Solving system of differential equations through Laplace transform

Example: Solve the initial value problem

$$\begin{aligned} x' - y &= 1 \qquad & x(0) &= 0, \\ y' + 9x &= 9 \qquad & y(0) &= 2. \end{aligned}$$

Solution: Apply the Laplace transform to the equations and obtain

$$\begin{aligned} s\mathscr{L}\{x(t)\} - x(0) - \mathscr{L}\{y(t)\} &= \frac{1}{s}, \\ s\mathscr{L}\{y(t)\} - y(0) + 9\mathscr{L}\{x(t)\} &= \frac{9}{s}. \end{aligned}$$

Plug in the initial conditions and rearrange that into

$$s\mathscr{L}\{x(t)\} - \mathscr{L}\{y(t)\} = \frac{1}{s}, \tag{1a}$$

$$9\mathscr{L}\{x(t)\} + s\mathscr{L}\{y(t)\} = \frac{9}{s} + 2. \tag{1b}$$

Solve this as an algebraic system of two equations in the two unknowns $\mathscr{L}\{x(t)\}$ and $\mathscr{L}\{y(t)\}$. The straightforward way is to multiply the first equation by s and then add it to the second equation. That eliminates $\mathscr{L}\{y(t)\}$ and we obtain

$$(s^2+9)\mathscr{L}\{x(t)\} = 1 + \left(\frac{9}{s} + 2\right) = 3 + \frac{9}{s} = \frac{3s+9}{s},$$

which we solve for $\mathscr{L}\{x(t)\}$:

$$\mathscr{L}\{x(t)\} = \frac{3s+9}{s(s^2+9)}. \tag{2a}$$

We return to the equations (1a) and (1b), and eliminate $\mathscr{L}\{x(t)\}$ between them by multiplying (1a) by -9 and (1b) by s and adding the results, whereby

$$(s^2+9)\mathscr{L}\{y(t)\} = -\frac{9}{s} + 9 + 2s = \frac{-9+9s+2s^2}{s},$$

which we solve for $\mathscr{L}\{y(t)\}$:

$$\mathscr{L}\{y(t)\} = \frac{-9+9s+2s^2}{s(s^2+9)}. \tag{2b}$$

Equations (2a) and (2b) express the Laplace transforms of the desired solutions $x(t)$ and $y(t)$. To find the solutions themselves, we need to invert those transforms.

Toward that end, we begin with the partial fraction decomposition of the right-hand side of (2a), as in

$$\frac{3s+9}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}.$$

We calculate A, B, and C through the usual procedure and obtain $A = 1$, $B = -1$, $C = 3$, and thus

$$\begin{aligned}\frac{3s+9}{s(s^2+9)} &= \frac{1}{s} + \frac{-s+3}{s^2+9} \\ &= \frac{1}{s} - \frac{s}{s^2+9} + \frac{3}{s^2+9},\end{aligned}$$

from which we conclude that

$$x(t) = 1 - \cos 3t + \sin 3t.$$

Similarly, we apply the partial fraction decomposition of the right-hand side of (2b):

$$\frac{-9+9s+2s^2}{s(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+9}.$$

We find that $A = -1$, $B = 3$, $C = 9$, and therefore

$$\begin{aligned}\mathscr{L}\{y(t)\} &= -\frac{1}{s} + \frac{3s+9}{s^2+9} \\ &= -\frac{1}{s} + \frac{3s}{s^2+9} + \frac{9}{s^2+9}.\end{aligned}$$

from which we conclude that

$$y(t) = -1 + 3\cos 3t + 3\sin 3t.$$

Exercises

Solve the following initial value problems.

1. $x' + y = 1 \qquad x(0) = 1,$
   $y' - 4x = 8 \qquad y(0) = 1.$

   Answer: $x(t) = -2 + 3\cos 2t, \quad y(t) = 1 + 6\sin 2t.$

2. $x' + y = 5e^{-t} \qquad x(0) = 0,$
   $y' - 4x = 8 \qquad y(0) = 0.$

   Answer: $x(t) = -2 - e^{-t} + 3\cos 2t + 2\sin 2t, \quad y(t) = 4e^{-t} - 4\cos 2t + 6\sin 2t.$

3. $x'' + 16y = 0 \qquad x(0) = 0, \quad x'(0) = 8$
   $y'' + x = 0 \qquad y(0) = 0, \quad y'(0) = 0.$

   Answer: $x(t) = e^{2t} - e^{-2t} + 2\sin 2t, \quad y(t) = -\frac{1}{4}e^{2t} + \frac{1}{4}e^{-2t} + \frac{1}{2}\sin 2t.$

4. $x'' + x' + 2y' = 0 \qquad x(0) = 0, \quad x'(0) = 3$
   $y' + x = 0 \qquad y(0) = 1.$

   Answer: $x(t) = e^{t} - e^{-2t}, \quad y(t) = \frac{5}{2} - e^{t} - \frac{1}{2}e^{-2t}.$