Math 225, Fall 2024 **Final Exam** **Name: ____________________**

- Please make an effort to write neatly, and insert a few words where necessary to get your ideas across. It's difficult to understand (and evaluate) mathematics in the absence of guiding words. ***I will award up to 2 bonus points*** if I find your work well-documented and easy to read.
- No books, notes, and electronic devices on this exam.
- Each of the five problems is worth 10 points.
- Use the reverse sides of the pages or the extra blank sheets at the end if you need them.

Variation of parameters

If the linearly independent functions $y_1(x)$ and $y_2(x)$ are solutions of the *homogeneous* differential equation $a(x)y'' + b(x)y' + c(x)y = 0$, then a particular solution of the *nonhomogeneous* equation $a(x)y'' + b(x)y' + c(x)y = f(x)$, may be obtained through $y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$, where

$$v_1'(x) = \frac{-y_2(x)f(x)}{a(x)W(y_1, y_2)}, \quad v_2'(x) = \frac{y_1(x)f(x)}{a(x)W(y_1, y_2)},$$

and where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2.

Definition and properties of the Laplace transform

1. $\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \equiv F(s)$
2. $\mathscr{L}\{c_1 f(t) + c_2 g(t)\} = c_1\mathscr{L}\{f(t)\} + c_2\mathscr{L}\{g(t)\}$
3. $\mathscr{L}\{e^{at} f(t)\} = F(s-a)$
4. $\mathscr{L}\{t f(t)\} = -\frac{d}{ds}F(s)$
5. $\mathscr{L}\{f'(t)\} = s\mathscr{L}\{f(t)\} - f(0)$
6. $\mathscr{L}\{f''(t)\} = s^2\mathscr{L}\{f(t)\} - sf(0) - f'(0)$
7. $\mathscr{L}\{u(t-c)f(t-c)\} = e^{-cs}\mathscr{L}\{f(t)\} = e^{-cs}F(s) \quad \text{where} \quad u(t-c) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t > c \end{cases}$

Laplace transforms of a few specific functions

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathscr{L}\{e^{at}\} = \frac{1}{s-a} \qquad \mathscr{L}\{\delta(t-c)\} = e^{-cs}$$

$$\mathscr{L}\{\cos bt\} = \frac{s}{s^2+b^2} \qquad \mathscr{L}\{\sin bt\} = \frac{b}{s^2+b^2}$$

Cheers!

1. The functions $y_1(x) = x$ and $y_2(x) = x^3$ are solutions of the homogeneous differential equation $x^2y'' - 3xy' + 3y = 0$. Apply the method of variation of parameters to find the general solution of the nonhomogeneous differential equation $x^2y'' - 3xy' + 3y = x^2 + 3$.

Solution: [Like Sec 3.8, #14, 15]

We apply the variation of parameters formula given on the cover sheet to find a particular solution $y_p(x)$ of the nonhomogeneous equation. We begin with calculating the Wronskian of y_1 and y_2:

$$W = \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} x & x^3 \\ 1 & 3x^2 \end{pmatrix} = 3x^3 - x^3 = 2x^3.$$

Then, considering that $a(x) = x^2$ and $f(x) = x^2 + 3$, we have

$$\begin{aligned} v_1'(x) &= \frac{-(x^3)(x^2+3)}{(x^2)(2x^3)} = -\frac{1}{2} - \frac{3}{2x^2} = -\frac{1}{2} - \frac{3}{2}x^{-2}, \\ v_2'(x) &= \frac{(x)(x^2+3)}{(x^2)(2x^3)} = \frac{1}{2x^2} + \frac{3}{2x^4} = \frac{1}{2}x^{-2} + \frac{3}{2}x^{-4}. \end{aligned}$$

It follows that

$$\begin{aligned} v_1(x) &= -\frac{1}{2}x + \frac{3}{2}x^{-1} = -\frac{1}{2}x + \frac{3}{2x}, \\ v_2(x) &= -\frac{1}{2}x^{-1} - \frac{1}{2}x^{-3} = -\frac{1}{2x} - \frac{1}{2x^3}. \end{aligned}$$

