Math 225, Fall 2024 **Exam #1** **Name: ____________________**

- Please make an effort to write neatly, and *insert a few words where necessary* to get your ideas across. It's difficult to understand (and evaluate) mathematics in the absence of guiding words.
  I will award up to 2 bonus points if I find your work well-documented and easy to read and understand.

- *No books, notes, calculators or other electronic devices on this exam.*

- Use the reverse sides of the pages or the extra blank sheets at the end if you need them.

- There are four problems, each worth 10 points.

Cheers!

1. A quantity $Q(t)$ of radioactive material decays according to $\frac{d}{dt}Q(t) = -kQ(t)$, where k is the material's *decay constant.*

   (a) (4 pts) Solve the decay equation along with the initial condition $Q(0) = Q_0$ to determine $Q(t)$.

   (b) (2 pts) Say, in a few words, what is meant by the material's *half-life* τ. Use the result of part (a) to determine a relationship between k and τ.

   (c) (4 pts) Carbon-14 has a half life of approximately 5600 years. How long will it take for an amount of carbon-14 to be reduced to 10% of its current amount due to radioactive decay?

Solution:

(a) The differential equation may be solved by both separation of variables and integrating factor methods. Let's do it with the integrating factor method. We have

$$\frac{d}{dt}Q(t) + kQ(t) = 0,$$

and therefore the integrating factor is $\mu(t) = e^{\int k\,dt} = e^{kt}$. We multiply the equation by the integrating factor

$$e^{kt}\frac{d}{dt}Q(t) + ke^{kt}Q(t) = 0,$$

simplify

$$\frac{d}{dt}\Big(e^{kt}Q(t)\Big) = 0,$$

and integrate:

$$e^{kt}Q(t) = C.$$

Evaluating this at $t = 0$ and applying the initial condition and obtain $C = Q_0$. We thus arrive at $e^{kt}Q(t) = Q_0$, whence

$$Q(t) = Q_0 e^{-kt}.$$

(b) The half-life τ is the length of time that is takes for half of the radioactive material to disintegrate. Therefore, if we start off with Q_0 amount, after time τ elapses we will be left with $\frac{1}{2}Q_0$, that is

$$\frac{1}{2}Q_0 = Q_0 e^{-k\tau}.$$

This simplifies to $e^{k\tau} = 2$, and therefore

$$k\tau = \ln 2.$$

(c) The time t needed to reduce carbon-14 to 10% of its initial amount is

$$\frac{1}{10}Q_0 = Q_0 e^{-kt}.$$

This implies that $e^{kt} = 10$, that is, $kt = \ln 10$. But from part (b) we have $k = \frac{\ln 2}{\tau}$. Therefore

$$\frac{\ln 2}{\tau}t = \ln 10,$$

whence

$$t = \frac{\ln 10}{\ln 2}\tau = 5600\frac{\ln 10}{\ln 2}\text{ years.}$$

That's approximately 18,603 years.

2. A cup of hot cocoa at the temperature of 220°F is served at the ski slope where the air temperature is 20°F. (That's cold!)

   (a) (2 pts) Write down the differential equation that expresses the rate of change of the temperature $T(t)$ of the cocoa according to *Newton's Law of Cooling*.

   (b) (4 pts) Solve the differential equation to determine $T(t)$ at any time t.

   (c) (4 pts) The drink's temperature drops to 120°F within two minutes. If not drunk, when will the cocoa reach the freezing temperature of 32°F?

Solution:

(a) Newton's Law of Cooling says

$$\frac{d}{dt}T(t) = -k\big(T(t) - M\big)$$

where M is the ambient temperature. In our case we have $M(t) = 20$, and therefore

$$\frac{d}{dt}T(t) = -k\big(T(t) - 20\big).$$

(b) We put the differential equation into the standard form

$$\frac{d}{dt}T(t) + kT(t) = 20k.$$

The integrating factor is $\mu(t) = e^{\int k\,dt} = e^{kt}$. We multiply the equation by the integrating factor

$$e^{kt}\frac{d}{dt}T(t) + ke^{kt}T(t) = 20ke^{kt},$$

simplify

$$\frac{d}{dt}\Big(e^{kt}T(t)\Big) = 20ke^{kt},$$

and integrate:

$$e^{kt}T(t) = 20e^{kt} + C.$$

Evaluating this at $t = 0$ and applying the initial condition $T(0) = 220$ we obtain $220 = 20 + C$, whence $C = 200$. We thus arrive at $e^{kt}T(t) = 20e^{kt} + 200$, whence

$$T(t) = 20 + 200e^{-kt}.$$

(c) We are given that

$$120 = 20 + 200e^{-2k}.$$

Therefore $200e^{-2k} = 100$, that is, $e^{2k} = 2$. We conclude that $2k = \ln 2$, and so

$$k = \frac{1}{2}\ln 2.$$

The time to reach the temperature of 32°F is obtained by solving the equation $32 = 20 + 200e^{-kt}$, that is, $12 = 200e^{-kt}$, and therefore $e^{kt} = 200/12$, and thus $kt = \ln(200/12)$. Substituting for the value of k calculated earlier, this become

$$\left(\frac{1}{2}\ln 2\right) t = \ln\frac{200}{12}.$$

We conclude that

$$t = \frac{2\ln\frac{200}{12}}{\ln 2} = \frac{2\ln\frac{50}{3}}{\ln 2}.$$

That's approximately 8 minutes.

