Math 225, Fall 2024 **Quiz #2** **Name:** ____________________

A tank initially contains 100 gallons of water and 10 pounds of salt. In order to reduce the salt concentration to half of its original amount, we add fresh water at the rate of 5 gallons per minute to the tank, and simultaneously drain the well-mixed brine at the same rate. How long will it take?

Grading rubric:

(a) 3 points for correctly stating the equation of conservation of mass and obtaining the differential equation for the mass of salt in the tank;
(b) 4 points for solving the differential equation;
(c) 3 points for calculating the requested time.

Solution: [This is like Exercises #4 and 7 of Section 2.4]

(a) Let $Q(t)$ be the amount of salt in the tank at any time t. The equation of conservation of mass is

$$\frac{d}{dt}Q(t) = F_{\text{in}}c_{\text{in}} - F_{\text{out}}c(t),$$

where F_{in} and F_{out} are the inflow and outflow rates, c_{in} is the concentration of the incoming salt, and $c(t)$ is the concentration of salt in the tank at time t. By the definition of concentration we have $Q(t) = c(t)V(t)$, where $V(t)$ is the volume of the water in the tank at time t. Since $c(t) = Q(t)/V(t)$, we may eliminate $c(t)$ in the conservation equation above and obtain:

$$\frac{d}{dt}Q(t) = F_{\text{in}}c_{\text{in}} - F_{\text{out}}\frac{Q(t)}{V(t)},$$

We are told $F_{\text{in}} = F_{\text{out}}$. Therefore the volume of the water in the tank remains constant at $V = 100$ gallons. Plugging the given data into this equation, we arrive at

$$\begin{aligned}&\frac{d}{dt}Q(t) = 5 \times 0 - 5 \times \frac{Q(t)}{100},\\ &Q(0) = 10.\end{aligned}$$

(b) The differential equation simplifies to $\frac{d}{dt}Q(t) = -\frac{1}{20}Q(t)$. This may be solved through separation of variables or integrating factors. Let's do it with integrating factors. We have

$$\frac{d}{dt}Q(t) + \frac{1}{20}Q(t) = 0,$$

and therefore the integrating factor is $\mu(t) = e^{t/20}$. Multiplying through by $\mu(t)$ we obtain

$$\frac{d}{dt}\Big[e^{t/20}Q(t)\Big] = 0,$$

and therefore $e^{t/20}Q(t) = K$. Applying the initial condition $Q(0) = 10$ yields $K = 10$. We conclude that $e^{t/20}Q(t) = 10$, and therefore

$$Q(t) = 10e^{-t/20}.$$

(c) To determine the time that it takes to cut the amount salt in the tank into half of its original amount, we solve the equation

$$\frac{1}{2} \times 10 = 10e^{-t/20}.$$

This simplifies to

$$e^{t/20} = 2,$$

and therefore $t/20 = \ln 2$. We conclude that

$$t = 20 \ln 2 \text{ minutes.}$$

Observations:

1. $20 \ln 2 = -20 \ln \frac{1}{2}$ [both forms of the answer are correct]
2. Just for the record: $t = 20 \ln 2 \approx 13.86$ minutes $=$ 13 minutes and 52 seconds