We conclude that

$$\begin{aligned} y_p(x) &= v_1(x)y_1(x) + v_2(x)y_2(x) \\ &= \left(-\frac{1}{2}x + \frac{3}{2x}\right)x + \left(-\frac{1}{2x} - \frac{1}{2x^3}\right)x^3 = -\frac{1}{2}x^2 + \frac{3}{2} - \frac{1}{2}x^2 - \frac{1}{2} = 1 - x^2, \end{aligned}$$

and therefore

$$y(x) = c_1x + c_2x^3 + 1 - x^2.$$

2. A bowling ball weighing $W = 16$ pounds is suspended from the ceiling through a spring. At equilibrium, it stretches the spring by $\Delta L = 8$ inches. The ball is then pulled down by an additional 3 inches an released with a downward velocity of $2\,\text{ft/sec}$. The coefficient of air resistance is $7\,\text{lb/(ft/sec)}$ and the gravitational acceleration is $g = 32\,\text{ft/sec}^2$. How long does it take for the ball to reach the lowest point during its motion?

Solution: [Like Sec 3.10, #14]

Let $u(t)$ be the ball's displacement downward from the equilibrium position. The equation of motion is

$$m\ddot{u} + b\dot{u} + ku = 0,$$

where m is the ball's mass, k is the spring constant, and b is the coefficient of air resistance.

We have

$$m = \frac{W}{g} = \frac{16}{32} = \frac{1}{2}\ \text{slug}, \qquad k = \frac{W}{\Delta L} = \frac{16}{8/12} = 24\ \text{lb/ft},$$

therefore the differential equation reads $\frac{1}{2}\ddot{u} + 7\dot{u} + 24u = 0$. We are thus lead to the initial value problem

$$\ddot{u} + 14\dot{u} + 48u = 0, \quad u(0) = \frac{3}{12} = \frac{1}{4}, \quad \dot{u}(0) = 2.$$

The characteristic equation is

$$r^2 + 14r + 48 = 0,$$

which factorizes as

$$(r+6)(r+8) = 0,$$

and therefore the roots are $r = -6, -8$.

Alternatively, if it does not occur to you to factor the equation, you may calculate the roots through the quadratic formula:

$$r = \frac{-14 \pm \sqrt{14^2 - (4)(48)}}{2} = \frac{-14 \pm \sqrt{196 - 192}}{2} = \frac{-14 \pm 2}{2} = -6, -8.$$

Either way, we conclude that the general solution of the differential equation is

$$u(t) = c_1 e^{-6t} + c_2 e^{-8t}.$$

To determine the coefficients c_1 and c_2, we calculate

$$\dot{u}(t) = -6c_1 e^{-6t} - 8c_2 e^{-8t}.$$

Then applying the initial conditions, we see that

$$c_1 + c_2 = \frac{1}{4}, \quad -6c_1 - 8c_2 = 2.$$

Solving this system of equations, we obtain

$$c_1 = 2, \qquad c_2 = -\frac{7}{4},$$

and therefore

$$u(t) = 2e^{-6t} - \frac{7}{4}e^{-8t}.$$

When the displacement is at its maximum, we have $\dot{u}(t) = 0$. But

$$\dot{u}(t) = -12e^{-6t} + 14e^{-8t},$$

therefore $\dot{u}(t) = 0$ implies that

$$12e^{-6t} = 14e^{-8t},$$

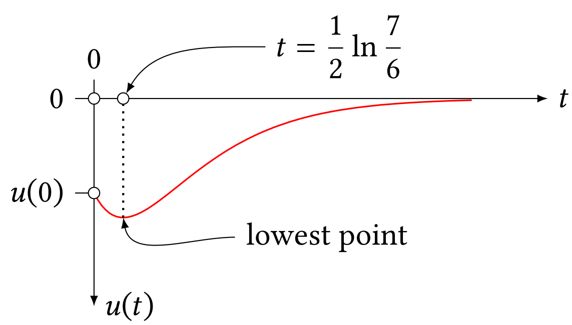
which simplifies to

$$e^{2t} = \frac{14}{12} = \frac{7}{6},$$

and therefore

$$t = \frac{1}{2}\ln\frac{7}{6}.$$

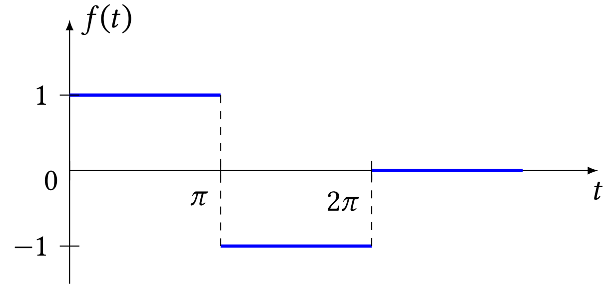
Here is what the motion looks like:



3. Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0,$$

where the forcing function $f(t)$ is as in the graph below:



Solution: [Like Sec 5.6, #7]

We have

$$f(t) = 1 - 2u(t-\pi) + u(t-2\pi),$$

therefore

$$\begin{aligned}\mathscr{L}\{f(t)\} &= \mathscr{L}\{1\} - 2\mathscr{L}\{u(t-\pi)\} + \mathscr{L}\{u(t-2\pi)\}\\ &= \frac{1}{s} - \frac{2e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s}\\ &= \frac{1}{s}\left[1 - 2e^{-\pi s} + e^{-2\pi s}\right].\end{aligned}$$

Applying the Laplace transform to the differential equation, we get

$$s^2\mathscr{L}\{y\} - sy(0) - y'(0) + \mathscr{L}\{y\} = \mathscr{L}\{f(t)\},$$

Substituting the given initial conditions and the Laplace transform computed above, we obtain

$$(s^2+1)\mathscr{L}\{y\} = \frac{1}{s}\left[1 - 2e^{-\pi s} + e^{-2\pi s}\right],$$

and therefore

$$\mathscr{L}\{y\} = \frac{1}{s(s^2+1)}\left[1 - 2e^{-\pi s} + e^{-2\pi s}\right]. \tag{1}$$

To evaluate the inverse Laplace transform, we split the fraction above into partial fractions, as in

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}. \tag{2}$$

To calculate A, we multiply equation (2) by s

$$\frac{1}{s^2+1} = A + \frac{Bs+C}{s^2+1} \times s,$$

and then let $s = 0$ and conclude that $A = 1$. To calculate B and C, we multiply equation (2) by s^2+1

$$\frac{1}{s} = \frac{A}{s} \times (s^2+1) + Bs + c,$$

and then let $s = i$, whereby

$$\frac{1}{i} = Bi + C.$$

Multiplying through by i, we see that

$$1 = -B + iC,$$

whence we conclude that $B = -1$, $C = 0$. Thus, equation (2) takes the form

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1} = \mathscr{L}\{1\} - \mathscr{L}\{\cos t\} = \mathscr{L}\{1 - \cos t\}.$$

Returning to equation (1), we now have

$$\mathscr{L}\big\{y(t)\big\} = \mathscr{L}\big\{1 - \cos t\big\} - 2\mathscr{L}\big\{1 - \cos t\big\}\Big|_{\text{delayed by }\pi} + \mathscr{L}\big\{1 - \cos t\big\}\Big|_{\text{delayed by }2\pi},$$

and therefore

$$y(t) = \big[1 - \cos t\big] - 2\big[1 - \cos(t-\pi)\big]u(t-\pi) + \big[1 - \cos(t-2\pi)\big]u(t-2\pi).$$

Optionally, this answer may be decoded as follows. We have $\cos(t-\pi) = -\cos t$ and $\cos(t-2\pi) = \cos t$. Therefore

$$y(t) = \begin{cases} (1-\cos t) & 0 < t < \pi, \\ (1-\cos t) - 2(1+\cos t) & \pi < t < 2\pi, \\ (1-\cos t) - 2(1+\cos t) + (1-\cos t) & t > 2\pi, \end{cases}$$

which simplifies to

$$y(t) = \begin{cases} 1-\cos t & 0 < t < \pi, \\ -1-3\cos t & \pi < t < 2\pi, \\ -4\cos t & t > 2\pi. \end{cases}$$

4. A bowling ball weighing $W = 8$ pounds is suspended from the ceiling through a spring. At equilibrium, it stretches the spring by $\Delta L = 6$ inches. The ball is initially at rest, but at time $t = \pi$ it is set into motion by striking it with a hammer which exerts a downward impulsive force of $f(t) = \delta(t - \pi)$, where δ is Dirac's delta function. What is the amplitude of the ball's oscillations after being struck? Ignore any damping effects. The gravitational acceleration is $g = 32\,\mathrm{ft/sec}^2$.

