this becomes

$$\begin{aligned} &v_t = kv_{xx} + \tilde{q}(x,t) && 0 < x < \ell, \quad t > 0 && (11.5a) \\ &v(0,t) = 0 && t > 0, && (11.5b) \\ &v(\ell,t) = 0 && t > 0, && (11.5c) \\ &v(x,0) = \tilde{f}(x) && 0 < x < \ell. && (11.5d) \end{aligned}$$

Comparing with the original initial boundary value problem (11.1), we see that the change of variable from u to v has had the effect of reducing the boundary conditions to zeros while modifying the source term q and the initial condition f. We may apply the methods that we learned in the previous chapters to solve (11.5), and then find u from (11.2).

Remark 11.1. As was noted above, there is quite a bit of flexibility in the choosing a function η that satisfies the requirements (11.3). Here are a couple of suggestions.

1. This choice[1]

$$\eta(x,t) = \Big(1 - \frac{x}{\ell}\Big)\alpha(t) + \frac{x}{\ell}\beta(t) \tag{11.6}$$

works in general, is simple and straightforward, but there can be better options in special cases, such as the one described next.

[1] Be sure to verify for yourself that $\eta(0,t) = \alpha(t)$, $\eta(\ell,t) = \beta(t)$.

2. In the special case where α and β are constants, and where the heat source q is independent of time, the IBVP (11.1) has a *steady-state* solution, $u^{(ss)}(x)$, which may be obtained by solving the boundary value problem for the second order ODE

$$\begin{aligned} &ku^{(ss)}_{xx} + q(x) = 0 && 0 < x < \ell, \\ &u^{(ss)}(0) = \alpha, \\ &u^{(ss)}(\ell) = \beta. \end{aligned}$$

Then $\eta(x,t) \equiv u^{(ss)}(x)$ satisfies the requirement (11.3). Furthermore, $\tilde{q}(x,t)$ evaluates to zero, and the IBVP (11.5) reduces to the remarkably simple

$$\begin{aligned} &v_t = kv_{xx} && 0 < x < \ell, \quad t > 0 \\ &v(0,t) = 0 && t > 0, \\ &v(\ell,t) = 0 && t > 0, \\ &v(x,0) = \tilde{f}(x) - u^{(ss)}(x) && 0 < x < \ell. \end{aligned}$$

Example 11.1. Solve the initial boundary value problem

$$\begin{aligned} &u_t = ku_{xx} && 0 < x < \ell, \quad t > 0, && (11.7a) \\ &u(0,t) = 0 && t > 0, && (11.7b) \\ &u(\ell,t) = \sigma \sin \omega t && t > 0, && (11.7c) \\ &u(x,0) = 0 && 0 < x < \ell. && (11.7d) \end{aligned}$$

Solution. This corresponds to problem (11.1) with

$$q(x,t) = 0, \quad \alpha(t) = 0, \quad \beta(t) = \sigma \sin \omega t, \quad f(x) = 0$$

We let $u = v + \eta$ as in (11.2), and pick η according to (11.6), that is,

$$\eta(x,t) = \frac{\sigma x}{\ell} \sin \omega t, \tag{11.8}$$

and calculate $\tilde{q}$ and $\tilde{f}$ according to (11.4):

$$\tilde{q}(x,t) = -\frac{\sigma \omega x}{\ell} \cos \omega t, \quad \tilde{f}(x) = 0.$$

Therefore, the initial boundary value problem (11.5) takes the form

$$\begin{aligned} &v_t = kv_{xx} - \frac{\sigma \omega x}{\ell} \cos \omega t, && 0 < x < \ell, \quad t > 0 && (11.9a) \\ &v(0,t) = 0, && t > 0, && (11.9b) \\ &v(\ell,t) = 0, && t > 0, && (11.9c) \\ &v(x,0) = 0. && 0 < x < \ell. && (11.9d) \end{aligned}$$

This is quite similar to Chapter 10's Example 10.5 on page 136 and we leave its solution to you as an exercise. When you work out the details, you will find that the solution u of (11.7) is

$$u(x,t) = \frac{\sigma x}{\ell}\sin\omega t + \sum_{n=1}^{\infty}\frac{Q_n}{\omega^2 + k^2\lambda_n^2}\Big[- k\lambda_n e^{-k\lambda_n t} + k\lambda_n\cos\omega t + \omega\sin\omega t\Big]\sin\gamma_n x, \quad (11.10)$$

where $Q_n = \frac{2\sigma\omega(-1)^n}{\gamma_n \ell}$.

As in Example 10.5, the solution consists of the sum of a transient (the exponentially decaying term) and steady-state oscillations of angular frequency ω. That may be made explicit by rearranging the solution into

$$\begin{aligned} u(x,t) &= -\sum_{n=1}^{\infty}\frac{k\lambda_n Q_n}{\omega^2 + k^2\lambda_n^2}e^{-k\lambda_n t}\sin\gamma_n x \\ &\quad + \left[\sum_{n=1}^{\infty}\frac{k\lambda_n Q_n}{\omega^2 + k^2\lambda_n^2}\sin\gamma_n x\right]\cos\omega t \\ &\quad + \left[\frac{\sigma x}{\ell} + \sum_{n=1}^{\infty}\frac{\omega Q_n}{\omega^2 + k^2\lambda_n^2}\sin\gamma_n x\right]\sin\omega t. \end{aligned}$$

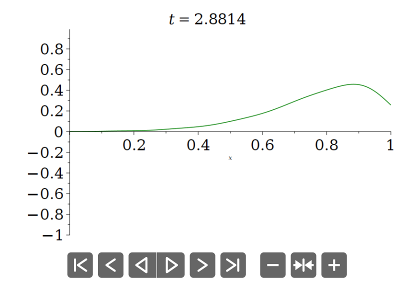


Figure 11.1: An animation of the solution (11.10) over the time range $0 < t < 34$.

The animation in Figure 11.1 was made over the time interval $0 \le t \le 34$ with the parameter values of

$$\ell = 1, \quad \sigma = 1, \quad \omega = 1, \quad k = 1/50$$

after truncating the series in (11.10) to its first 10 terms.

Figure 11.2 depicts the graph of $u(1/2,t)$, that is, the temperature at $x = 1/2$, corresponding to the animation in Figure 11.1. We see that the solution settles to a steady-state oscillation after a brief transient.

□

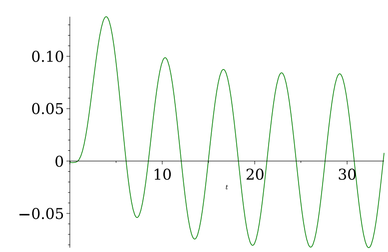


Figure 11.2: The graph of $u(1/2,t)$ of the solution (11.10).

11.2 *General boundary conditions*

We now turn to an IBVP with general boundary conditions:

$$\begin{aligned} &u_t = ku_{xx} + q(x,t) && 0 < \ell < x, \quad t > 0 && (11.11a)\\ &\alpha_1 u(0,t) + \alpha_2 u_x(0,t) = \alpha(t) && t > 0, && (11.11b)\\ &\beta_1 u(\ell,t) + \beta_2 u_x(\ell,t) = \beta(t) && t > 0, && (11.11c)\\ &u(x,0) = f(x) && 0 < \ell < x, && (11.11d) \end{aligned}$$

where α_1, α_2, β_1, β_2, are given constants, and $\alpha(t)$, $\beta(t)$, $q(x,t)$, and $f(x)$ are prescribed functions. We introduce the new unknown $v(x,t)$ which is related to the original unknown $u(x,t)$ through

$$u(x,t) = v(x,t) + \kappa_0(t) + \kappa_1(t)x, \quad (11.12)$$

where $\kappa_0(t)$ and $\kappa_1(t)$ will be determined later. Substituting (11.12) into IBVP (11.11) results in

$$
\begin{aligned}
&v_t(x,t) + \kappa_0'(t) + \kappa_1'(t)x = kv_{xx}(x,t) + q(x,t),\\
&\alpha_1\big[v(0,t) + \kappa_0(t)\big] + \alpha_2\big[v_x(0,t) + \kappa_1(t)\big] = \alpha(t),\\
&\beta_1\big[v(\ell,t) + \kappa_0(t) + \ell\kappa_1(t)\big] + \beta_2\big[v_x(\ell,t) + \kappa_1(t)\big] = \beta(t),\\
&v(x,0) + \kappa_0(0) + \kappa_1(0)x = f(x),
\end{aligned}
$$

which we rearrange as

$$
\begin{aligned}
&v_t(x,t) = kv_{xx}(x,t) + q(x,t) - \big[\kappa_0'(t) + \kappa_1'(t)x\big],\\
&\alpha_1 v(0,t) + \alpha_2 v_x(0,t) = \alpha(t) - \big[\alpha_1\kappa_0(t) + \alpha_2\kappa_1(t)\big],\\
&\beta_1 v(\ell,t) + \beta_2 v_x(\ell,t) = \beta(t) - \Big[\beta_1\big(\kappa_0(t) + \ell\kappa_1(t)\big) + \beta_2\kappa_1(t)\Big],\\
&v(x,0) = f(x) - \big[\kappa_0(0) + \kappa_1(0)x\big].
\end{aligned}
$$

We force homogeneous boundary conditions on v by setting the right-hand sides of the boundary conditions to zero:

$$
\begin{aligned}
&\alpha(t) - \big[\alpha_1\kappa_0(t) + \alpha_2\kappa_1(t)\big] = 0,\\
&\beta(t) - \Big[\beta_1\big(\kappa_0(t) + \ell\kappa_1(t)\big) + \beta_2\kappa_1(t)\Big] = 0.
\end{aligned}
$$

We solve this for $\kappa_0(t)$ and $\kappa_1(t)$ and obtain

$$\kappa_0(t) = \frac{(\ell\beta_1 + \beta_2)\alpha(t) - \alpha_2\beta(t)}{(\ell\beta_1 + \beta_2)\alpha_1 - \alpha_2\beta_1}, \tag{11.13a}$$

$$\kappa_1(t) = \frac{\alpha_1\beta(t) - \beta_1\alpha(t)}{(\ell\beta_1 + \beta_2)\alpha_1 - \alpha_2\beta_1}. \tag{11.13b}$$

With this choice of $\kappa_0(t)$ and $\kappa_1(t)$ we arrive at an IBVP for v with homogeneous boundary conditions:

$$v_t(x,t) = kv_{xx}(x,t) + q(x,t) - \big[\kappa_0'(t) + \kappa_1'(t)x\big], \tag{11.14a}$$

$$\alpha_1 v(0,t) + \alpha_2 v_x(0,t) = 0, \tag{11.14b}$$

$$\beta_1 v(\ell,t) + \beta_2 v_x(\ell,t) = 0, \tag{11.14c}$$

$$v(x,0) = f(x) - \big[\kappa_0(0) + \kappa_1(0)x\big], \tag{11.14d}$$

which can be solved through Chapter 10's eigenfunction expansion method. After calculating v, we obtain the solution u of the original IBVP (11.11) through (11.12).

Remark 11.2. The method described above is not completely fool-proof. The calculation of $\kappa_0(t)$ and $\kappa_1(t)$ can fail if the denominators in (11.13) happen to be zero. In particular, that is guaranteed to happen if $\alpha_1 = \beta_1 = 0$.[2] We will see in the exercises how to get around that issue.

[2] When $\alpha_1 = \beta_1 = 0$, the boundary conditions (11.11b) and (11.11c) amount the interesting case of prescribed fluxes at the ends of the heat conducting rod.

11.3 Exercises

11.1. Provide the details that lead to the solution (11.10) of Example (11.1).

11.2. It was pointed out that the change of variable (11.12) fails when $\alpha_1 = \beta_1 = 0$. Show that replacing (11.12) with

$$u(x,t) = v(x,t) + \kappa_1(t)x + \kappa_2(t)x^2$$

gets around that problem.

11.3. Show that

$$\eta(x,t) = \frac{1}{6k\ell}\Big[x(\ell-x)(2\ell-x)\big(q(0,t)-\alpha'(t)\big)\Big] + \frac{1}{6k\ell}\Big[x(\ell-x)(\ell+x)\big(q(\ell,t)-\beta'(t)\big)\Big] + \Big(1-\frac{x}{\ell}\Big)\alpha(t) + \frac{x}{\ell}\beta(t)$$

satisfies the requirements (11.3) and therefore the change of variable $u = v + \eta$ transforms the nonhomogeneous IBVP (11.1) to the homogeneous one (11.5). Additionally, show that with this choice of η, the function $\tilde{q}$ defined in (11.4) is zero at the boundary, that is,

$$\tilde{q}(0,t) = 0, \quad \tilde{q}(\ell,0) = 0.$$

This is a very desirable result. In order to solve the IBVP (11.5) through eigenfunction expansion, we need to expand $\tilde{q}$ into a series of eigenfunctions. The function $\tilde{q}$ being zero at the boundary results in a fast converging series since the eigenfunctions are zero at the boundary.

11.4. Solve the IBVP

$$\begin{aligned}
&u_t = ku_{xx} && 0 < x < \ell, \quad t > 0,\\
&u(0,t) = 0 && t > 0,\\
&u_x(\ell,t) = \sigma \sin \omega t && t > 0,\\
&u(x,0) = 0 && 0 < x < \ell.
\end{aligned}$$

This corresponds to heat conduction in a bar with one end maintained at temperature zero, while the other end is subjected to sinusoidal heat flux.

11.5. Consider the IBVP

$$\begin{aligned}
&u_t = ku_{xx} + 1 && 0 < x < \ell, \quad t > 0,\\
&u(0,t) = \alpha && t > 0,\\
&u(\ell,t) = \beta && t > 0,\\
&u(x,0) = f(x) && 0 < x < \ell,
\end{aligned}$$

where α and β are constants. Pick $\eta(x,t)$ according to (11.6) and reduce the IBVP to one of homogeneous boundary conditions. No need to solve the reduced IBVP.

11.6. Consider the previous exercise's IBVP, but now select an alternative η according to the idea introduced in Remark 11.1 (page 144). Use that η to reduce the IBVP to one of homogeneous boundary conditions. No need to solve the reduced IBVP. Do you see an advantage of this choice of η versus that in the previous exercise?

