Math 225, Fall 2024 **Exam #2** **Name:** ____________________

- Please make an effort to write neatly, and *insert a few words where necessary* to get your ideas across. It's difficult to understand (and evaluate) mathematics in the absence of guiding words.
  I will award up to 2 bonus points if I find your work well-documented and easy to read and understand.
- *No books, notes, calculators or other electronic devices on this exam.*
- Use the reverse sides of the pages or the extra blank sheets at the end if you need them.
- There are four problems, each worth 10 points.

The method of undetermined coefficients. Consider the second order, linear, constant coefficients, nonhomogeneous equation

$$ay'' + by' + cy = e^{\alpha x}\Big[P_n(x)\cos\beta x + Q_n(x)\sin\beta x\Big], \tag{1}$$

where P_n and Q_n are polynomials of up to nth degree. Then the differential equation has a particular solution of the form

$$y_p(x) = x^s e^{\alpha x}\Big[(A_0 + A_1 x + \cdots + A_n x^n)\cos\beta x + (B_0 + B_1 x + \cdots + B_n x^n)\sin\beta x\Big], \tag{2}$$

where A_0 through A_n and B_0 through B_n are constants to be determined.

As to the exponent s, let r_1 and r_2 be the roots of the characteristic equation, and $z \equiv \alpha + i\beta$. If z equals neither of the roots r_1 and r_2, then $s = 0$. If z equals only one of the roots, then $s = 1$. If z equals both roots, then $s = 2$.

Cheers!

1. The function $y_1(x) = e^x$ is solution of the differential equation $xy'' - 2(x-1)y' + (x-2)y = 0$. Apply the method of *reduction of order* to find a second, linearly independent solution of the form $y_2(x) = v(x)y_1(x)$, and then find the equation's *general solution.*

   Note: For full credit, go through the usual steps that lead to the calculation of $v(x)$. If you choose to apply the textbook's formula for $v(x)$ which you may have memorized, you need to show how that formula is obtained.

   Solution: [Like Exercise #6 of Section 3.3 and Quiz #4]

   We look for a second solution of the form $y_2(x) = v(x)y_1(x)$, where $v(x)$ is to be determined. There are two ways to proceed. In one approach we calculate with $y_2 = vy_1$ and only at the very end we insert $y_1 = e^x$. In the other approach, we let $y_2 = e^x v$ up front and do the calculations with that. You decide which you prefer.

   Method 1: Let $y_2 = vy_1$ and calculate

$$y_2' = v'y_1 + vy_1', \qquad y_2'' = v''y_1 + 2v'y_1' + vy_1''.$$

   Plugging these into the differential equation we get

$$x\Big[v''y_1 + 2v'y_1' + vy_1''\Big] - 2(x-1)\Big[v'y_1 + vy_1'\Big] + (x-2)vy = 0,$$

   which we rearrange into

$$xy_1v'' + \Big[2xy_1' - 2(x-1)y_1\Big]v' + \Big[xy_1'' - 2(x-1)y_1' + (x-2)y\Big]v = 0.$$

   As y_1 is a solution of the DE, the last term on the left-hand side is zero and the equation reduces to

$$xy_1v'' + \Big[2xy_1' - 2(x-1)y_1\Big]v'.$$

   Substituting $y_1 = e^x$ this further simplifies to

$$xv'' + 2v' = 0.$$

   Method 2: Let $y_2 = e^x v$ and calculate

$$y_2' = e^x v + e^x v', \qquad y_2'' = e^x v + 2e^x v' + e^x v''$$

   Plugging these into the differential equation we get

$$x\big(v + 2v' + v''\big)e^x - 2(x-1)\big(v + v'\big)e^x + (x-2)e^x v = 0,$$

   which simplifies to

$$xv'' + 2v' = 0.$$

   Both methods result to the same equation, as expected.

