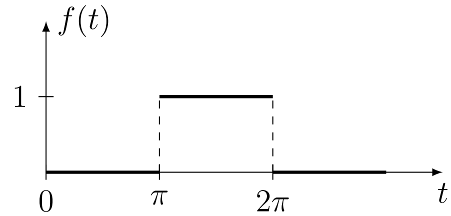
please write name legibly

Math 225, Fall 2024 **Quiz #11** **Name:** ____________________

1. [3 pts] The graph of the function $f(t)$ is shown below.
   Express $f(t)$ in terms of the unit step function and find $\mathscr{L}\{f(t)\}$.



2. [7 pts] Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 1.$$

Solution of question 1: We have

$$f(t) = u(t-\pi) - u(t-2\pi).$$

Therefore

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{u(t-\pi)\} - \mathscr{L}\{u(t-2\pi)\} = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s} = \frac{1}{s}\Big[e^{-\pi s} - e^{-2\pi s}\Big].$$

Solution of question 2: Applying the Laplace transform to the differential equation, we get

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \mathscr{L}\{f(t)\},$$

Substituting the given initial conditions and the Laplace transform computed above, we get

$$(s^2+1)\mathscr{L}\{y\} - 1 = \frac{1}{s}\Big[e^{-\pi s} - e^{-2\pi s}\Big]$$

when we obtain

$$\mathscr{L}\{y\} = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)}\Big[e^{-\pi s} - e^{-2\pi s}\Big]. \tag{1}$$

We recognize the first term on the right-hand side as the Laplace transform of $\sin t$. To evaluate the second term, we split it into partial fractions, as in

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}. \tag{2}$$

To calculate A, we multiply equation (2) by s

$$\frac{1}{s^2+1} = A + \frac{Bs+C}{s^2+1} \times s,$$

and then let $s = 0$ and conclude that $A = 1$. To calculate B and C, we multiply equation (2) by $s^2 + 1$

$$\frac{1}{s} = \frac{A}{s} \times (s^2 + 1) + Bs + c,$$

and then let $s = i$, whereby

$$\frac{1}{i} = Bi + C.$$

Multiplying through by i, we see that

$$1 = -B + iC,$$

whence we conclude that $B = -1$, $C = 0$. Thus, equation (2) takes the form

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} = \mathscr{L}\{1\} - \mathscr{L}\{\cos t\} = \mathscr{L}\{1 - \cos t\}.$$

Returning to equation (1), we now have

$$\mathscr{L}\{y\} = \mathscr{L}\{\sin t\} + \mathscr{L}\Big\{1 - \cos t\Big\}\Big|_{\text{delayed by } \pi} - \mathscr{L}\Big\{1 - \cos t\Big\}\Big|_{\text{delayed by } 2\pi},$$

and therefore

$$y(t) = \sin t + \big[1 - \cos(t - \pi)\big]u(t - \pi) - \big[1 - \cos(t - 2\pi)\big]u(t - 2\pi).$$

Optionally, this answer may be decoded as follows. We have $\cos(t - \pi) = -\cos t$ and $\cos(t - 2\pi) = \cos t$. Therefore

$$y(t) = \begin{cases} \sin t & 0 < t < \pi, \\ \sin t + 1 + \cos t & \pi < t < 2\pi, \\ \sin t + 1 + \cos t - 1 + \cos t & t > 2\pi, \end{cases}$$

which simplifies to

$$y(t) = \begin{cases} \sin t & 0 < t < \pi, \\ 1 + \sin t + \cos t & \pi < t < 2\pi, \\ \sin t + 2\cos t & t > 2\pi. \end{cases}$$