11.7. Solve the following IBVP for the wave equation which is forced at the boundary. Is resonance possible?

$$
\begin{aligned}
&u_{tt} = c^2 u_{xx} && 0 < x < \ell, \quad t > 0, \\
&u(0,t) = 0 && t > 0, \\
&u_x(\ell,t) = \sigma \sin \omega t && t > 0, \\
&u(x,0) = 0 && 0 < x < \ell, \\
&u_t(x,0) = 0 && 0 < x < \ell.
\end{aligned}
$$

12
The Poisson equation

Poisson's equation is ubiquitous in all areas of physics, encompassing the studies of diverse subjects such as electromagnetism, celestial mechanics, fluid dynamics, aerodynamics, and structural mechanics. Depending on the application, it may be posed in two three spatial dimensions. In this chapter we limit our study to two spatial dimensions, and therefore the unknown will be a function u on a domain Ω in $\mathbf{R}^2$. If Ω is embedded in the Cartesian xy plane, Poisson's equation is expressed as

$$u_{xx} + u_{yy} + q = 0 \quad \text{in } \Omega,$$

where $q = q(x, y)$ is a prescribed and $u = u(x, y)$ is to be determined.

Applications that give rise to Poisson's equation provide supplementary data in the form of boundary conditions that enables one to identify a unique solution. A typical formulation is

$$u_{xx} + u_{yy} + q = 0 \qquad \text{in } \Omega, \tag{12.1a}$$

$$u = f \qquad \text{on } \partial\Omega. \tag{12.1b}$$

where $\partial\Omega$ is the customary notation for *the boundary of the domain* Ω. Equations (12.1) constitute a *Poisson boundary value problem*.

In a Poisson boundary value problem (BVP) the analysis of the PDE is intricately intertwined with the geometry of Ω. The challenge in solving a Poisson problem is to successfully handle these two aspects simultaneously. In this textbook we limit Ω to simple geometric shapes, such as rectangles, disks, sectors of disks, and annuli. More complex geometric shapes are typically treated through numerical methods, such as finite differences and finite elements.

12.1 Membranes and the Poisson equation

As was noted earlier, Poisson's equation is ubiquitous in applications. To provide a concrete perspective and motivate the study of the rest of this chapter, we sketch an easy-to-grasp instance that gives rise to Poisson's equation.

In Chapter 5 we saw that an application of Newton's law of motion to a taut string results in the wave equation $u_{tt} = c^2 u_{xx} + q(x,t)$, where $u(x,t)$ is the string's transverse displacement, and $q(x,t)$ is an externally applied force per unit mass. A similar reasoning, applied to uniformly stretched *membrane*, such as that of the surface of a drum, leads to the equation

$$u_{tt} = c^2(u_{xx} + u_{yy}) + q(x,y,t), \tag{12.2}$$

where $u = u(x,y,t)$ is the membrane's transverse displacement and q is is an externally applied force per unit mass.

Consider the membrane's *static equilibrium*, that is, the situation where motion has ceased and the membrane has reached a rest state under the influence of the applied force q. Then, u and q no longer change with time, and so now we have $u = u(x,y)$ and $q = q(x,y)$. In particular, the acceleration u_{tt} is zero, and therefore (12.2) reduces to the Poisson equation

$$u_{xx} + u_{yy} + q = 0 \quad \text{in } \Omega,$$

were in a typical drum the domain Ω is a disk but our formulation applies to general shapes. A rectangular drum, for instance, is quite admissible.

Since the membrane is fixed along its edges to the drum, the displacement is zero at the edges and therefore the boundary condition is $u = 0$ on $\partial\Omega$.

12.2 *Solving the Poisson equation on a rectangle*

Here we solve the Poisson boundary value problem

$$u_{xx} + u_{yy} + q(x,y) = 0 \qquad \text{in } \Omega, \tag{12.3a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{12.3b}$$

where Ω is the rectangle

$$\Omega = \big\{(x,y) \mid 0 < x < a, \quad 0 < y < b\big\}, \tag{12.3c}$$

of some positive dimensions a and b. This is achieved by expanding $u(x,y)$ and $q(x,y)$ into series of suitably chosen eigenfunctions.

Let us introduce the pair of eigenvalue problems

$$X''(x) + \lambda X(x) = 0, \qquad Y''(y) + \mu Y(y) = 0, \tag{12.4a}$$

$$X(0) = X(a) = 0, \qquad Y(0) = Y(b) = 0. \tag{12.4b}$$

We know that each of these has infinitely many eigenvalues and eigenfunctions, given through

$$\gamma_m = \frac{m\pi}{a}, \qquad \nu_n = \frac{n\pi}{b}, \tag{12.5a}$$

$$\lambda_m = \gamma_m^2 = \left(\frac{m\pi}{a}\right)^2, \qquad \mu_n = \nu_n^2 = \left(\frac{n\pi}{b}\right)^2, \tag{12.5b}$$

$$X_m(x) = \sin\gamma_m x = \sin\frac{m\pi x}{a}, \quad Y_n(y) = \sin\nu_n y = \sin\frac{n\pi y}{b}, \tag{12.5c}$$

$$\|X_m\|^2 = \int_0^a X_m(x)^2\,dx = \frac{1}{2}a, \quad \|Y_n\|^2 = \int_0^b Y_n(y)^2\,dy = \frac{1}{2}b. \tag{12.5d}$$

For any fixed y, the function $q(x,y)$ may be regarded as as function of x defined over the interval $[0,a]$ and therefore may be expanded into an infinite series of the eigenfunctions X_m:

$$q(x,y) = \sum_{m=1}^{\infty} c_m(y)X_m(x), \tag{12.6}$$

where the coefficients c_m depend on the choice of y in that expansion.

Now, c_m is a function of y where $y \in [0,b]$, and therefore it can be expanded into an infinite series of the eigenfunctions Y_n:

$$c_m(y) = \sum_{n=1}^{\infty} Q_{mn}Y_n(y), \quad m = 1,2,\ldots.$$

The coefficients Q_{mn} depend on m since c_m depends on m. Substituting this expression for $c_m(y)$ into (12.6), we get

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn}X_m(x)Y_n(y). \tag{12.7}$$

To find the coefficients Q_{mn}, we multiply that expression by $X_i(x)Y_j(y)$ (with arbitrary i and j) and integrate the result over the rectangle

$$\iint_\Omega q(x,y)X_i(x)Y_j(y)\,dx\,dy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\iint_\Omega Q_{mn}X_m(x)Y_n(y)X_i(x)Y_j(y)\,dx\,dy.$$

The double integrals on the right-hand side can be evaluated through Fubini's theorem:

$$\begin{aligned}\iint_\Omega X_m(x)Y_n(y)X_i(x)Y_j(y)\,dx\,dy &= \int_0^b\Big[\int_0^a X_m(x)X_i(x)\,dx\Big]Y_n(y)Y_j(y)\,dy\\ &= \Big[\int_0^a X_m(x)X_i(x)\,dx\Big]\Big[\int_0^b Y_n(y)Y_j(y)\,dy\Big],\end{aligned}$$

and therefore

$$\iint_\Omega q(x,y)X_i(x)Y_j(y)\,dx\,dy = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn}\Big[\int_0^a X_m(x)X_i(x)\,dx\Big]\Big[\int_0^b Y_n(y)Y_j(y)\,dy\Big].$$

Due to the orthogonality of the eigenfunctions, the integral $\int_0^b Y_n(y)Y_j(y)\,dy$ is zero for all n and j except when $n = j$ in which case it evaluates to $b/2$. Therefore, as the summation index n runs from 1 to infinity, only the jth term survives and the inner summation yields $b/2$. We thus obtain

$$\iint_\Omega q(x,y)X_i(x)Y_j(y)\,dx\,dy = \frac{b}{2}\sum_{m=1}^{\infty} Q_{mj}\Big[\int_0^a X_m(x)X_i(x)\,dx\Big].$$

Similarly, the integral $\int_0^a X_m(x)X_j(x)\,dy$ is zero for all m and i except when $m = i$ in which case it evaluates to $a/2$. Therefore, as the summation index m runs from 1 to infinity, only the ith term survives and the summation yields $a/2$. We thus arrive at

$$\iint_\Omega q(x,y)X_i(x)Y_j(y)\,dx\,dy = \frac{ab}{4}Q_{ij},$$

and therefore

$$Q_{ij} = \frac{4}{ab}\iint_\Omega q(x,y)X_i(x)Y_j(y)\,dx\,dy, \quad i,j = 1,2,\dots. \tag{12.8}$$

In summary, any function[1] q defined on the rectangle Ω may be expanded into the doubly-infinite series of eigenfunctions (12.7). The coefficients of the expansion may be computed through (12.8).

[1] Well, almost any. We need q to be square-integrable over the rectangle, that is,

$$\iint_\Omega |q(x,y)|^2\,dx\,dy < \infty.$$

To solve the Poisson BVP (12.3), we look for the solution u in the form of a doubly-infinite series of eigenfunctions, as in

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}X_m(x)Y_n(y). \tag{12.9}$$

To find the coefficients U_{mn}, we substitute this representation into (12.3a):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}\Big[X_m''(x)Y_n(y) + X_m(x)Y_n''(y)\Big] + q(x,y) = 0. \tag{12.10}$$

In view of (12.4) and (12.5) we have $X_m'' = -\lambda_m X_m$ and $Y_n'' = -\mu_n Y_n$, and therefore (12.10) leads to

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}(\lambda_m + \mu_n)X_m(x)Y_n(y).$$

Comparing this with (12.7) we see that $Q_{mn} = U_{mn}(\lambda_m + \mu_n)$, whence

$$U_{mn} = \frac{Q_{mn}}{\lambda_m + \mu_n}, \quad m,n = 1,2,\dots. \tag{12.11}$$

Substituting this in (12.9) we arrive at the solutions of the Poisson's BVP.

Example 12.1. Solve Poisson's BVP (12.3a)–(12.3c) when $q(x,y)$ is identically 5 over Ω.

Solution. We begin with calculating the coefficients Q_{mn} from (12.8) with $q \equiv 5$:

$$Q_{mn} = \frac{4}{ab}\iint_\Omega 5\, X_m(x)Y_n(y)\,dx\,dy = \frac{20}{ab}\int_0^b \Big[\int_0^a X_m(x)\,dx\Big] Y_n(y)\,dx.$$

Referring to (12.5c), we have

$$\begin{aligned}\int_0^a X_m(x)\,dx = \int_0^a \sin\gamma_m x\,dx = -\frac{1}{\gamma_m}\cos\gamma_m x\Big|_0^a \\ = \frac{1}{\gamma_m}\Big[1-\cos\gamma_m a\Big] = \frac{1}{\gamma_m}\Big[1-(-1)^m\Big],\end{aligned}$$

and similarly,

$$\int_0^b Y_n(y)\,dy = \frac{1}{\nu_n}\Big[1-(-1)^n\Big],$$

where γ_m and ν_n are given in (12.5a). It follows that

$$Q_{mn} = \frac{20}{ab\gamma_m\nu_n}\big[1-(-1)^m\big]\big[1-(-1)^n\big],$$

and therefore

$$U_{mn} = \frac{20}{ab\gamma_m\nu_n(\lambda_m+\mu_n)}\big[1-(-1)^m\big]\big[1-(-1)^n\big].$$

We conclude that

$$u(x,y) = \frac{20}{ab}\sum_{m=1}^\infty\sum_{n=1}^\infty \frac{\big[1-(-1)^m\big]\big[1-(-1)^n\big]}{\gamma_m\nu_n(\lambda_m+\mu_n)} X_m(x)Y_n(y).$$

Figure 12.1 shows the graph of u with the choices $a=1$ and $b=1$.

□

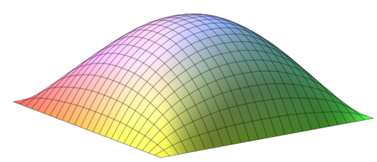


Figure 12.1: The graph of the solution $u(x,y)$ of Example 12.1 with $a=1$ and $b=1$.

12.3 Exercises

12.1. Consider rectangular membrane spanning the horizontal rectangle Ω defined in (12.3c). Suppose the membrane is glued to the rectangle's left, bottom, and right edges, but is displaced vertically along the top edge by $f(x) = 2x(a-x)$. Find the membrane's shape, that is, the vertical displacement $u(x,y)$ at any $(x,y)\in\Omega$. Suppose there is no external force acting on the membrane.

Hint: Verify that the function $\eta(x,y) = \frac{y}{b}f(x)$ satisfies the same boundary conditions as the membrane. Consequently, if we let $u(x,y) = v(x,t)+\eta(x,y)$, then v will satisfy zero boundary conditions all around Ω's boundary.

12.2. Find the steady-state temperature distribution on the rectangular lamina $\Omega = \big\{(x,y) \mid 0<x<a,\quad 0<y<b\big\}$ where the boundary temperature is prescribed as zero on the bottom half $(0<y<b/2)$ and one on the top half.

13
PDEs in polar coordinates

Our study of the Poisson equation in Chapter 12 was limited to rectangular domains. Here we extend that study to domains whose boundaries are circles or arcs of circles.

13.1 Laplace's operator in polar coordinates

The geometry of domains with circular boundaries is naturally expressed in polar coordinates. To solve PDEs in such domains, we would want to express the unknown and its derivatives in polar coordinates as well. Figure 13.1 shows us the customary representation of the polar coordinates (r, θ) of a point P as they relate to the point's Cartesian coordinates (x, y) where it's evident that

$$x = r\cos\theta, \quad y = r\sin\theta. \tag{13.1}$$

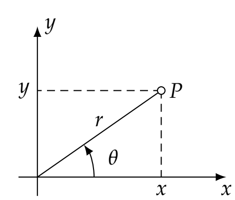


Figure 13.1: The point P has Cartesian coordinates (x, y) and polar coordinates (r, θ).

