Math 225, Fall 2024 **Quiz #1** **Name:** ____________________

1. Solve the differential equation $xy' + y = 3x^2$ for the unknown $y(x)$.

   Solution: [Like Exercise #9 of Section 2.1]

   Solution 1 (the quick way): This is a linear first order differential equation. We observe that in view of the *product rule*, the two terms on the left-hand side collapse into a single term, as in:

   $$\big(xy\big)' = 3x^2,$$

   and therefore *no integrating factor is needed.* We integrate the above and get

   $$xy = x^3 + c,$$

   for an arbitrary constant c. We conclude that

   $$y = x^2 + \frac{c}{x}.$$

   Solution 2 (the roundabout way): We divide the equation through by x to put it in the standard form:

   $$y' + \frac{1}{x}y = 3x,$$

   and then calculate the integrating factor:

   $$\mu(x) = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$$

   We multiply the equation through by the integrating factor and arrive at:

   $$xy' + y = 3x^2.$$

   We have returned to the original equation! The integrating factor was unnecessary. The left-hand side could be collapsed into a single term without the help from an integrating factor, as in:

   $$\big(xy\big)' = 3x^2.$$

   We integrate and obtain

   $$xy = x^3 + c,$$

   and therefore

   $$y = x^2 + \frac{c}{x}.$$

2. Solve the initial value problem $y' = xy, \;\; y(0) = -1$.

Solution: [Like Example 4 and Exercise #13 of Section 2.2]

Solution 1 (integrating factor): We put this linear first order equation into the standard form $y' - xy = 0$. It follows that the integrating factor is

$$\mu(x) = e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}.$$

Multiplying through by $\mu(x)$, the DE collapses to $\left(e^{-\frac{1}{2}x^2}y(x)\right)' = 0$, whence $e^{-\frac{1}{2}x^2}y(x) = c$. Applying the initial condition leads to $c = -1$, and therefore the solution is

$$y(x) = -e^{\frac{1}{2}x^2}.$$

Solution 2 (separation of variables): Upon separating the variables we obtain $\frac{dy}{y} = x\,dx$, and therefore $\ln|y(x)| = \frac{1}{2}x^2 + \tilde{c}$, where $\tilde{c}$ is an arbitrary constant. It follows that

$$\big|y(x)\big| = e^{\frac{1}{2}x^2+\tilde{c}} = e^{\tilde{c}} \cdot e^{\frac{1}{2}x^2}.$$

Removing the absolute value from $y(x)$ we arrive at

$$y(x) = \pm e^{\tilde{c}} e^{\frac{1}{2}x^2}.$$

The expression $\pm e^{\tilde{c}}$ is an arbitrary constant. We call it c, and arrive at

$$y(x) = ce^{\frac{1}{2}x^2}.$$

Applying the initial condition $y(0) = -1$ we see that $-1 = c$, and therefore the solution of the initial value problem is

$$y(x) = -e^{\frac{1}{2}x^2}.$$