Math 225, Fall 2024 **Quiz #7** **Name:** ____________________

According to the method of variation of parameters, if $y_1(x)$ and $y_2(x)$ is a pair of linearly independent solutions of the differential equation $a(x)y'' + b(x)y' + c(x)y = f(x)$, then $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$ is a particular solution provided that

$$v_1'(x) = \frac{-y_2(x)f(x)}{a(x)W(y_1, y_2)}, \quad v_2'(x) = \frac{y_1(x)f(x)}{a(x)W(y_1, y_2)},$$

where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2.

Apply that formulation to the differential equation $y'' + 2y' + y = e^{-x}\ln x$. Specifically, determine y_1 and y_2, evaluate $W(y_1, y_2)$, and then find $v_1'(x)$ and $v_2'(x)$ and simplify.

Aside: It's possible to integrate $v_1'(x)$ and $v_2'(x)$ to find $v_1(x)$ and $v_2(x)$, but you are not required to do that in the short time allocated to this quiz.

Solution: The characteristic equation $r^2 + 2r + 1 = 0$, that is $(r+1)^2 = 0$, has a repeated root of $r = -1$. It follows that $y_1(x) = e^{-x}$ and $y_2(x) = xe^{-x}$ are linearly independent solutions of the homogeneous equation. The corresponding Wronskian is

$$W(y_1, y_2) = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \det\begin{pmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{pmatrix} = e^{-x}\big(e^{-x} - xe^{-x}\big) + xe^{-2x} = e^{-2x}.$$

Considering that $a(x) = 1$ in this case, we arrive at

$$v_1'(x) = \frac{-\big(xe^{-x}\big)\big(e^{-x}\ln x\big)}{e^{-2x}} = -x\ln x,$$
$$v_2'(x) = \frac{\big(e^{-x}\big)\big(e^{-x}\ln x\big)}{e^{-2x}} = \ln x.$$