As x and y vary, so do r and θ accordingly, that is to say, r and θ may be viewed as functions of x and y. What are the derivatives of those functions? To find out, differentiate the two equations in (13.1) with respect to x with the help of the chain rule. We get

$$1 = r_x\cos\theta - r\theta_x\sin\theta, \quad 0 = r_x\sin\theta + r\theta_x\cos\theta.$$

Solving this system of two equations for the unknowns r_x and θ_x, we get

$$r_x = \cos\theta, \quad \theta_x = -\frac{1}{r}\sin\theta. \tag{13.2a}$$

A similar computation, where we differentiate the two equations in (13.1) with respect to y, leads to

$$r_y = \sin\theta, \quad \theta_y = \frac{1}{r}\cos\theta. \tag{13.2b}$$

A function $u(x, y)$ defined in Cartesian coordinates may be evaluated at the corresponding polar coordinates as $u(r\cos\theta, r\sin\theta)$. Such evaluation defines a function $U(r, \theta)$, where

$$u(x, y) = u(r\cos\theta, r\sin\theta) = U(r, \theta).$$

Then by the chain rule, and with the help of (13.2a) and (13.2b), we get

$$u_x = U_r r_x + U_\theta \theta_x = U_r \cos\theta - \frac{1}{r} U_\theta \sin\theta,$$
$$u_y = U_r r_y + U_\theta \theta_y = U_r \sin\theta + \frac{1}{r} U_\theta \cos\theta.$$

Carrying on to second derivatives, we calculate

$$\begin{aligned} u_{xx} &= \Big(U_r \cos\theta - \frac{1}{r} U_\theta \sin\theta\Big)_x \\ &= \Big(U_r \cos\theta - \frac{1}{r} U_\theta \sin\theta\Big)_r r_x + \Big(U_r \cos\theta - \frac{1}{r} U_\theta \sin\theta\Big)_\theta \theta_x \\ &= \Big(U_{rr} \cos\theta + \frac{1}{r^2} U_\theta \sin\theta - \frac{1}{r} U_{r\theta} \sin\theta\Big)\, r_x \\ &\qquad + \Big(U_{r\theta} \cos\theta - U_r \sin\theta - \frac{1}{r} U_{\theta\theta} \sin\theta - \frac{1}{r} U_\theta \cos\theta\Big)\, \theta_x. \end{aligned}$$

We substitute for r_x and θ_x from (13.2a), multiply through and group the terms, and arrive at

$$\begin{aligned} u_{xx} = U_{rr} \cos^2\theta - \frac{2}{r} U_{r\theta} \sin\theta\cos\theta + \frac{1}{r^2} U_{\theta\theta} \sin^2\theta \\ + \frac{1}{r} U_r \sin^2\theta + \frac{2}{r^2} U_\theta \sin\theta\cos\theta. \end{aligned}$$

As similar computation shows that

$$\begin{aligned} u_{yy} = U_{rr} \sin^2\theta + \frac{2}{r} U_{r\theta} \sin\theta\cos\theta + \frac{1}{r^2} U_{\theta\theta} \cos^2\theta \\ + \frac{1}{r} U_r \cos^2\theta - \frac{2}{r^2} U_\theta \sin\theta\cos\theta. \end{aligned}$$

As we add up the u_{xx} and u_{yy} calculated above, terms cancel/simplify and we are led to a somewhat simple expression for the Laplacian in polar coordinates:

$$\Delta u = u_{xx} + u_{yy} = U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\theta\theta}. \tag{13.3}$$

13.2 *Separation of variables*

In Chapter (7) we derived the equation of heat conduction $u_t = k u_{xx} + q$ in a one-dimensional homogeneous medium through an analysis of the balance of thermal energy. A very similar analysis leads to the equation of heat conduction in 2D:

$$u_t = k\Delta u + q, \tag{13.4}$$

where Δ is the Laplacian operator, k is medium's thermal diffusivity, and q is the internal heat source (if there is one) per unit area per unit time. If q and the boundary conditions are independent of time, them the temperature approaches steady-state that satisfies the Poisson equation $k\Delta u + q = 0$. Normally one divides the equation by k and renames q/k to q, and considers the reduced equation

$$\Delta u + q = 0 \tag{13.5}$$

The separation of variables approach works quite well with this equation in polar coordinates. We illustrate the approach through a few examples.

Example 13.1 (Quarter-disk)**.** Consider a thin lamina in the shape of a quarter-disk of radius a defined in polar coordinates as

$$\Omega = \Big\{(r,\theta) \mid 0 \le r \le a,\ \ 0 \le \theta \le \frac{\pi}{2}\Big\}.$$

The lamina consists of a homogeneous heat-conducting material. It is thermally insulated on its top and bottom surfaces (see Figure 13.2) but it may exchange heat with the outside through its edges.

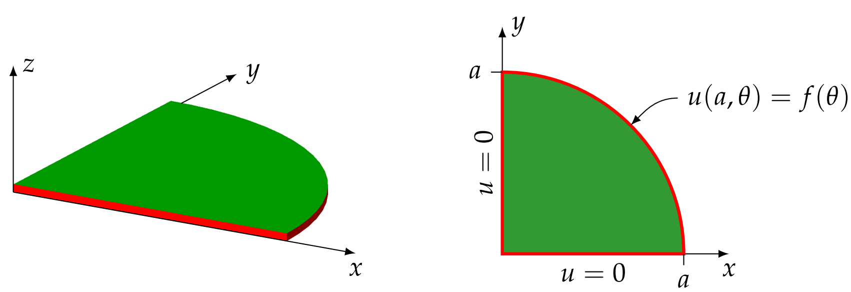


Figure 13.2: On the left, the lamina is shown as a three-dimensional object for visualization purposes. The mathematical model treats the lamina as a flat two-dimensional object. The top and bottom faces (drawn in green) are thermally insulated. The edges (drawn in red) can exchange heat with the outside.

In the diagram on the right, we have specified $u = 0$ on the edges along the x and y axes, and a variable temperature along the circular arc.

We specify an arbitrary temperature distribution along the lamina's curved edge, while maintaining its straight edges at zero temperature. Assuming that there is no internal heat source, the steady-state temperature $u(r,\theta)$ in the lamina solves the boundary value problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{in } \Omega, \tag{13.6a}$$

$$u(r,0) = 0 \qquad 0 \le r \le a, \tag{13.6b}$$

$$u\Big(r,\frac{\pi}{2}\Big) = 0 \qquad 0 \le r \le a, \tag{13.6c}$$

$$u(a,\theta) = f(\theta) \qquad 0 \le \theta \le \frac{\pi}{2}. \tag{13.6d}$$

To determine $u(r,\theta)$, we begin by looking for functions $u(r,\theta) = R(r)\Psi(\theta)$ that satisfy the first three of the equations (13.6). From (13.6a) we get

$$R''(r)\Psi(\theta) + \frac{1}{r}R'(r)\Psi(\theta) + \frac{1}{r^2}R(r)\Psi''(\theta) = 0,$$

which separates into

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\Psi''(\theta)}{\Psi(\theta)} = \lambda, \tag{13.7}$$

where λ is the separation constant, while the boundary conditions (13.6b) and (13.6c) yield

$$R(r)\Psi(0) = 0, \quad R(r)\Psi\Big(\frac{\pi}{2}\Big) = 0.$$

We thus arrive at a familiar eigenvalue problem for Ψ:

$$\Psi''(\theta) + \lambda\Psi(\theta) = 0, \quad \Psi(0) = 0, \quad \Psi\Big(\frac{\pi}{2}\Big) = 0. \tag{13.8}$$

The eigenvalues and eigenfunctions are

$$\gamma_n = 2n, \quad \lambda_n = \gamma_n^2 = 4n^2,$$
$$\Psi_n(\theta) = \sin\gamma_n\theta = \sin 2n\theta, \qquad n = 1, 2, \ldots. \tag{13.9}$$

Then from (13.7) we get

$$r^2 R_n''(r) + rR_n'(r) - \lambda_n R_n(r) = 0. \tag{13.10}$$

This is the well-know Euler's differential equation whose solutions are of the form $R(r) = r^p$. To find the value of the exponent p, we calculate $R'(r) = pr^{p-1}$, and $R''(r) = p(p-1)r^{p-2}$ and substitute into the differential equation. We get $p(p-1)r^p + pr^p - \lambda_n r^p = 0$, and therefore $p(p-1) + p - \lambda_n = 0$, which simplifies to $p^2 = \lambda_n$. We conclude that $p = \pm\sqrt{\lambda_n} = \pm\gamma_n = \pm 2n$ and therefore

$$R_n(r) = A_n r^{\gamma_n} + \frac{B_n}{r^{\gamma_n}}, \tag{13.11}$$

where $\gamma_n = 2n$, and A_n and B_n are arbitrary constants. We are forced to set $B_n = 0$ because otherwise the solution will blow up at $r = 0$.

Having constructed the functions $u_n(r,\theta) = r^{\gamma_n}\sin\gamma_n\theta$, $n = 1, 2, \ldots$ that satisfy the first three of the equations in (13.6), we form the infinite linear combination

$$u(r,\theta) = \sum_{n=1}^{\infty} c_n r^{\gamma_n}\sin\gamma_n\theta \tag{13.12}$$

and determine the coefficients c_n so that to enforce the fourth equation, that is, $u(a,\theta) = f(\theta)$. Substituting $r = a$ in (13.12), we get

$$f(\theta) = \sum_{n=1}^{\infty} c_n a^{\gamma_n}\sin\gamma_n\theta.$$

This the usual sine series representation of the function f. The coefficients are obtained from

$$c_n a^{\gamma_n} = \frac{2}{\pi/2}\int_0^{\pi/2} f(\theta)\sin\gamma_n\theta\,d\theta,$$

and therefore

$$c_n = \frac{4}{a^{\gamma_n}\pi} \int_0^{\pi/2} f(\theta) \sin \gamma_n \theta \, d\theta.$$

This, together with (13.12), completely determines the solution of the original BVP.

As a concrete case, let's say $f(\theta) = 1$ for all $\theta \in [0, \frac{\pi}{2}]$. Then

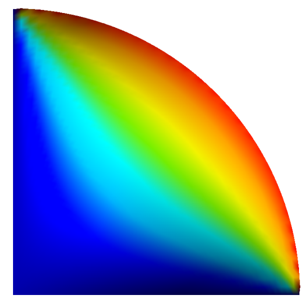
$$\int_0^{\pi/2} f(\theta) \sin \gamma_n \theta \, d\theta = \int_0^{\pi/2} \sin 2n\theta \, d\theta$$
$$= -\frac{1}{2n} \cos 2n\theta \Big|_0^{\pi/2} = \frac{1}{2n}\Big[1 - \cos n\pi\Big] = \frac{1-(-1)^n}{2n},$$

and therefore

$$c_n = \frac{2}{\pi} \cdot \frac{1-(-1)^n}{na^{2n}},$$

whence

$$u(r,\theta) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \Big(\frac{r}{a}\Big)^{2n} \sin 2n\theta.$$



The adjacent diagram shows the temperature distribution in the lamina. Red is hot, blue is cold.

Example 13.2 (Partial annulus)**.** Let's solve the heat equation in the domain

$$\Omega = \Big\{(r,\theta) \mid a \le r \le b, \;\; 0 \le \theta \le \frac{\pi}{2}\Big\}$$

shown in Figure 13.3. We fix the boundary temperature at zero along the straight edges, and $f(\theta)$ and $g(\theta)$ along the curved edges, as indicated in the figure. There are no internal heat sources. This leads to the boundary value problem

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{in } \Omega, \tag{13.13a}$$
$$u(r,0) = 0 \qquad a \le r \le b, \tag{13.13b}$$
$$u\Big(r, \frac{\pi}{2}\Big) = 0 \qquad a \le r \le b, \tag{13.13c}$$
$$u(a,\theta) = f(\theta) \qquad 0 \le \theta \le \frac{\pi}{2}, \tag{13.13d}$$
$$u(b,\theta) = g(\theta) \qquad 0 \le \theta \le \frac{\pi}{2}. \tag{13.13e}$$

The search for solutions of the form $u(r,\theta) = R(r)\Psi(\theta)$ works exactly as in Example (13.1) and leads to the eigenvalue problem (13.8) for Ψ whose solutions are given in (13.9), and the Euler equation (13.10) for R whose general solution is given in (13.11).

At this point, the solution of the current problem diverges from that of Example (13.1). In the previous case we set the constant B in (13.11) to zero to avoid the singularity at $r = 0$, that is, at the origin. In the current case that is not an issue since Ω *does not include the*

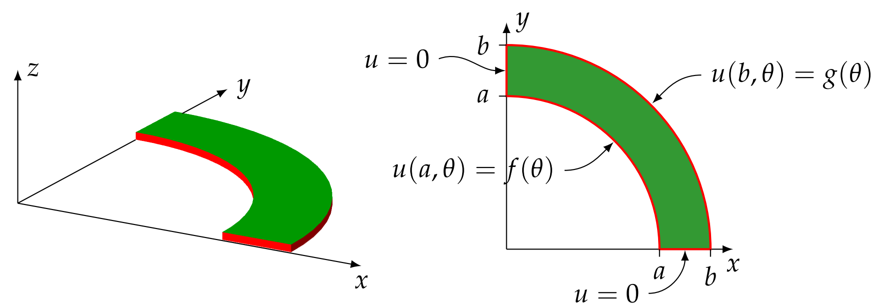


Figure 13.3: The temperature is zero along the lamina's straight edges, and $f(\theta)$ and $g(\theta)$ along the curved edges.

origin. Therefore we retain the full form of (13.11) and thus arrive at a sequence of separated solutions

$$u_n(r,\theta) = \left(A_n r^{\gamma_n} + \frac{B_n}{r^{\gamma_n}}\right)\sin\gamma_n\theta, \quad n = 1, 2, \dots,$$

where $\gamma_n = 2n$. Each of these satisfies the first three of the equations (13.13), and therefore so does the infinite linear combination

$$u(r,\theta) = \sum_{n=1}^{\infty}\left(A_n r^{\gamma_n} + \frac{B_n}{r^{\gamma_n}}\right)\sin\gamma_n\theta. \tag{13.14}$$

We now determine the coefficients A_n and B_n so that the remaining two conditions hold as well, that is,

$$f(\theta) = \sum_{n=1}^{\infty}\left(A_n a^{\gamma_n} + \frac{B_n}{a^{\gamma_n}}\right)\sin\gamma_n\theta,$$
$$g(\theta) = \sum_{n=1}^{\infty}\left(A_n b^{\gamma_n} + \frac{B_n}{b^{\gamma_n}}\right)\sin\gamma_n\theta.$$

