please write name legibly

Math 225, Fall 2024 **Quiz #4** **Name:** ____________________

The function $y_1(x) = x^2$ is a solution of the differential equation $x^2y'' + xy' - 4y = 0$. Apply the method of *reduction of order* to find a second linearly independent solution $y_2(x)$.

Solution: [This is like Exercise #8 of Section 3.3]

We look for a second solution in the form $y_2(x) = v(x)y_1(x) = x^2v(x)$, where $v(x)$ is to be determined. We plug that into the differential equation in order to determine v. There are two different approaches to doing that. In one approach we calculate with $y_2 = vy_1$ and only at the very end we insert $y_1 = x^2$. In the other approach, we let $y_2 = x^2v$ up front and do the calculations with that. You decide which you prefer.

Method 1: Let $y_2 = vy_1$. We substitute

$$y_2' = v'y_1 + vy_1', \qquad y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

into the differential equation

$$x^2(v''y_1 + 2v'y_1' + vy_1'') + x(v'y_1 + vy_1') - 4vy_1 = 0,$$

and rearrange:

$$x^2y_1v'' + (2x^2y_1' + xy_1)v' + (x^2y_1'' + xy_1' - 4y_1)v = 0.$$

The last term on the left-hand side is zero because y_1 is a solution of the DE. That leaves us with

$$x^2y_1v'' + (2x^2y_1' + xy_1)v' = 0.$$

We set $v' = w$ and arrive at

$$x^2y_1w' + (2x^2y_1' + xy_1)w = 0.$$

It's time now to substitute $y_1 = x^2$ and $y_1' = 2x$. We get

$$(x^2)(x^2)w' + \Big[(2x^2)(2x) + (x)(x^2)\Big]w = 0,$$

which simplifies to

$$x^4w' + 5x^3w = 0.$$

to solve this first order linear differential equation with the method of integrating factors,[1] we put it in the standard form

$$w' + \frac{5}{x}w = 0,$$

The integrating factor is $\mu = e^{\int 5/x\,dx} = e^{5\ln x} = e^{\ln x^5} = x^5$. Multiplying the equation by μ and combining the two terms as usual, we arrive at

$$\Big(x^5w\Big)' = 0.$$

Consequently, $x^5w = K_1$, and therefore $w = K_1x^{-5}$, or equivalently, $v' = K_1x^{-5}$. We integrate to obtain $v = -\frac{1}{4}K_1x^{-4} + K_2$. We pick $K_1 = -4$ and $K_2 = 0$ and arrive at $v = x^{-4}$. Then

$$y_2 = vy_1 = \Big(x^{-4}\Big)\Big(x^2\Big) = \frac{1}{x^2}.$$

[1] The method of separation of variables will work equally well.

Method 2: Let $y_2 = vy_1 = x^2 v$.

$$y_2' = x^2 v' + 2xv, \qquad y_2'' = x^2 v'' + 4xv' + 2v.$$

Plugging these into the differential equation we get

$$x^2\Big[x^2 v'' + 4xv' + 2v\Big] + x\Big[x^2 v' + 2xv\Big] - 4x^2 v = 0,$$

which simplifies to $x^4 v'' + 5x^3 v' = 0$. Dividing through by x^4 further reduces this to $v'' + \frac{5}{x}v' = 0$. We let $w = v'$, whereby the equation takes the form $w' + \frac{5}{x}w = 0$. We solve this first order equation through the method of integrating factors, although it can equally well be solved through separation of variables.

The integrating factor is $\mu = e^{\int 5/x\,dx} = e^{5\ln x} = e^{\ln(x^5)} = x^5$. Multiplying the differential equation through by μ we get $x^5 w' + 5x^4 w = 0$ which collapses into $\big(x^5 w\big)' = 0$. Consequently, $x^5 w = K_1$, that is, $w = K_1 x^{-5}$, where K_1 is an arbitrary constant. But $w = v'$, so we have arrived at $v' = K_1 x^{-5}$, and therefore $v = -\frac{1}{4}K_1 x^{-4} + K_2$, where K_2 is yet another arbitrary constant. We choose $K_1 = -4$ and $K_2 = 0$ and obtain $v = x^{-4}$. We conclude that

$$y_2 = x^2 v = \Big(x^2\Big)\Big(x^{-4}\Big) = \frac{1}{x^2}.$$