3. A projectile is thrown upward with the initial velocity v_0. The force of air resistance is $-kv$ where v is the velocity during the flight.

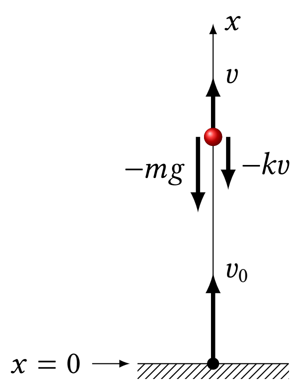
   (a) (3 pts) Draw a diagram that shows a (vertical) coordinate axis x along which the motion takes place. Mark the location of the origin ($x = 0$), the direction of increasing x, and the vectors of forces that act on the projectile during its upward flight. Write down the differential equation that expresses Newton's law of motion.

   (b) (4 pts) Solve the initial value problem and thus obtain the velocity $v(t)$ of the projectile as a function of time.

   (c) (3 pts) How long does it take for the projectile to reach the maximum height before it reverses its motion and begins to fall down?

Solution: [This is the first half of Exercise 16 of Section 2.6.]

We take the x axis pointing up, with the origin set at the launch point. The forces acting on the projectile are the force of gravity $-mg$, and the air resistance $-kv$. As the projectile ascends, both of those forces point down as shown in the following diagram.



According to Newton's Law of Motion, the resultant of the applied forces equals mass times acceleration, that is,

$$m\frac{d}{dt}v(t) = -mg - kv(t),$$

where $v(t)$ is the velocity at any time t.

(b) We normalize the equation motion

$$\frac{d}{dt}v(t) + \frac{k}{m}v(t) = -g,$$

and calculate the integrating factor

$$\mu(t) = e^{\int k/m\,dt} = e^{kt/m}.$$

We multiply the equation by $\mu(t)$

$$e^{kt/m}\frac{d}{dt}v(t) + e^{kt/m}kv(t) = -ge^{kt/m}$$

and combine the terms on the left and get

$$\frac{d}{dt}\left(e^{kt/m}v(t)\right) = -ge^{kt/m}.$$

Integrating this we obtain

$$e^{kt/m}v(t) = -\frac{mg}{k}e^{kt/m} + c,$$

where c is the integration constant. We determine c by applying the initial condition $v(0) = v_0$:

$$v_0 = -\frac{mg}{k} + c,$$

and therefore

$$c = v_0 + \frac{mg}{k}.$$

We thus arrive at

$$e^{kt/m}v(t) = -\frac{mg}{k}e^{kt/m} + v_0 + \frac{mg}{k},$$

and therefore

$$v(t) = \left(v_0 + \frac{mg}{k}\right)e^{-kt/m} - \frac{mg}{k}.$$

(c) The projectile's velocity is zero at the maximum height. That enables us to calculate the time of that event:

$$0 = (v_0 + \frac{mg}{k})e^{-kt/m} - \frac{mg}{k}.$$

We rearrange that into

$$(v_0 + \frac{mg}{k})e^{-kt/m} = \frac{mg}{k},$$

and isolate the exponential term

$$e^{kt/m} = \frac{v_0 + \frac{mg}{k}}{\frac{mg}{k}} = 1 + \frac{kv_0}{mg}.$$

Taking the logarithm of both sides we get

$$\frac{kt}{m} = \ln\left(1 + \frac{kv_0}{mg}\right),$$

and conclude that

$$t = \frac{m}{k}\ln\left(1 + \frac{kv_0}{mg}\right).$$

4. Apply Euler's method with a step size of $h = 2$ to calculate the solution of the initial value problem $y' = x - y^2$, $y(0) = 1$, at $x = 2, 4, 6$.

Euler's method calculates an approximate solution to the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$, through the recursion

$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + hf(x_n, y_n), \quad n = 0, 1, \dots,$$

so it amounts to filling the missing numbers in the table below.

n	x_n	y_n	$f(x_n, y_n)$
0	$x_0 = 0$	$y_0 = 1$	$f(x_0, y_0) =$
1	$x_1 =$	$y_1 =$	$f(x_1, y_1) =$
2	$x_2 =$	$y_2 =$	$f(x_2, y_2) =$
3	$x_3 =$	$y_3 =$	

Solution: [This is like Exercises #20 of Section 2.7.]

n	x_n	y_n	$f(x_n, y_n)$
0	$x_0 = 0$	$y_0 = 1$	$f(x_0, y_0) = x_0 - y_0^2 = 0 - 1^2 = -1$
1	$x_1 = x_0 + h = 0 + 2 = 2$	$y_1 = y_0 + hf(x_0, y_0) = 1 + (2)(-1) = -1$	$f(x_1, y_1) = x_1 - y_1^2 = 2 - (-1)^2 = 1$
2	$x_2 = x_1 + h = 2 + 2 = 4$	$y_2 = y_1 + hf(x_1, y_1) = -1 + (2)(1) = 1$	$f(x_2, y_2) = x_2 - y_2^2 = 4 - (1)^2 = 3$
3	$x_3 = x_2 + h = 4 + 2 = 6$	$y_3 = y_2 + hf(x_2, y_2) = 1 + (2)(3) = 7$	