These are the Fourier sine series expansions of the functions f and f. The coefficients are obtained in the usual way:

$$A_n a^{\gamma_n} + \frac{B_n}{a^{\gamma_n}} = \frac{2}{\pi/2}\int_0^{\pi/2} f(\theta)\sin\gamma_n\,d\theta,$$
$$A_n b^{\gamma_n} + \frac{B_n}{b^{\gamma_n}} = \frac{2}{\pi/2}\int_0^{\pi/2} g(\theta)\sin\gamma_n\,d\theta.$$

Solving this system of two equations for the two unknowns A_n and B_n, we get

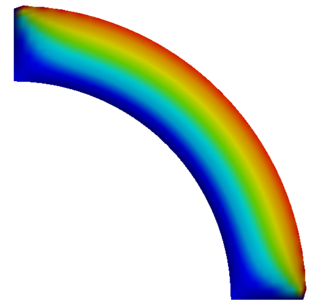
$$A_n = \frac{b^{\gamma_n}Q_n - a^{\gamma_n}P_n}{b^{2\gamma_n} - a^{2\gamma_n}}, \quad B_n = \frac{a^{\gamma_n}b^{\gamma_n}(b^{\gamma_n}P_n - a^{\gamma_n}Q_n}{b^{2\gamma_n} - a^{2\gamma_n}},$$

where P_n and Q_n are the right-hand sides of the system, that is

$$P_n = \frac{4}{\pi}\int_0^{\pi/2} f(\theta)\sin\gamma_n\,d\theta, \quad Q_n = \frac{4}{\pi}\int_0^{\pi/2} g(\theta)\sin\gamma_n\,d\theta.$$

We conclude that

$$A_n r^{\gamma_n} + \frac{B_n}{r^{\gamma_n}} = \frac{1-\left(\frac{r}{b}\right)^{2\gamma_n}}{1-\left(\frac{a}{b}\right)^{2\gamma_n}} \cdot \left(\frac{a}{r}\right)^{\gamma_n} P_n + \frac{1-\left(\frac{a}{r}\right)^{2\gamma_n}}{1-\left(\frac{a}{b}\right)^{2\gamma_n}} \cdot \left(\frac{r}{b}\right)^{\gamma_n} Q_n,$$



and therefore

$$u(r,\theta) = \sum_{n=1}^{\infty} \left[\frac{1-\left(\frac{r}{b}\right)^{2\gamma_n}}{1-\left(\frac{a}{b}\right)^{2\gamma_n}} \cdot \left(\frac{a}{r}\right)^{\gamma_n} P_n + \frac{1-\left(\frac{a}{r}\right)^{2\gamma_n}}{1-\left(\frac{a}{b}\right)^{2\gamma_n}} \cdot \left(\frac{r}{b}\right)^{\gamma_n} Q_n \right] \sin \gamma_n \theta.$$

The adjacent diagram shows the temperature distribution in the lamina. Red is hot, blue is cold.

Example 13.3 (Quarter-disk with heat source). Consider the steady-state heat conduction equation (13.5) in the quarter-disk domain Ω of Example 13.1 in the presence of a heat source $q(r,\theta)$. Let's take, for the sake of simplicity, that the temperature is fixed at zero all around Ω's boundary. We wish to determine the temperature $u(r,\theta)$ in the lamina. Here is the formulation of the BVP:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + q(r,\theta) = 0 \qquad \text{in } \Omega, \tag{13.15a}$$

$$u(r,0) = 0 \qquad 0 \le r \le a, \tag{13.15b}$$

$$u\left(r, \frac{\pi}{2}\right) = 0 \qquad 0 \le r \le a, \tag{13.15c}$$

$$u(a,\theta) = 0 \qquad 0 \le \theta \le \frac{\pi}{2}. \tag{13.15d}$$

We expand the unknown temperature distribution $u(r,\theta)$ and the known heat source $q(r,\theta)$ into series of the eigenfunction $\Psi_n(\theta)$ that we found in Example (13.1)—see equations (13.8) and (13.9)—as in

$$u(r,\theta) = \sum_{n=1}^{\infty} R_n(r)\Psi_n(\theta), \tag{13.16a}$$

$$q(r,\theta) = \sum_{n=1}^{\infty} \chi_n(r)\Psi_n(\theta). \tag{13.16b}$$

The coefficients $\chi_n(r)$ are readily found through the usual formula

$$\chi_n(r) = \frac{2}{\pi/2}\int_0^{\pi/2} q(r,\theta)\Psi_n(\theta)\,d\theta = \frac{4}{\pi}\int_0^{\pi/2} q(r,\theta)\sin 2n\theta\,d\theta. \tag{13.17}$$

To find the coefficients $R_n(r)$, we substitute (13.16a) in the PDE (13.15a). We get

$$\sum_{n=1}^{\infty} \left[R_n''(r)\Psi_n(\theta) + \frac{1}{r}R_n'(r)\Psi_n(\theta) + \frac{1}{r^2}R_n(r)\Psi_n''(\theta) \right] + q(r,\theta) = 0.$$

We make the substitution[1] $\Psi_n''(\theta) = -\lambda_n \Psi_n(\theta) = -\gamma_n^2 \Psi_n(\theta)$, group

[1] $\Psi_n(\theta)$, being an eigenfunction of (13.8), satisfies $\Psi_n''(\theta) + \lambda_n \Psi_n(\theta) = 0$.

the terms, isolate $q(r,\theta)$, and arrive at

$$q(r,\theta) = -\sum_{n=1}^{\infty}\left[R_n''(r) + \frac{1}{r}R_n'(r) - \frac{\gamma_n^2}{r^2}R_n(r)\right]\Psi_n(\theta).$$

Comparing this against (13.16b), leads to

$$\chi_n(r) = -\left[R_n''(r) + \frac{1}{r}R_n'(r) - \frac{\gamma_n^2}{r^2}R_n(r)\right],$$

which is better expressed as

$$r^2R_n''(r) + rR_n'(r) - \gamma_n^2 R_n(r) = -r^2\chi_n(r), \quad n = 1,2,\ldots. \tag{13.18}$$

These are nonhomogeneous versions of Euler's equation. The values of the right-hand sides are available in (13.17). The general solution of the corresponding *homogeneous equation* for each n is given in (13.11). All there remains is to find a particular solution of the nonhomogeneous equation and add to the solution of the homogeneous equation.

Let's work out a concrete case. Suppose that the heat source is uniformly distributed over the lamina. Thus, take $q \equiv 1$. We compute $\chi_n(r)$ through (13.17):

$$\chi_n(r) = \frac{4}{\pi}\int_0^{\pi/2}\sin 2n\theta\,d\theta = -\frac{2}{n\pi}\cos 2n\theta\Big|_0^{\pi/2}$$
$$= \frac{2}{n\pi}\big[1 - \cos n\pi\big] = \frac{2}{n\pi}\big[1 - (-1)^n\big].$$

We see that $\chi_n(r)$ is independent of r, and therefore as far as the ODEs (13.18) are concerned, it is a constant. Let

$$Q_n = \frac{2}{n\pi}\big[1 - (-1)^n\big]$$

be a temporary placeholder for $\chi_n(r)$ and write the ODEs as

$$r^2R_n''(r) + rR_n'(r) - \gamma_n^2 R_n(r) = -Q_n r^2. \tag{13.19}$$

Considering the special structure of the equation, it make sense to look for a particular solution of the form Cr^2. Plugging that guess into the ODE we obtain $4Cr^2 - \gamma_n^2 Cr^2 = -Q_n r^2$, whence $C = \frac{Q_n}{\gamma_n^2 - 4}$. We conclude that the general solution of (13.19) is

$$R_n(r) = A_n r^{\gamma_n} + \frac{B_n}{r^{\gamma_n}} + \frac{Q_n}{\gamma_n^2 - 4}r^2. \tag{13.20}$$

But wait! Since $\gamma_n = 2n$, we have $\gamma_1^2 = 4$ and therefore the evaluation of $R_1(r)$ through (13.20) fails!

So we go back to the drawing board and take a close look at (13.19) when $n = 1$, that is,

$$r^2R_1''(r) + rR_1'(r) - 4R_1(r) = -Q_1 r^2.$$

We can see clearly why a particular solution of the form $R_1(r) = Cr^2$ is not feasible—plugging $R_1(r) = Cr^2$ into that equation results in $0 = -Q_1 r^2$. So we try something like $R_1(r) = r^2\phi(r)$ for a ϕ that is to be determined.[2] Substituting that guess into the ODE, expanding the terms and simplifying, we arrive at

$$\phi''(r) + \frac{5}{r}\phi'(r) = -\frac{Q_1}{r^2},$$

which we then multiply this through integrating factor[3] r^5 to get

$$\left(r^5\phi'(r)\right)' = -Q_1 r^3.$$

and then integrate:

$$r^5\phi'(r) = -\frac{Q_1}{4}r^4.$$

We have thus obtained $\phi'(r) = -\frac{Q_1}{4r}$, and therefore $\phi(r) = -\frac{1}{4}Q_1 \ln r$. Recalling that $\gamma_1 = 2$, we conclude that

$$R_1(r) = A_1 r^2 + \frac{B_1}{r^2} - \frac{1}{4}Q_1 r^2 \ln r. \tag{13.21}$$

Note that this is quite different from $R_n(r)$ in (13.20). That expression is good for $n \geq 2$. This one is for $n = 1$.

As in Example (13.1), we take $B_n = 0$ both in (13.20) and in (13.21) because otherwise the solution will blow up at the origin. We thus arrive at the solution candidate

$$u(r,\theta) = \left[A_1 r^2 - \frac{1}{4}Q_1 r^2 \ln r\right] \sin\gamma_1\theta + \sum_{n=2}^{\infty}\left[A_n r^{\gamma_n} + \frac{Q_n}{\gamma_n^2 - 4}r^2\right] \sin\gamma_n\theta. \tag{13.22}$$

This satisfies the PDE, and the boundary conditions on the lamina's straight edges. The boundary condition (13.15d) on the curved edge implies that

$$\left[A_1 a^2 - \frac{1}{4}Q_1 a^2 \ln a\right] \sin\gamma_1\theta + \sum_{n=2}^{\infty}\left[A_n a^{\gamma_n} + \frac{Q_n}{\gamma_n^2 - 4}a^2\right] \sin\gamma_n = 0.$$

From the orthogonality of the eigenfunctions it follows that the coefficients of the sine functions are zero, that is,

$$A_1 a^2 - \frac{1}{4}Q_1 a^2 \ln a = 0,$$
$$A_n a^{\gamma_n} + \frac{Q_n}{\gamma_n^2 - 4}a^2 = 0, \quad n = 2, 3, \ldots$$

whence we obtain

$$A_1 = \frac{1}{4}Q_1 \ln a, \quad A_n = -\frac{Q_n a^2}{(\gamma_n^2 - 4)a^{\gamma_n}}.$$

[2] You must have seen something similar in your course in ODEs. To solve the equation $y''(t) + y(t) = \sin\omega t$, we look for a particular solution of the form $y_p(x) = A\sin\omega t$. Substituting y_p into the ODE we get $\left(-A\omega^2 + A\right)\sin\omega t = \sin\omega t$, whence $A = \frac{1}{1-\omega^2}$, which works as long as $\omega^2 \neq 1$. When $\omega^2 = 1$ (the resonance case) we look for a particular solution of the form $y_p(t) = t\sin\omega t$.

[3] Here and in the next integration we discard the constant of integration since any particular solution will do.

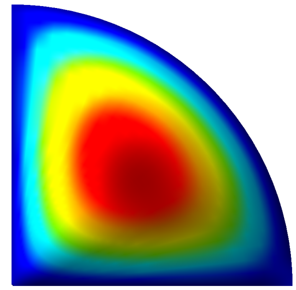
We then evaluate the coefficients in the series (13.22):

$$A_1 r^2 - \frac{1}{4} Q_1 r^2 \ln r = \frac{1}{4} Q_1 r^2 \ln \frac{a}{r}$$
$$A_n r^{\gamma_n} + \frac{Q_n}{\gamma_n^2 - 4} r^2 = \frac{Q_n a^2}{\gamma_n^2 - 4} \left[\left(\frac{r}{a}\right)^2 - \left(\frac{r}{a}\right)^{\gamma_n} \right]$$

We conclude that

$$u(r, \theta) = \frac{1}{4} Q_1 r^2 \ln \frac{a}{r} \sin 2n\theta + \sum_{n=2}^{\infty} \frac{Q_n a^2}{\gamma_n^2 - 4} \left[\left(\frac{r}{a}\right)^2 - \left(\frac{r}{a}\right)^{\gamma_n} \right] \sin \gamma_n \theta.$$

The adjacent diagram shows the temperature distribution in the lamina. Red is hot, blue is cold.



13.2.1 Fourier series with periodic boundary conditions

When solving Poisson's equation on a quarter-disk (see Figure 13.2 on page 159), the boundary conditions $u = 0$ along the radii $\theta = 0$ and $\theta = \frac{\pi}{2}$, lead to the boundary condition $\Psi(0) = 0$ and $\Psi\left(\frac{\pi}{2}\right) = 0$ in the eigenvalue problem (13.8). The domain's 90-degree angle is not particularly significant in the calculations' logic; these could have been done on a sector of any opening angle α, $0 < \alpha < 2\pi$, and would have merely resulted in changing Ψ's boundary conditions to $\Psi(0) = 0$ and $\Psi(\alpha) = 0$. The same goes for the partial annulus of Example 13.2. A few exercises at the end of this chapter illustrate such variations.

Solving Poisson's equation on the *whole disk* or a *whole annulus* is a different story—there are no straight edges and therefore no radial boundary data, so it seems that we lose the boundary conditions on Ψ's eigenvalue problem. Or do we?