   Reminder: The ball's displacement $y(t)$ relative to the equilibrium position obeys Newton's law of motion: $m\ddot{y} + ky = f(t)$.

   Solution: [Like Sec 5.7, #7]

   Let m be the ball's mass, and k be the spring's constant. Then we have $W = mg$ and $W = k\Delta L$. Therefore

$$m = \frac{W}{g} = \frac{8}{32} = \frac{1}{4}\ \text{slug}, \qquad k = \frac{W}{\Delta L} = \frac{8}{6/12} = 16\,\mathrm{lb/ft}.$$

   Therefore, Newton's equation of motion takes the form

$$\frac{1}{4}\ddot{y} + 16y = \delta(t - \pi),$$

   which we rearrange into

$$\ddot{y} + 64y = 4\delta(t - \pi).$$

   The application of the Laplace transform to this equation results in

$$s^2\mathscr{L}\{y(t)\} - sy(0) - \dot{y}(0) + 64\mathscr{L}\{y(t)\} = 4e^{-\pi s}.$$

   Due to the initial conditions $y(0) = 0$, $\dot{y}(0) = 0$, this reduces to

$$(s^2 + 64)\mathscr{L}\{y(t)\} = 4e^{-\pi s},$$

   and consequently

$$\mathscr{L}\{y(t)\} = \frac{4}{s^2 + 64}e^{-\pi s} = \frac{1}{2} \cdot \frac{8}{s^2 + 8^2}e^{-\pi s} = \frac{1}{2}\mathscr{L}\{\sin 8t\}e^{-\pi s} = \mathscr{L}\left\{\frac{1}{2}\sin 8t\right\}e^{-\pi s}$$

   We conclude that $y(t)$ is obtained from the function $\frac{1}{2}\sin 8t$ by delaying it by π, that is

$$y(t) = \left[\frac{1}{2}\sin 8(t - \pi)\right]u(t - \pi) = \begin{cases} 0 & \text{if } t < \pi, \\ \frac{1}{2}\sin 8(t - \pi) & \text{if } t > \pi. \end{cases}$$

   Considering that $\sin 8(t - \pi) = \sin(8t - 8\pi) = \sin t$, this simplifies to

$$y(t) = \begin{cases} 0 & \text{if } t < \pi, \\ \frac{1}{2}\sin 8t & \text{if } t > \pi. \end{cases}$$

   It is evident that the amplitude of motion is $\frac{1}{2}\,\mathrm{ft} = 6\,\mathrm{in}$.

5. Solve the following initial value problem

$$\begin{cases} x' + 4x - 6y = 0, & x(0) = 2, \\ y' + x - y = 1, & y(0) = 0. \end{cases}$$

Solution: [Like any of the exercises in the last homework]

Applying the Laplace transform to the equations, we get

$$\begin{cases} s\mathscr{L}\{x(t)\} - x(0) + 4\mathscr{L}\{x(t)\} - 6\mathscr{L}\{y(t)\} = 0, \\ s\mathscr{L}\{y(t)\} - y(0) + \mathscr{L}\{x(t)\} - \mathscr{L}\{y(t)\} = \frac{1}{s}. \end{cases}$$

We insert the initial conditions and rearrange that into

$$\begin{cases} (s+4)\mathscr{L}\{x(t)\} - 6\mathscr{L}\{y(t)\} = 2, \\ \mathscr{L}\{x(t)\} + (s-1)\mathscr{L}\{y(t)\} = \frac{1}{s}. \end{cases} \tag{1}$$