   Continuing from there, we let $v' = w$, and conclude that

$$xw' + 2w = 0,$$

which may be solved through separation of variables or the integrating factor method. Let's do it with the latter. We rearrange the equation into the standard form

$$w' + \frac{2}{x} w = 0,$$

and calculate the integrating factor

$$\mu(x) = e^{\int 2/x \, dx} = e^{2 \ln x} = e^{\ln x^2} = x^2.$$

Multiplying the equation through by the integrating factor, we arrive at

$$x^2 w' + 2xw = 0,$$

which collapses to

$$\left(x^2 w\right)' = 0.$$

It follows that $x^2 w = K_1$, whence $w = \frac{K_1}{x^2}$, that is, $v' = \frac{K_1}{x^2}$, whence $v = -\frac{K_1}{x} + K2$. We take $K_1 = -1$, $K_2 = 0$, and arrive at $v = \frac{1}{x}$, and conclude that

$$y_2 = \frac{1}{x} e^x.$$

The solutions $y_1 = e^x$ and $y_2 = \frac{1}{x} e^x$ are linearly independent since one is not a constant multiple of the other. It follows that the sought general solution is

$$y(x) = c_1 e^x + \frac{c_2}{x} e^x.$$

2. Find the general solution of each of the following DEs:

   (a) $y'' - 4y = 3e^x$

   (b) $4y'' + 4y' + y = x$

   (c) $y'' + 3y' + 2y = 10\sin x$.

   Solution:

   (a) The characteristic equation is $r^2 - 4 = 0$, which has roots $r = \pm 2$. Therefore the general solution of the homogeneous equation is

$$y_h = c_1 e^{2x} + c_1 e^{-2x},$$

   where c_1 and c_2 are arbitrary constants.

   We look for a particular solution of the form $y_p = Ae^x$. We have $y_p' = Ae^x$ and $y_p'' = Ae^x$. Therefore

$$Ae^x - 4Ae^x = 3e^x,$$

   which simplifies to $-3A = 3$, whence $A = -1$. We conclude that $y_p = -e^x$, and therefore the general solution is

$$y = y_h + y_p = c_1 e^{2x} + c_1 e^{-2x} - e^x.$$

   (b) The characteristic equation is $4r^2 + 4r + 1 = 0$, which factors as $(2r+1)^2 = 0$, whence $r = -1/2$ is the only root. We conclude that the general solution of the homogeneous equation is

$$y_h = c_1 e^{-x/2} + c_2 x e^{-x/2}.$$

   We look for a particular solution of the form $y_p = Ax + B$. We have $y_p = A$ and $y_p'' = 0$. Therefore $4A + (Ax + B) = x$, that is, $Ax + (4A + B) = x$. We conclude that $A = 1$ and $4A + B = 0$, that is, $B = -4A = -4$. We thus arrive at $y_p = x - 4$, and the general solution

$$y = y_h + y_p = c_1 e^{-x/2} + c_2 x e^{-x/2} + x - 4.$$

   (c) The characteristic equation is $r^2 + 3r + 2 = 0$, which factors as $(r+1)(r+2) = 0$. The roots are $r = -1$ and $r = -2$, and therefore the general solution of the homogeneous equation is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}.$$

   We look for a particular solution of the form $y_p = A\cos x + B\sin x$. We have $y_p' = -A\sin x + B\cos x$, $y_p'' = -A\cos x - B\sin x$. Plugging these into the differential equation we get

$$(-A\cos x - B\sin x) + 3(-A\sin x + B\cos x) + 2(A\cos x + B\sin x) = 10\sin x,$$

   which simplifies to

$$(-A + 3B + 2A)\cos x + (-B - 3A + 2B)\sin x = 10\sin x,$$

   that is

$$(A + 3B)\cos x + (-3A + B)\sin x = 10\sin x.$$

This will hold provided that

$$A + 3B = 0,$$
$$-3A + B = 10.$$

Solving this system we obtain $A = -3$, $B = 1$, which results in

$$y_p = -3\cos x + \sin x.$$

We conclude that the general solution is

$$y = y_h + y_p = c_1 e^{-x} + c_2 e^{-2x} - 3\cos x + \sin x.$$

3. (a) [5 pts] Find the general solution of the differential equation $y'' + 3y' + 2y = e^{-x}$.
   (b) [5 pts] Solve the initial value problem of that differential equation with $y(0) = 1$, $y'(0) = 2$.