The answer is both yes and no. Yes, we do lose what would have been boundary conditions $\Psi(0) = 0$ and $\Psi(2\pi) = 0$ as we no longer have radial data. And no, we still have data—of a different kind— on $\Psi(0)$ and $\Psi(2\pi)$. To see that, observe that the beginning and ending angles 0 and 2π on a disk are mere mathematical artifacts; the radii along $\theta = 0$ and $\theta = 2\pi$ have no distinguished roles among all other radii on a physical disk. The solution $u(r, \theta)$ of the Poisson problem is not "aware" of any special feature at $\theta = 0$ and $\theta = 2\pi$; it just sees a continuous, if fact differentiable, temperature $u(r, \theta)$ which is *periodic* of period 2π in θ. Consequently, $\Psi(\theta)$ that emerges in the separation of variables process is continuously differentiable and 2π-periodic over $\theta = (-\infty, \infty)$. We may express this as $\Psi(0) = \Psi(2\pi)$, $\Psi'(0) = \Psi'(2\pi)$, or in an equivalent and a more appealing symmetric fashion, $\Psi(-\pi) = \Psi(\pi)$, $\Psi'(-\pi) = \Psi'(\pi)$, and thus arrive at the eigenvalue problem

$$\Psi''(\theta) + \lambda \Psi(\theta) = 0, \quad \Psi(-\pi) = \Psi(\pi), \quad \Psi'(-\pi) = \Psi'(\pi). \tag{13.23}$$

We proceed to analyze this in the usual manner, by considering three possible cases of signs of λ.

Case 1: $\lambda < 0$. To enforce the negativity of λ, take $\lambda = -\gamma^2$, where $\gamma > 0$. Then $\Psi''(-\theta) - \gamma^2\Psi(\theta) = 0$, and therefore

$$\Psi(\theta) = a\cosh\gamma\theta + b\sinh\gamma\theta,$$
$$\Psi'(\theta) = a\gamma\sinh\gamma\theta + b\gamma\cosh\gamma\theta.$$

Applying the boundary conditions results in the equations

$$a\cosh(-\gamma\pi) + b\sinh(-\gamma\pi) = a\cosh(\gamma\pi) + b\sinh(\gamma\pi),$$
$$a\gamma\sinh(-\gamma\pi) + b\gamma\cosh(-\gamma\pi) = a\gamma\sinh(\gamma\pi) + b\gamma\cosh(\gamma\pi),$$

which simplify to

$$b\sinh(\gamma\pi) = 0, \quad a\gamma\sinh(\gamma\pi) = 0.$$

Since $\gamma \neq 0$, these imply $a = b = 0$ and lead to the trivial solution. We thus abandon the $\lambda < 0$ case.

Case 2: $\lambda = 0$. The differential equation is $\Psi''(\theta) = 0$ whose general solution is $\Psi(\theta) = a\theta + b$. The first of the boundary conditions in (13.23) yields $a = 0$ and reduces the solution to $\Psi(\theta) = b$. Then $\Psi'(\theta) = 0$, and therefore the second boundary condition in (13.23) is automatically satisfied. We conclude that $\lambda = 0$ is an eigenvalue and the corresponding eigenfunction may be taken as $\Psi_0(\theta) = 1$.

Case 3: $\lambda > 0$. To enforce the negativity of λ, take $\lambda = \gamma^2$, where $\gamma > 0$. Then $\Psi''(-\theta) + \gamma^2\Psi(\theta) = 0$, and therefore

$$\Psi(\theta) = a\cos\gamma\theta + b\sin\gamma\theta,$$
$$\Psi'(\theta) = -a\gamma\sin\gamma\theta + b\gamma\cos\gamma\theta.$$

Applying the boundary conditions results in the equations

$$a\cos(-\gamma\pi) + b\sin(-\gamma\pi) = a\cos(\gamma\pi) + b\sin(\gamma\pi),$$
$$-a\gamma\sin(-\gamma\pi) + b\gamma\cos(-\gamma\pi) = -a\gamma\sin(\gamma\pi) + b\gamma\cos(\gamma\pi),$$

which simplify to

$$b\sin(\gamma\pi) = 0, \quad a\gamma\sin(\gamma\pi) = 0.$$

These hold for arbitrary a and b as long as $\sin(\gamma\pi) = 0$, that is, if $\gamma\pi$ is an integer multiple of π. So we set $\gamma_n = n$, and conclude that there are infinitely many eigenvalues $\lambda_n = \gamma_n^2 = n^2$, $n = 1, 2, \ldots$. Corresponding to each eigenvalue λ_n, there are *two eigenfunctions*, $\cos\gamma_n\theta = \cos n\theta$ and $\sin\gamma_n\theta = \sin n\theta$, that is, λ_n is an *eigenvalue of multiplicity* 2 when $n = 1, 2, \ldots$.

It can be shown that the eigenfunctions that we have just computed form an orthogonal basis for the space of square-integrable function $L^2(-\pi, \pi)$ and consequently a function $f \in L^2(-\pi, \pi)$ may be expanded into the infinite series

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta. \tag{13.24}$$

This is called the (full) *Fourier series* of the function f. The coefficients may be evaluated through the orthogonality properties of the the eigenfunctions:

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta, \tag{13.25a}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta, \tag{13.25b}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta. \tag{13.25c}$$

13.3 The Poisson problem on the disk

Let us solve the Poisson problem

$$\Delta u + q = 0 \qquad \text{in } \Omega, \tag{13.26a}$$

$$u = f \qquad \text{on } \partial\Omega \tag{13.26b}$$

on the disk

$$\Omega = \{(r, \theta) \mid 0 \le r < a, \quad -\pi \le \theta < \pi\}$$

in polar coordinates, for given forcing function q and boundary data f.

We expand u, q, and f in the series eigenfunctions of the problem (13.23):

$$u(r, \theta) = a_0(r) + \sum_{n=1}^{\infty} a_n(r) \cos n\theta + \sum_{n=1}^{\infty} b_n(r) \sin n\theta, \tag{13.27a}$$

$$q(r, \theta) = \eta_0(r) + \sum_{n=1}^{\infty} \eta_n(r) \cos n\theta + \sum_{n=1}^{\infty} \xi_n(r) \sin n\theta, \tag{13.27b}$$

$$f(\theta) = \phi_0 + \sum_{n=1}^{\infty} \phi_n \cos n\theta + \sum_{n=1}^{\infty} \psi_n \sin n\theta. \tag{13.27c}$$

First, we calculate the coefficients of the expansions of q and f through the formulas (13.25):

$$\eta_0(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} q(r, \theta)\, d\theta, \qquad \phi_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta, \tag{13.28a}$$

$$\eta_n(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} q(r, \theta) \cos n\theta\, d\theta, \quad \phi_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta\, d\theta, \tag{13.28b}$$

$$\xi_n(r) = \frac{1}{\pi} \int_{-\pi}^{\pi} q(r, \theta) \sin n\theta\, d\theta, \quad \psi_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta. \tag{13.28c}$$

Next, we calculate the Laplacian of u in (13.27a) and substitute that, and also the expansion (13.27b), into (13.26a):

$$\begin{aligned}a_0''(r) + \frac{1}{r}a_0'(r) + \sum_{n=1}^{\infty}\Big(a_n''(r) + \frac{1}{r}a_n'(r) - \frac{1}{r^2}a_n(r)n^2\Big)\cos n\theta \\ + \sum_{n=1}^{\infty}\Big(b_n''(r) + \frac{1}{r}b_n'(r) - \frac{1}{r^2}b_n(r)n^2\Big)\sin n\theta \\ + \eta_0(r) + \sum_{n=1}^{\infty}\eta_n(r)\cos n\theta + \sum_{n=1}^{\infty}\xi_n(r)\sin n\theta = 0,\end{aligned}$$

and regroup:

$$\begin{aligned}a_0''(r) + \frac{1}{r}a_0'(r) + \eta_0(r) \\ + \sum_{n=1}^{\infty}\Big(a_n''(r) + \frac{1}{r}a_n'(r) - \frac{1}{r^2}a_n(r)n^2 + \eta_n(r)\Big)\cos n\theta \\ + \sum_{n=1}^{\infty}\Big(b_n''(r) + \frac{1}{r}b_n'(r) - \frac{1}{r^2}b_n(r)n^2 + \xi_n(r)\Big)\sin n\theta = 0.\end{aligned}$$

It follows that

$$a_0''(r) + \frac{1}{r}a_0'(r) + \eta_0(r) = 0, \tag{13.29a}$$

$$a_n''(r) + \frac{1}{r}a_n'(r) - \frac{1}{r^2}a_n(r)n^2 + \eta_n(r) = 0, \quad n = 1, 2, \dots, \tag{13.29b}$$

$$b_n''(r) + \frac{1}{r}b_n'(r) - \frac{1}{r^2}b_n(r)n^2 + \xi_n(r) = 0, \quad n = 1, 2, \dots. \tag{13.29c}$$

These are linear, second order, nonhomogeneous ODEs. Their corresponding homogeneous parts are Euler equations. The general solution of each equation involves two arbitrary constants. We insert the solution, along with the arbitrary constants, into (13.27a), evaluate the result at $r = a$, and match the resulting series against that of the boundary condition f in (13.27c). That determines the values of the arbitrary constants and thus completes the solution of the Poisson problem.

The calculations are mostly mechanical/straightforward. The laborious parts are (a) evaluating the integrals in (13.28), and (b) finding particular solutions of the ODEs (13.29). We illustrate the process by solving a very simple special case in the next section.

13.4 *Disk heated on half of its boundary*

Let us solve the steady-state heat conduction problem on a homogeneous disk of radius a, where the temperature on the disk's lower-half and upper-half boundaries are fixed at 0 and 1, respectively. Assume there is no internal heat source.

This amounts to solving the previous section's BVP with $q \equiv 0$ and

$$f(\theta) = \begin{cases} 0 & \text{if } -\pi \le \theta < 0, \\ 1 & \text{if } 0 \le \theta < \pi. \end{cases} \tag{13.30}$$

Referring to (13.28), we see that $\eta_0(r)$, $\eta_n(r)$, and $\xi_n(r)$ are zero, and

$$\phi_0 = \frac{1}{2\pi}\int_0^\pi 1\, d\theta = \frac{1}{2}, \tag{13.31a}$$

$$\phi_n = \frac{1}{\pi}\int_0^\pi \cos n\theta\, d\theta = \frac{1}{n\pi}\sin n\pi = 0, \tag{13.31b}$$

$$\psi_n = \frac{1}{\pi}\int_0^\pi \sin n\theta\, d\theta = \frac{1}{n\pi}\Big[1 - \cos n\pi\Big] = \frac{1}{n\pi}\Big[1-(-1)^n\Big]. \tag{13.31c}$$

Since $\eta_0(r) = 0$, equation (13.29a) reduces to $a_0''(r) + \frac{1}{r}a_0'(r) = 0$ whose general solution is $a_0(r) = A_0 + \tilde{A}_0 \ln r$. To avoid blowup at $r = 0$, we take $\tilde{A}_0$ and arrive at $a_0(r) = A_0$.

Since $\eta_n(r) = 0$, equation (13.29b) reduces to $a_n''(r) + \frac{1}{r}a_n'(r) - \frac{1}{r^2}a_n(r)n^2 = 0$ whose general solution is $a_n(r) = A_n r^n + \tilde{A}_n r^{-n}$. To avoid blowup at $r = 0$, we take $\tilde{A}_n$ and arrive at $a_n(r) = A_n r^n$. Similarly, we solve (13.29c) and obtain $b_n(r) = B_n r^n$. Plugging these into (13.27a) we arrive at

$$u(r,\theta) = A_0 + \sum_{n=1}^\infty A_n r^n \cos n\theta + \sum_{n=1}^\infty B_n r^n \sin n\theta. \tag{13.32}$$

We evaluate this at $r = a$ and equate the result to (13.27c)

$$A_0 + \sum_{n=1}^\infty A_n a^n \cos n\theta + \sum_{n=1}^\infty B_n a^n \sin n\theta = \phi_0 + \sum_{n=1}^\infty \phi_n \cos n\theta + \sum_{n=1}^\infty \psi_n \sin n\theta,$$

and, taking (13.31) into consideration, conclude that

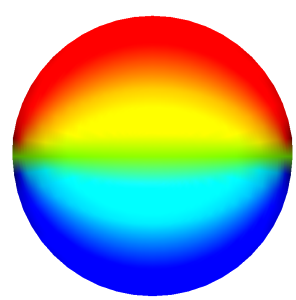
$$A_0 = \phi_0 = \frac{1}{2}, \quad A_n a^n = \phi_n = 0, \quad B_n a^n = \psi_n = \frac{1}{n\pi}\Big[1-(-1)^n\Big].$$

At last, plugging these into (13.32) we arrive at the final solution:

$$u(r,\theta) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^\infty \frac{1}{n}\Big[1-(-1)^n\Big]\Big(\frac{r}{a}\Big)^n \sin n\theta.$$

The expression in the square brackets alternates between zero and two, so it suffices to sum over the odd integers only. We get

$$u(r,\theta) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{2n-1}\Big(\frac{r}{a}\Big)^{2n-1} \sin\big((2n-1)\theta\big).$$



The adjacent diagram shows the temperature distribution in the disk. Red is hot, blue is cold.

13.5 Exercises

13.1. Along the lines of Section 13.1's calculations, show that

$$u_{xy} = \frac{1}{2}\Big(U_{rr} - \frac{1}{r}U_r - \frac{1}{r^2}U_{\theta\theta}\Big)\sin 2\theta + \frac{1}{r}\Big(U_{r\theta} - \frac{1}{r}U_\theta\Big)\cos 2\theta.$$

13.2. Consider a homogeneous lamina in the shape of half a disk of radius a defined in polar coordinates as

$$\Omega = \Big\{(r,\theta) \mid 0 \le r \le a, \;\; 0 \le \theta \le \pi\Big\}.$$

The temperature is fixed at zero along the domain's straight edge but it is prescribed as $u(a,\theta) = f(\theta)$ along the curved edge. Find the steady-state temperature distribution $u(r,\theta)$ in the lamina. Assume that there is no internal heat source.

13.3. Consider a lamina in the form of a half-annulus:

$$\Omega = \Big\{(r,\theta) \mid a \le r \le b, \;\; 0 \le \theta \le \pi\Big\}$$

The temperature along the outer curved boundary is $u(b,\theta) = \sin\theta$, but it's zero on the remaining parts of the boundary. Find the steady-state temperature distribution $u(r,\theta)$ in the lamina. Assume that there is no internal heat source.