We multiply the first equation by $s-1$ and the second one by 6,

$$\begin{cases} (s-1)(s+4)\mathscr{L}\{x(t)\} - 6(s-1)\mathscr{L}\{y(t)\} = 2(s-1), \\ 6\mathscr{L}\{x(t)\} + 6(s-1)\mathscr{L}\{y(t)\} = \frac{6}{s}. \end{cases}$$

and add up the results. Then $\mathscr{L}\{y(t)\}$ drops out and we are left with

$$\big[(s-1)(s+4)+6\big]\mathscr{L}\{x(t)\} = 2(s-1) + \frac{6}{s},$$

which simplifies to

$$(s^2+3s+2)\mathscr{L}\{x(t)\} = \frac{2s^2-2s+6}{s},$$

whence

$$\mathscr{L}\{x(t)\} = \frac{2s^2-2s+6}{s(s^2+3s+2)} = \frac{2s^2-2s+6}{s(s+1)(s+2)}. \tag{2}$$

Returning to the equations (1), we multiply the second equation by $s+4$,

$$\begin{cases} (s+4)\mathscr{L}\{x(t)\} - 6\mathscr{L}\{y(t)\} = 2, \\ (s+4)\mathscr{L}\{x(t)\} + (s-1)(s+4)\mathscr{L}\{y(t)\} = \frac{s+4}{s}. \end{cases}$$

and subtract the results. Then $\mathscr{L}\{x(t)\}$ drops out and we are left with

$$\big[(s-1)(s+4)+6\big]\mathscr{L}\{y(t)\} = \frac{s+4}{s} - 2,$$

which simplifies to

$$(s^2+3s+2)\mathscr{L}\{y(t)\} = \frac{-s+4}{s},$$

whence

$$\mathscr{L}\{y(t)\} = \frac{-s+4}{s(s^2+3s+2)}\mathscr{L}\{y(t)\} = \frac{-s+4}{s(s+1)(s+2)}. \tag{3}$$

To find $x(t)$, we expand the right-hand side of equation (2) into partial fractions:

$$\frac{2s^2-2s+6}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiplying this by s we see that

$$\frac{2s^2-2s+6}{(s+1)(s+2)} = A + \frac{Bs}{s+1} + \frac{Cs}{s+2}.$$

Setting $s = 0$ we get $A = 3$.

Returning to the original fraction, we multiply through by $s+1$ and obtain

$$\frac{2s^2-2s+6}{s(s+2)} = \frac{A(s+1)}{s} + B + \frac{C(s+1)}{s+2}.$$

Setting $s = -1$ we obtain $B = -10$.

Returning to the original fraction, we multiply through by $s+2$ and obtain Setting $s = -1$ we obtain $B = -10$.

$$\frac{2s^2-2s+6}{s(s+1)} = \frac{A(s+2)}{s} + \frac{B(s+2)}{s+1} + C.$$

Setting $s = -2$ we obtain $C = 9$. We conclude that

$$\mathscr{L}\{x(t)\} = \frac{2s^2-2s+6}{s(s+1)(s+2)} = \frac{3}{s} - \frac{10}{s+1} + \frac{9}{s+2},$$

and therefore

$$x(t) = 3 - 10e^{t} + 9e^{2t}.$$

To find $y(t)$, we expand the right-hand side of equation (3) into partial fractions:

$$\frac{-s+4}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}.$$

Multiplying this by s we see that

$$\frac{-s+4}{(s+1)(s+2)} = A + \frac{Bs}{s+1} + \frac{Cs}{s+2}.$$

Setting $s = 0$ we get $A = 2$.

Returning to the original fraction, we multiply through by $s+1$ and obtain

$$\frac{-s+4}{s(s+2)} = \frac{A(s+1)}{s} + B + \frac{C(s+1)}{s+2}.$$

Setting $s = -1$ we obtain $B = -5$.

Returning to the original fraction, we multiply through by $s + 2$ and obtain

$$\frac{-s+4}{s(s+1)} = \frac{A(s+2)}{s} + \frac{B(s+2)}{s+1} + C.$$

Setting $s = -2$ we obtain $C = 3$. We conclude that

$$\frac{-s+4}{s(s+1)(s+2)} = \frac{2}{s} - \frac{5}{s+1} + \frac{3}{s+2},$$

and therefore

$$y(t) = 2 - 5e^{-t} + 3e^{-2t}.$$