Solution:

(a) The characteristic equation is $r^2 + 3r + 2 = 0$ which factors as $(r + 1)(r + 2) = 0$, and which has roots $r = -1$ and $r = -2$. The solution of the homogeneous equation is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}.$$

We see that the DE's right-hand side, that is e^{-x}, is a solution of the homogeneous equation. Therefore we look for a particular solution of the form

$$y_p = Axe^{-x}.$$

We have

$$y_p' = Ae^{-x} - Axe^{-x}, \qquad y_p'' = -2Ae^{-x} + Axe^{-x}.$$

We plug these into the DE:

$$\left(-2Ae^{-x} + Axe^{-x}\right) + 3\left(Ae^{-x} - Axe^{-x}\right) + 2Axe^{-x} = e^{-x},$$

and simplify to get $Ae^{-x} = e^{-x}$, and conclude that $A = 1$, and therefore $y_p = xe^{-x}$. It follows that the DE's general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} + xe^{-x}.$$

(b) To apply the initial conditions, we calculate

$$y'(x) = -c_1 e^{-x} - 2c_2 e^{-2x} + e^{-x} - xe^{-x}.$$

Then

$$\begin{aligned} y(0) = 1 \qquad &\Rightarrow \qquad 1 = c_1 + c_2, \\ y'(0) = 2 \qquad &\Rightarrow \qquad 2 = -c_1 - 2c_2 + 1. \end{aligned}$$

Adding these up, c_1 drops out and we obtain $3 = -c_2 + 1$, and consequently, $c_2 = -2$. Then from $1 = c_1 + c_2$ we get $c_1 = 3$. We conclude that the solution of the initial value problem is

$$y(x) = 3e^{-x} - 2e^{-2x} + xe^{-x}.$$

4. Determine *the form* of the particular solution $y_p(x)$ in each of the following equations. No need find the undermined coefficients. Each of the four questions is worth 2.5 points.

   (a) $y'' + 5y' + 6y = xe^x$
   (b) $y'' + 5y' + 6y = (1 + x^2)e^{-3x}$
   (c) $y'' + 6y' + 9y = (1 + x^2)e^{-3x}$
   (d) $y'' + 6y' + 10y = e^{-3x} \sin x$

Solution: (a) The characteristic equation $r^2 + 5r + 6 = 0$ has roots $r = -2$ and $r = -3$. The right-hand side matches the general template with $\alpha = 1$, $\beta = 0$, and $n = 1$. We let $z = \alpha + i\beta = 1$. Since z matches neither of the characteristic equation's roots, we have $s = 0$ and therefore a particular solution has the form

$$y_p(x) = x^0(A_0 + A_1x)e^x = (A_0 + A_1x)e^x.$$

(b) The characteristic equation $r^2 + 5r + 6 = 0$ has roots $r = -2$ and $r = -3$. The right-hand side matches the general template with $\alpha = -3$, $\beta = 0$, and $n = 2$. We let $z = \alpha + i\beta = -3$. Since z matches only one of the characteristic equation's roots, we have $s = 1$ and therefore a particular solution has the form

$$y_p(x) = x^1(A_0 + A_1x + A_2x^2)e^{-3x} = (A_0x + A_1x^2 + A_2x^3)e^{-3x}.$$

(c) The characteristic equation $r^2 + 6r + 9 = 0$ has the repeated root $r = -3$. The right-hand side matches the general template with $\alpha = -3$, $\beta = 0$, and $n = 2$. We let $z = \alpha + i\beta = -3$. Since z matches both of the characteristic equation's roots, we have $s = 2$ and therefore a particular solution has the form

$$y_p(x) = x^2(A_0 + A_1x + A_2x^2)e^{-3x} = (A_0x^2 + A_1x^3 + A_2x^4)e^{-3x}.$$

(d) The characteristic equation $r^2 + 6r + 10 = 0$ has roots $r = -3 \pm i$. The right-hand side matches the general template with $\alpha = -3$, $\beta = 1$, and $n = 0$. We let $z = \alpha + i\beta = -3 + i$. Since z matches only one of the characteristic equation's roots, we have $s = 1$ and therefore a particular solution has the form

$$y_p(x) = x^1e^{-3x}\left[A_0 \cos x + B_0 \sin x\right] = xe^{-3x}\left[A_0 \cos x + B_0 \sin x\right].$$