13.4. Consider the half-disk lamina of Exercise (13.2). Suppose that the temperature is fixed at zero all around Ω's boundary but there is a uniformly distributed internal heat source $q \equiv 1$ in Ω. Find the steady-state temperature distribution.

13.5. In Section 13.2.1 it was asserted that the eigenfunctions 1, $\cos n\theta$, and $\sin n\theta$ are mutually orthogonal in $L^2(-\pi,\pi)$. Verify that assertion, that is, show that

$$\begin{aligned} &\int_{-\pi}^{\pi} 1 \cdot \cos n\theta \, d\theta = 0, \\ &\int_{-\pi}^{\pi} 1 \cdot \sin n\theta \, d\theta = 0, \\ &\int_{-\pi}^{\pi} \sin m\theta \sin n\theta \, d\theta = 0, \quad m \ne n. \end{aligned}$$

13.6. Find the full Fourier series expansion of the function f, defined over $[-\pi,\pi]$ such that

$$f(\theta) = \begin{cases} 1 & \text{if } 0 \le \theta \le \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

13.7. Find the steady-state temperature distribution on a homogeneous disk of radius a, where the temperature on the quarter-boundary $0 \le \theta \le \frac{\pi}{2}$ is set to 1, and zero on the rest of the boundary. Assume there is no internal heat source.

13.8. Consider the steady-state conduction of heat in the annulus

$$\Omega = \Big\{(r,\theta) \mid a < r < b, \quad -\pi \leq \theta < \pi\Big\}$$

Suppose that the temperature of the inner boundary is fixed at zero, while that of the outer boundary is 0 and 1 on the lower and upper halves, respectively. Find the temperature distribution within the annulus, assuming that there are no internal heat sources.

14
Time-dependent PDEs in 2D

In Section 12.2 we learned how to expand a function q defined over the rectangle into a doubly-infinite Fourier series. For ease of reference, we duplicate the the results here.

For any $a > 0$, $b > 0$, consider pair of eigenvalue problems

$$X''(x) + \lambda X(x) = 0, \qquad Y''(y) + \mu Y(y) = 0, \tag{14.1a}$$

$$X(0) = X(a) = 0, \qquad Y(0) = Y(b) = 0, \tag{14.1b}$$

each of which has infinitely many eigenvalues and eigenfunctions given by

$$\gamma_m = \frac{m\pi}{a}, \qquad \nu_n = \frac{n\pi}{b}, \tag{14.2a}$$

$$\lambda_m = \gamma_m^2 = \left(\frac{m\pi}{a}\right)^2, \qquad \mu_n = \nu_n^2 = \left(\frac{n\pi}{b}\right)^2, \tag{14.2b}$$

$$X_m(x) = \sin \gamma_m x = \sin \frac{m\pi x}{a}, \qquad Y_n(y) = \sin \nu_n y = \sin \frac{n\pi y}{b}, \tag{14.2c}$$

$$\|X_m\|^2 = \int_0^a X_m(x)^2 \, dx = \frac{1}{2}a, \qquad \|Y_n\|^2 = \int_0^b Y_n(y)^2 \, dy = \frac{1}{2}b. \tag{14.2d}$$

Let q be a function defined over the rectangle

$$\Omega = \{(a,b) \mid 0 < x < a, \quad 0 < y < b\}. \tag{14.3}$$

If q is square-integrable[1], then it can be expanded into a doubly-infinite series of the eigenfunctions

[1] Meaning $\iint_\Omega q(x,y)^2 \, dx \, dx < \infty$.

$$q(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} X_m(x) Y_n(y), \tag{14.4}$$

where the coefficients Q_{mn} are given by

$$Q_{mn} = \frac{4}{ab} \iint_\Omega q(x,y) X_m(x) Y_n(y) \, dx \, dy, \quad m,n = 1,2,\ldots. \tag{14.5}$$

This is the key to solving time-dependent PDEs in 2D. We illustrate this through examples.

14.1 The vibration of a rectangular membrane

It was noted in Section 12.1 that the The transverse displacement u of a taut membrane solves the PDE (12.2). If the membrane is stretched over the rectangle Ω defined in (14.3), is glued/fixed along the rectangle's edges, and no external force is applied to it, then the the displacement $u(x,y,t)$ is the solution of the following IBVP:

$$u_{tt} = c^2(u_{xx}+u_{yy}) \qquad (x,y)\in\Omega, \quad t>0, \tag{14.6a}$$

$$u(x,y,t)=0 \qquad (x,y)\in\partial\Omega, \quad t>0, \tag{14.6b}$$

$$u(x,y,0)=f(x,y) \qquad (x,y)\in\Omega, \tag{14.6c}$$

$$u_t(x,y,0)=g(x,y) \qquad (x,y)\in\Omega, \tag{14.6d}$$

where f and g are the membrane's initial displacement and velocity.

At any time t, the displacement is a function of $(x,y)\in\Omega$, and according to (14.4) it has the series representation

$$u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{mn}(t)X_m(x)Y_n(y). \tag{14.7}$$

To determine the coefficients $T_{mn}(t)$, we substitute (14.7) into the PDE (14.6a):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T''_{mn}(t)X_m(x)Y_n(y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c^2\big(T_{mn}(t)X''_m(x)Y_n(y)+T_{mn}(t)X_m(x)Y''_n(y)\big).$$

But according to (14.1) and (14.2) we have

$$X''_m(x) = -\lambda_m X_m(x) = -\gamma_m^2 X_m(x),$$
$$Y''_n(y) = -\mu_n Y_n(y) = -\nu_n^2 Y_n(y),$$

and therefore the previous result reduces to

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T''_{mn}(t)X_m(x)Y_n(y) = -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c^2(\gamma_m^2+\nu_n^2)T_{mn}(t)X_m(x)Y_n(y).$$

We introduce

$$\sigma_{mn}^2 = c^2(\gamma_m^2+\nu_n^2), \tag{14.8}$$

and rearrange the result into

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(T''_{mn}(t)+\sigma_{mn}^2 T_{mn}(t)\Big)X_m(x)Y_n(y)=0.$$

Then based on the orthogonality of the eigenfunctions, we conclude that

$$T''_{mn}(t)+\sigma_{mn}^2T_{mn}(t)=0, \quad m,n=1,2,\dots,$$

and therefore

$$T_{mn}(t) = A_{mn} \cos \sigma_{mn} t + B_{mn} \sin \sigma_{mn} t,$$

where the constants A_{mn} and B_{mn} are to be determined.

The series expansion (14.7) of the displacement u now takes the form

$$u(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big(A_{mn} \cos \sigma_{mn} t + B_{mn} \sin \sigma_{mn} t \Big) X_m(x) Y_n(y), \quad (14.9a)$$

which we differentiate with respect to t to find the velocity

$$u_t(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Big(-A_{mn}\sigma_{mn} \sin \sigma_{mn} t + B_{mn}\sigma_{mn} \cos \sigma_{mn} t \Big) X_m(x) Y_n(y).$$

Evaluating the displacement and velocity at $t = 0$ and applying the initial conditions (14.6c) and (14.6d) lead to

$$f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} X_m(x) Y_n(y),$$
$$g(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}\sigma_{mn} X_m(x) Y_n(y).$$

The coefficients A_{mn} and B_{mn} then may be calculated in accordance with (14.5):

$$A_{mn} = \frac{4}{ab} \iint_\Omega f(x,y) X_m(x) Y_n(y)\, dx\, dy, \quad (14.9b)$$
$$B_{mn} = \frac{4}{ab\sigma_{mn}} \iint_\Omega g(x,y) X_m(x) Y_n(y)\, dx\, dy. \quad (14.9c)$$

Equations (14.9a) through (14.9c) completely determine the membrane's motion.

Example 14.1. Let us work out the details of a concrete case of a membrane with the parameters $a = b = 8$, $c = 1$, zero initial displacement f, and the initial velocity g is given by

$$g(x,y) = \begin{cases} 1 & \text{if } |x-4| < 1 \text{ and } |y-4| < 1, \\ 0 & \text{otherwise.} \end{cases}$$

This corresponds to hitting an 8×8 membrane at its center with a square mallet of 2×2 cross-section.

Solution. According to (14.2), we have

$$\gamma_m = \frac{m\pi}{8}, \quad \nu_n = \frac{n\pi}{8},$$
$$X_m(x) = \sin \gamma_m x = \sin \frac{m\pi x}{8}, \quad Y_n(y) = \sin \nu_n y = \sin \frac{n\pi y}{8}.$$

The coefficients A_{mn} are all zero because $f = 0$. In calculating B_{mn} from (14.9c), the integration of g over $[0, 8]$ reduces to integration over $[3, 5]$ and we obtain

$$B_{mn} = \frac{4}{ab\sigma_{mn}} \left(\int_3^5 \sin \gamma_m x \, dx \right) \left(\int_3^5 \sin \nu_n y \, dy \right).$$

We evaluate the integral of X_m

$$\begin{aligned} \int_3^5 \sin \gamma_m x \, dx &= -\frac{1}{\gamma_m} \cos \gamma_m x \Big|_3^5 \\ &= -\frac{1}{\gamma_m} \Big[\cos 5\gamma_m - \cos 3\gamma_m \Big] = \frac{2}{\gamma_m} \sin \gamma_m \sin 4\gamma_m, \end{aligned}$$

and similarly the integral of Y_n, and arrive at

$$B_{mn} = \frac{16}{ab\sigma_{mn}\gamma_m \nu_n} \sin \gamma_m \sin 4\gamma_m \sin \gamma_n \sin 4\gamma_n.$$

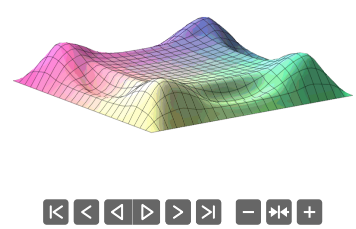
Finally, substituting for the known values of a, b, γ_m, γ_n, and σ_{mn}, we arrive at

$$B_{mn} = \frac{128}{\pi^3 mn\sqrt{m^2+n^2}} \sin \frac{m\pi}{8} \sin \frac{m\pi}{2} \sin \frac{n\pi}{8} \sin \frac{n\pi}{2}.$$

Then, in view of (14.9a), we arrive at

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{8} \sin \frac{n\pi y}{8} \sin \frac{\pi\sqrt{m^2+n^2}}{8} t. \qquad \square$$

On the right we have an animation of the solution produced by taking $m = n = 30$, over the time period $0 \le t \le 60$.



Remark 14.1. The solution is a sum of sinusoidal vibrations of angular frequencies $\frac{\pi\sqrt{m^2+n^2}}{8}$. The ratios of these frequencies are irrational numbers in general, and therefore the solution *is not* periodic in time. That's why a drum is not a suitable instrument for producing musical tones.

14.2 *The vibration of a circular membrane*

Let us repeat the previous section's calculation replacing the rectangular member by a circular one. So we replace the domain Ω by the disk of radius a expressed in polar coordinates as

$$\Omega = \big\{ (r, \theta) \mid 0 \le r < a, \quad -\pi \le \theta < \pi \big\}.$$

Then the membrane's transverse displacement $u(r,\theta,t)$ solves the IBVP

$$u_{tt} = c^2\Big[u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\Big] \qquad \text{in } \Omega, \tag{14.10a}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{14.10b}$$

$$u(r,\theta,0) = f(r,\theta) \qquad t > 0, \tag{14.10c}$$

$$u_t(r,\theta,0) = g(r,\theta) \qquad t > 0. \tag{14.10d}$$

As usual, we begin searching for a separable solution of the form

$$u(r,\theta,t) = R(r)\Psi(\theta)T(t) \tag{14.11}$$

that satisfies the PDE (14.10a) and the boundary condition (14.10b). Substituting that into the PDE we get

$$R(r)\Psi(\theta)T''(t) = c^2\Big[R''(r)\Psi(\theta)T(t) + \frac{1}{r}R'(r)\Psi(\theta)T(t) + \frac{1}{r^2}R(r)\Psi''(\theta)T(t)\Big],$$

which we separate into

$$\frac{T''(t)}{c^2T(t)} = \frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Psi''(\theta)}{r^2\Psi(\theta)} = -\gamma^2,$$

where $\gamma > 0$ is the separation constant, and thus we arrive at the pair of equations

$$T''(t) + \gamma^2c^2T(t) = 0, \tag{14.12a}$$

and

$$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} + \frac{\Psi''(\theta)}{r^2\Psi(\theta)} = -\gamma^2.$$

We multiply through the second equation by r^2 and rearrange it into separated form

$$-\frac{\Psi''(\theta)}{\Psi(\theta)} = \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} + r^2\gamma^2 = \nu^2,$$

where $\nu > 0$ is a second separation constant. It follows that

$$\Psi''(\theta) + \nu^2\Psi(\theta) = 0, \tag{14.12b}$$

and

$$r^2R''(r) + rR'(r) + (\gamma^2r^2 - \nu^2)R(r) = 0. \tag{14.12c}$$

We now turn our attention to solving the ODEs (14.12a), (14.12b), and (14.12c). Equation (14.12b) is the easiest to handle. The displacement $u(r,\theta,t$ should be a 2π-periodic function in θ, and therefore we have the periodicity conditions

$$\Psi(-\pi) = \Psi(\pi), \quad \Psi'(-\pi) = \Psi'(\pi),$$

The eigenvalue problem consisting of the ODE (14.12b) and this periodicity condition was thoroughly analyzed in Section (13.2.1). We concluded there that and concluded that corresponding to each $\nu = \nu_n = n$, $n = 1, 2, \ldots$, there exist two eigenfunctions $\cos n\theta$ and $\sin n\theta$. Additionally, corresponding to $\nu = \nu_0 = 0$ there exists an eigenfunction which is just a constant, say 1 as a function of θ.

Having determined that ν is an integer, we substitute $\nu = n$ into the ODE (14.12c), where $n = 0, 1, 2, \ldots$:

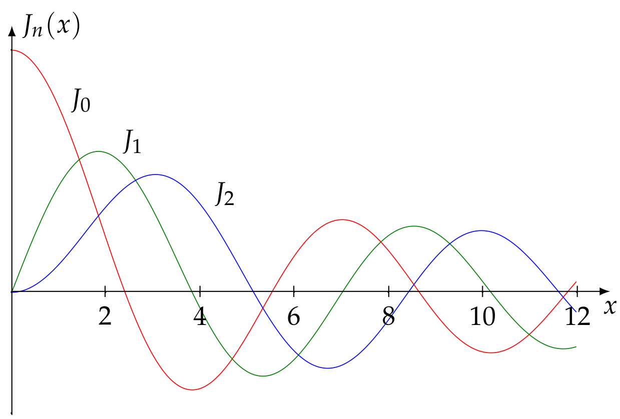
$$r^2 R''(r) + rR'(r) + (\gamma^2 r^2 - n^2)R(r) = 0. \tag{14.13}$$

This is the well-known *Bessel's differential equation*. It has no solution in terms of elementary functions but its properties have been studies thoroughly as it arises in a slew of applications in mathematics and physics. As with all second order differential equations, its solution are presented as linear combination of an independent pair fundamental solutions

$$R(r) = c_1 J_n(\gamma r) + c_2 Y_n(\gamma r). \tag{14.14}$$

where J_n and Y_n are known as *Bessel functions of the first and second kind* respectively.[2] In the old times, one looked up the values of J_n and Y_n in extensively detailed numerical tables. Nowadays one looks up their values on a computer. Programming languages such as Fortran and C, and software such as Maple, Mathematica, and Matlab, have built-in functions that provide the values of the Bessel function for any desired index and argument. Figure 14.1 shows the graphs of the Bessel functions J_n and Y_n for $n = 0, 1, 2$. Both functions oscillate indefinitely and gradually converge to zero as their arguments grow. The J_n functions are bounded while the Y_n functions go to $-\infty$ near zero.

[2] The indices n in J_n and Y_n refer to the n that appears in (14.13). The index is an integer in our application, but it need not be on other applications. The Bessel functions J_α and Y_α are defined for all α where α can be any real of complex number.



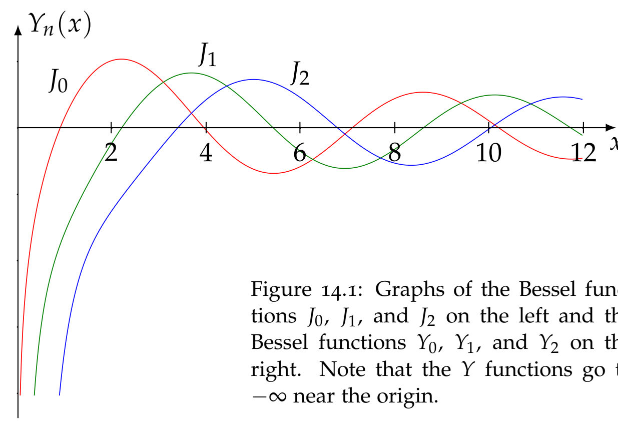


Figure 14.1: Graphs of the Bessel functions J_0, J_1, and J_2 on the left and the Bessel functions Y_0, Y_1, and Y_2 on the right. Note that the Y functions go to $-\infty$ near the origin.

There is no explicit formula for the roots of the Bessel functions. They have been tabulated, and are also available in most software that know about the Bessel functions. The first two roots of J_0, for instance, are approximately 2.4048 and 5.5201. Compare those with what you can discern in Figure 14.1. In what follows we write j_{mn} for the mth root of J_n, where $m = 1, 2, \ldots$.

Let us now return to the differential equation (14.13) and its general solution (14.14). Since the Y_n part blows up at the origin, we set c_2 to zero to avoid the singularity. The solution (14.11) now has the form

$$u(r, \theta, t) = J_n(\gamma r)\Psi(\theta)T(t).$$

Since the displacement on the boundary is zero—see (14.10b)—we need $u(a, \theta, t) = 0$ for all θ and t. That necessitates $J_n(\gamma a) = 0$, which in turn requires γa to be a root of J_n, that is, $\gamma a = j_{mn}$ for any m. We define

$$\gamma_{mn} = \frac{j_{mn}}{a}, \quad m = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots. \tag{14.15}$$

Having determined γ, we turn to the differential equation (14.12a) which now has the form

$$T''(t) + \gamma mn^2 c^2 T(t) = 0,$$

and whose general solution is

$$T_{mn}(t) = A_{mn} \cos \gamma_{mn} ct + B_{mn} \sin \gamma_{mn} ct.$$

. . . details to be filled in later . . .

We conclude that

$$\begin{aligned} u(r, \theta, t) = &\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{mn} J_n(\gamma_{mn} r) \cos n\theta \cos \gamma_{mn} ct \\ &+ \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} B_{mn} J_n(\gamma_{mn} r) \cos n\theta \sin \gamma_{mn} ct \\ &+ \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{mn} J_n(\gamma_{mn} r) \sin n\theta \cos \gamma_{mn} ct \\ &+ \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} D_{mn} J_n(\gamma_{mn} r) \sin n\theta \sin \gamma_{mn} ct. \end{aligned}$$

The coefficients A_{mn}, B_{mn}, C_{mn}, D_{mn} are to be determined by enforcing the initial conditions (14.10c) and (14.10d).

14.3 *Exercises*

14.1. Let's write $u^{(1)}$ for the solution of the IBVP (14.6) corresponding to the initial data $f = f_1$, $g = g_1$, and write $u^{(2)}$ for the solution of the IBVP (14.6) corresponding to the initial data $f = f_2$, $g = g_2$. Show that $u^{(1)} + u^{(2)}$ is the solution of that IBVP corresponding to the initial data $f = f_1 + f_2$, $g = g_1 + g_2$.

14.2. Evaluate the solution (14.9) of the IBVP (14.6) on the domain $\Omega = \left\{(x, y) \mid 0 < x < \pi, \quad 0 < y < \pi\right\}$ and the initial data $f(x, y) = \sin x \sin y$ and $g \equiv 0$. Take $c = 1$. Is the solution periodic in time?

14.3. Repeat Exercise 14.2 with the initial data $f(x, y) = \sin 2x \sin 3y$ and $g \equiv 0$. Is the solution periodic in time?

14.4. Repeat Exercise 14.2 with the initial data $f(x, y) = \sin x \sin y + \sin 2x \sin 3y$ and $g \equiv 0$. Is the solution periodic in time?

Hint: The conclusions of exercises 14.1 through 14.3 can be of help.

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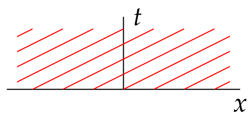
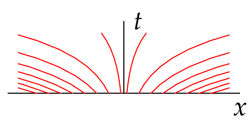
Answers to selected exercises

Chapter 1

1.1. $u(x,t) = f(x-2t)$



1.2. $u(x,t) = f\left(xe^{-t}\right)$



1.3. $u(x,t) = f(x-2t)e^{-t}$

1.4. $u(x,t) = f(xe^{-t}) + t$

1.5. $u(x,t) = f(xe^{-t})e^{-2t}$

1.6. $u(x,t) = f(xe^{-t^2})$

1.7. $u(x,t) = f(xe^{-t^2})\, e^{-t}$

1.8. $u(x,t) = f(x-t^2) + t^2$

1.9. $u(x,t) = f(x+2t^3)e^{-t^2/2}$

1.10. $u(x,t) = f(x+4t^3)e^{-t(x+3t^3)}$

1.11. $u(x,t) = f\left(x - \frac{1}{2}t^2\right) + tx - \frac{1}{3}t^3$

1.12. $u(x,t) = f\left(1 + (x-t-1)e^{-t}\right) - (x-t-1)e^{-t} + \frac{1}{2}t^2 + x - 1$

1.13. $u(x,t) = \dfrac{2-t^2}{2+2x+2t}$

1.15. $u(x,t) = \dfrac{2t^3+3t^2+6x+6}{6(t+1)}$

1.16. $u(x,t) = \dfrac{xe^{2t}-2e^t+2x}{e^{2t}-2}.$

1.17. $u(x,t) = \dfrac{2x\left(e^{4t}-3\right)}{e^{4t}+3}$

1.18. $u(x,t) = \dfrac{3xe^{4t}+x}{e^{4t}-2}$

Chapter 2

2.2.

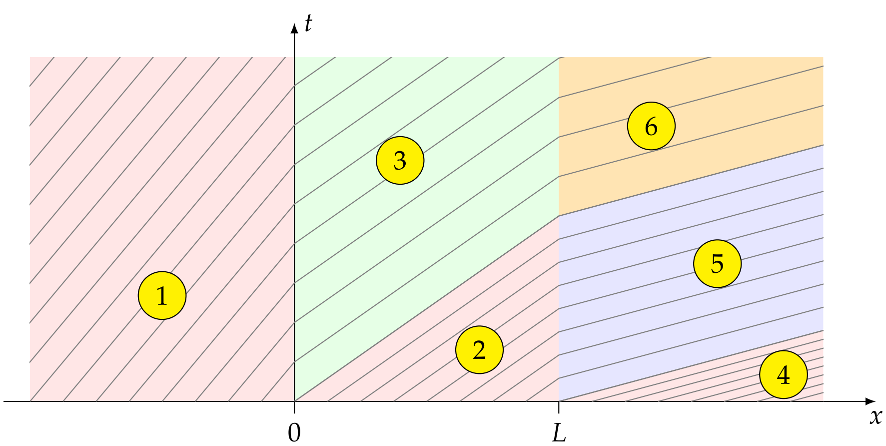
$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) = -\beta\rho(x,t)$$

2.4.

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x}\phi(x,t) = -\frac{A'(x)}{A(x)}\phi(x,t).$$

2.5.

$$\rho(x,t) = \begin{cases} f(x-c_1t) & \text{in region 1,} \\ f(x-c_2t) & \text{in region 2,} \\ \frac{c_1}{c_2}f\left(\frac{c_1}{c_2}(x-c_2t)\right) & \text{in region 3,} \\ f(x-c_3t) & \text{in region 4,} \\ \frac{c_2}{c_3}f\left(L+\frac{c_2}{c_3}(x-c_3t)\right) & \text{in region 5,} \\ c_1f\left(\frac{c_1}{c_2}\left(L+\frac{c_2}{c_3}(x-c_3t)\right)\right) & \text{in region 6.} \end{cases}$$



Chapter 3

3.1. $\rho(x,t) = \begin{cases} 5 & \text{if } x < -t, \\ 4-\frac{x}{t} & \text{if } -t < x < t, \\ 3 & \text{if } x > t. \end{cases}$

3.2. $\rho(x,t) = \begin{cases} 3 & \text{if } x < t, \\ 4 - \frac{x}{t} & \text{if } t < x < 3t, \\ 1 & \text{if } x > 3t. \end{cases}$

3.3. $\rho(x,t) = \begin{cases} 7 & \text{if } x < -3t, \\ 4 - \frac{x}{t} & \text{if } -3t < x < -t, \\ 5 & \text{if } x > -t. \end{cases}$

3.4. $\rho(x,t) = \begin{cases} 3 & \text{if } x < t, \\ \frac{8t+3-2x}{1+2t} & \text{if } t < x < 3t+1, \\ 1 & \text{if } x > 3t+1. \end{cases}$

3.5. $\rho(x,t) = \begin{cases} 7 & \text{if } x < -3t, \\ \frac{8t+7-2x}{1+2t} & \text{if } -3t < x < 1-t, \\ 5 & \text{if } x > 1-t. \end{cases}$

3.6.

$$\text{When } 0 \le t < 1: \qquad \rho(x,t) = \begin{cases} 2 & x < 2t, \\ \frac{x+2-4t}{1-t} & 2t < x < 3-t, \\ 5 & x > 3-t. \end{cases}$$

$$\text{When } t \ge 1: \qquad \rho(x,t) = \begin{cases} 2 & x < \frac{1}{2}t + \frac{3}{2}, \\ 5 & x > \frac{1}{2}t + \frac{3}{2}. \end{cases}$$

3.7.

$$\text{When } 0 \le t < 1: \qquad \rho(x,t) = \begin{cases} 2 & x < 2t, \\ \frac{2x+2-8t}{1-2t} & 2t < x < 3-4t, \\ 8 & x > 3-4t. \end{cases}$$

$$\text{When } t \ge 1: \qquad \rho(x,t) = \begin{cases} 2 & x < \frac{3}{2} - t, \\ 8 & x > \frac{3}{2} - t. \end{cases}$$

3.8.

$$\text{When } 0 \le t < 2: \qquad \rho(x,t) = \begin{cases} 5 & x \le -\frac{1}{2}t, \\ \frac{2x+10-4t}{2-t} & -\frac{1}{2}t < x \le 2 - \frac{3}{2}t, \\ 7 & x > 2 - \frac{3}{2}t. \end{cases}$$

$$\text{When } t \ge 2: \qquad \rho(x,t) = \begin{cases} 5 & x < 1-t, \\ 7 & x > 1-t. \end{cases}$$

3.10. $c(\rho) = 1 - 3\rho^2, \quad \rho(x,t) = \begin{cases} 1 & x < -2t, \\ \sqrt{\frac{1}{3}\left(1 - \frac{x}{t}\right)} & -2t < x < t, \\ 0 & x > t. \end{cases}$

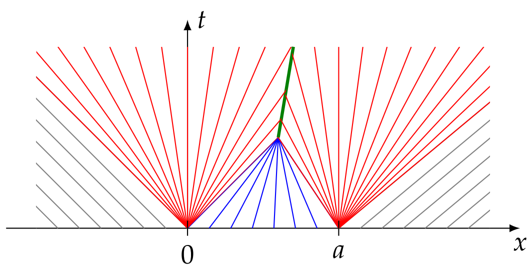
3.11. If $t \le 6/5$, then

$$\rho(x,t) = \begin{cases} 2 & x < \frac{3}{2}t, \\ 3 & \frac{3}{2}t < x < 3 - t, \\ 7 & x > 3 - t, \end{cases}$$

and if $t > 6/5$, then

$$\rho(x,t) = \begin{cases} 2 & x < \frac{12}{5} - \frac{1}{2}t, \\ 7 & x > \frac{12}{5} - \frac{1}{2}t. \end{cases}$$

3.12. The shock is initiated at $x = \dfrac{1-2\alpha}{2(\beta-\alpha)}a,\ \ t = \dfrac{a}{2(\beta-\alpha)v_{\max}}$, and its equation is $x = \dfrac{a}{2} + (1-\alpha-\beta)v_{\max}t$.



3.13. The equation of the curved shock:

$$x(t) = v_{\max}t - 2\sqrt{Lv_{\max}t}, \quad t \ge L/v_{\max}$$

3.14. $$\rho(x,t) = \begin{cases} 0 & \text{if } x > t, \\ \dfrac{1}{2} - \dfrac{1}{8}\left(\dfrac{x}{t}\right)^2 - \dfrac{1}{8}\dfrac{x}{t}\sqrt{8 + \left(\dfrac{x}{t}\right)^2}, & \text{if } x < t. \end{cases}$$

3.17. $$\rho(x,t) = \begin{cases} 0 & \text{if } x < -\frac{1}{4}v_{\max}\,t, \\ \frac{1}{3}\rho_{\max}\left[2 - \sqrt{1 + \frac{3}{v_{\max}} \cdot \frac{x}{t}}\right] & \text{if } -\frac{1}{4}v_{\max} < x < v_{\max}\,t, \\ 0 & \text{if } x > v_{\max}\,t \end{cases}$$

3.18. $$\rho(x,t) = \begin{cases} \rho_{\max} & \text{if } x < -\frac{1}{2}v_{\max}\,t, \\ \frac{4}{9}\,\rho_{\max}\left(1 - \frac{x}{v_{\max}\,t}\right)^2 & \text{if } -\frac{1}{2}v_{\max}\,t < x < v_{\max}\,t, \\ 0 & \text{if } x > v_{\max}\,t. \end{cases}$$

Chapter 4

4.1. (a) linear; (b) linear; (c) not linear

4.2. (a) hyperbolic; (b) elliptic; (c) hyperbolic

4.3. Elliptic when $x > 0$, hyperbolic when $x < 0$.

4.1. Elliptic in the in the subsonic ($v < c$) region, hyperbolic in the supersonic ($v > c$) region.

Chapter 5

5.2. $$u(x,t) = \begin{cases} 0 & \text{if } x < -ct, \\ \frac{1}{2c}(x+ct) & \text{if } -ct < x < ct, \\ t & \text{if } x > ct, \end{cases}$$

5.4. $g(x) = u_t(x,0) = -cf'(x)$

5.5. $u_{tt} = g\big(xu_x\big)_x$ or equivalently $u_{tt} = g\big(xu_{xx} + u_x\big)$, where g is the gravitational acceleration constant.

5.7. $$u(x,t) = \frac{1}{2c}\Big[\arctan(x+ct) - \arctan(x-ct)\Big]$$

5.8. $$u(x,t) = \frac{1}{c}\sin ct \cos x$$

Chapter 6

6.2. $$T = \frac{2a}{c}$$

6.3. $$T = \frac{2a}{c}$$

6.4. $$u(x,t) = \begin{cases} 0 & x > ct \\ A\sin\big(\frac{\omega}{c}(ct-x)\big) & x \le ct \end{cases}$$

6.5. $$u(x,t) = \begin{cases} 0 & x > ct \\ \phi\Big(\frac{ct-x}{c}\Big) & x \le ct \end{cases}$$

6.6. $u(x,t) = f(ct+x) + f(ct-x)$

6.7. $$U(t) = \frac{2T}{mc^2}\int_0^{ct} e^{-\frac{T}{mc}(t-\frac{\xi}{c})} f(\xi)\,d\xi$$
$$g(\xi) = U\Big(\frac{\xi}{c}\Big) - f(\xi)$$

6.8. $$U(t) = \begin{cases} 0 & \text{if } ct < a \\ 2\Big(1 - e^{-\frac{T}{mc^2}(ct-a)}\Big) & \text{if } ct > a \end{cases}$$

6.11. $U(t) = \int_0^{ct} e^{-\frac{k}{2T}(ct-\xi)} f'(\xi)\,d\xi$
or after an integration by parts:
$U(t) = f(ct) - \frac{k}{2T}\int_0^{ct} e^{\frac{k}{2T}(ct-\xi)} f(\xi)\,d\xi$

6.13. Case 1: $\begin{cases} 3H(3t+x-6)e^{-2t-2x/3+4} - 3H(3t+x-6)e^{-t-x/3+2} + H(3t+x-6) & \text{if } x<0 \\ H(3t+x-6) + 3H(3t-x-6)e^{-2t+2x/3+4} - 3H(3t-x-6)e^{-t+x/3+2} & \text{if } x>0 \end{cases}$

Case 2: $\begin{cases} H(2t+x-6) - 2H(2t+x-6)e^{-t-x/2+3}\sin(t+x/2-3) & \text{if } x<0 \\ H(2t+x-6) - 2H(2t-x-6)e^{-t+x/2+3}\sin(t-x/2-3) & \text{if } x>0 \end{cases}$

Chapter 7

7.9. $u(x,t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{x+1}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{4kt}}\right)\right].$

7.10. $u(x,t) = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-2kt}{\sqrt{4kt}}\right)\right] e^{kt-x}$

7.11. $u(x,t) = \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{x-2kt}{\sqrt{4kt}}\right)\right] e^{kt-x} + \frac{1}{2}\left[1+\operatorname{erf}\left(\frac{-x-2kt}{\sqrt{4kt}}\right)\right] e^{kt+x}$

7.12. $u(x,t) = \frac{e^{ht}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}} f(s)\,ds$

7.13. $u(x,t) = \frac{e^{\frac{cx}{2k}-\frac{c^2t}{4k}}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}-\frac{c}{2k}s} f(s)\,ds$

7.14. $u(x,t) = \frac{e^{\frac{cx}{2k}-\left(\frac{c^2}{4k}-h\right)t}}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-s)^2}{4kt}-\frac{c}{2k}s} f(s)\,ds$

7.16. $u(x,t) = u_0\operatorname{erf}\left(\frac{x}{\sqrt{4kt}}\right)$

Chapter 8

8.5. $c = -\frac{\pi}{2}$

8.6. $f(x) = 1, \quad g(x) = x - \frac{1}{2}, \quad h(x) = x^2 - x + \frac{1}{6}$

8.7.

$$\Phi_{m,n} = \begin{cases} \pi & m=n=0, \\ \frac{\pi}{2} & m=n\neq 0, \\ 0 & \text{otherwise} \end{cases}$$

8.8. $\gamma_n = \frac{(2n-1)\pi}{2\ell}, \quad \lambda_n = \gamma_n^2, \quad y_n(x) = \sin\gamma_n x = \sin\frac{(2n-1)\pi x}{2\ell}, \quad n = 1,2,\ldots.$

8.9. $\gamma_n = \frac{n\pi}{\ell}, \quad \gamma_n = \gamma_n^2, \quad y_n(x) = \cos\gamma_n x = \cos\frac{n\pi x}{\ell}, \quad n = 0,1,2,\ldots.$

Chapter 9

9.1. $u(x,t) = e^{-\pi^2 t} \sin \pi x$

9.2. $u(x,t) = -\dfrac{8}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^n\, n}{4n^2-1} e^{-n^2 t} \sin nx$

9.3. $u(x,t) = \dfrac{8}{\pi} \sum_{n=1}^{\infty} \dfrac{n}{4n^2-1} e^{-n^2 t} \sin nx$

9.4. $u(x,t) = \cos \pi t \sin \pi x$

9.5. $u(x,t) = \dfrac{1}{\pi} \sin \pi t \sin \pi x$

9.6. $u(x,t) = -\dfrac{8}{\pi} \sum_{n=1}^{\infty} \dfrac{(-1)^n\, n}{4n^2-1} \cos nt \sin nx$

9.7. $u(x,t) = \dfrac{8}{\pi} \sum_{n=1}^{\infty} \dfrac{n}{4n^2-1} \cos nt \sin nx$

9.8. $u(x,t) = \dfrac{8}{\pi} \sum_{n=1}^{\infty} \dfrac{1}{4n^2-1} \sin nt \sin nx$

9.9. $u(x,t) = \dfrac{8}{\pi^2} \sum_{n=1}^{\infty} \dfrac{1}{n^3} \sin \dfrac{n\pi}{2} \sin nt \sin nx$

Chapter 10

10.6. $\big(K(x)X'(x)\big)' = -\lambda \rho(x) c_p(x) X(x), \quad X(0) = 0, \quad X(\ell) = 0$

10.10.

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} \frac{1}{c\gamma_n} \psi_n \sin \gamma_n x \sin \gamma_n ct \\
\psi_n &= \frac{2}{\ell} \int_0^{\ell} g(x) \sin \gamma_n x \, dx \\
\gamma_n &= (2n-1) \frac{\pi}{2\ell}
\end{aligned}
$$

10.11. $u(x,t) = \sum_{n=1}^{\infty} \dfrac{Q_n}{c^2\gamma_n^2 - \omega^2} \left(\sin \omega t - \dfrac{\omega}{c\gamma_n} \sin c\gamma_n t \right) \sin \gamma_n x$

where $Q_n = \dfrac{2\sigma\big(1-(-1)^n\big)}{n\pi}, \quad \gamma_n = \dfrac{n\pi}{\ell}$

Chapter 11

11.4.

$$
\begin{aligned}
u(x,t) &= \sigma x \sin \omega t \\
&\qquad + \sum_{n=1}^{\infty} \frac{k\lambda_n Q_n}{\omega^2 + k^2\lambda_n^2} \left[-e^{-k\lambda_n t} + \cos \omega t + \frac{\omega}{k\lambda_n} \sin \omega t \right] X_n(x),
\end{aligned}
$$

where

$$Q_n = \frac{2\sigma\omega}{\ell\gamma_n^2}(-1)^n, \quad X_n(x) = \sin\gamma_n x, \quad \lambda_n = \gamma_n^2, \quad \gamma_n = \frac{(2n-1)\pi}{2\ell}.$$

11.5.

$$\begin{aligned} &v_t = kv_{xx} + 1 && 0 < x < \ell, \quad t > 0 \\ &v(0,t) = 0 && t > 0 \\ &v(\ell,t) = 0 && t > 0 \\ &v(x,0) = f(x) - \left(1 - \frac{x}{\ell}\right)\alpha - \frac{x}{\ell}\beta && 0 < x < \ell \end{aligned}$$

11.6.

$$\begin{aligned} &v_t = kv_{xx} && 0 < x < \ell, \quad t > 0, \\ &v(0,t) = 0 && t > 0, \\ &v(\ell,t) = 0 && t > 0, \\ &v(x,0) = f(x) + \frac{1}{2k}x^2 - \left(\frac{1}{\ell}(\beta-\alpha) + \frac{1}{2k}\ell\right)x - \alpha && 0 < x < \ell. \end{aligned}$$

11.7.

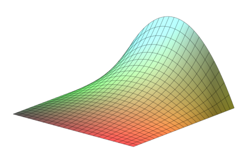
$$\begin{aligned} u(x,t) = \sigma x \sin\omega t + \sum_{n=1}^{\infty}\Bigg[\frac{1}{\gamma_n c}\left(h_n - \frac{\omega Q_n}{c^2\gamma_n^2 - \omega^2}\right)\sin\gamma_n ct \\ + \frac{Q_n}{c^2\gamma_n^2-\omega^2}\sin\omega t\Bigg] X_n(x), \end{aligned}$$

where

$$Q_n = -\frac{2\sigma\omega^2}{\ell\gamma_n^2}(-1)^n, \quad h_n = \frac{2\sigma\omega}{\ell\gamma_n^2}(-1)^n, \quad X_n(x) = \sin\gamma_n x,$$
$$\lambda_n = \gamma_n^2, \quad \gamma_n = \frac{(2n-1)\pi}{2\ell}.$$

Chapter 12

12.1. $u(x,y) = \dfrac{2y}{b}x(a-x) + \dfrac{16}{\pi^4}\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\left[1-(-1)^m\right](-1)^n}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$



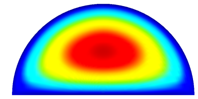
Chapter 13

13.2. $u(r,\theta) = \displaystyle\sum_{n=1}^{\infty} c_n r^n \sin n\theta, \quad c_n = \frac{2}{a^n\pi}\int_0^{\pi} f(\theta)\sin n\theta\, d\theta$

13.3. $u(r,\theta) = \dfrac{ab}{b^2-a^2}\left(\dfrac{r}{a} - \dfrac{a}{r}\right)\sin\theta$

13.4. $u(r,\theta) = \dfrac{4a^2}{3\pi}\dfrac{r}{a}\left(1 - \dfrac{r}{a}\right)\sin\theta + \dfrac{2a^2}{\pi}\sum_{n=3}^{\infty}\dfrac{1-(-1)^n}{n(n^2-4)}\left[\left(\dfrac{r}{a}\right)^2 - \left(\dfrac{r}{a}\right)^n\right]\sin n\theta$

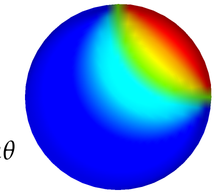
Note the summation's starting index!



#13.4

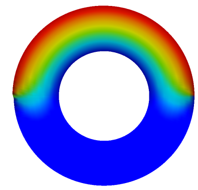
13.6. $f(\theta) = \dfrac{1}{4} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin\dfrac{n\pi}{2}\cos n\theta + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\left[1 - \cos\dfrac{n\pi}{2}\right]\sin n\theta$

13.7. $u(r,\theta) = \dfrac{1}{4} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\left(\dfrac{r}{a}\right)^n\sin\dfrac{n\pi}{2}\cos n\theta + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\left(\dfrac{r}{a}\right)^n\left[1 - \cos\dfrac{n\pi}{2}\right]\sin n\theta$



#13.7

13.8. $u(r,\theta) = \dfrac{\ln(r/a)}{2\ln(b/a)} + \dfrac{1}{\pi}\sum_{n=1}^{\infty}\dfrac{1-(-1)^n}{n}\left[\dfrac{(r/b)^n}{1-(a/b)^{2n}} - \dfrac{(b/r)^n}{(b/a)^{2n}-1}\right]\sin n\theta$



#13.8

Chapter 14

14.2. $u(x,y,t) = \sin x \sin y \cos\sqrt{2}\,t$

14.3. $u(x,y,t) = \sin 2x \sin 3y \cos\sqrt{13}\,t$