THE GAUSS–BONNET THEOREM

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1. The geodesic curvature

Consider a parametric surface M parametrized as $\boldsymbol{x}(u, v)$, and an arbitrary (smooth) curve C on M expressed as a unit speed curve through $\boldsymbol{\alpha}(s) = \boldsymbol{x}\big(u(s), v(s)\big)$. We write $\boldsymbol{t}(s) = \boldsymbol{\alpha}'(s)$ for the unit tangent the curve at the point $p = \boldsymbol{\alpha}(s)$, and and $\boldsymbol{N}$ for the unit normal to the surface at p.

Let $\boldsymbol{u} = \boldsymbol{N} \times \boldsymbol{t}$. Then $\boldsymbol{u}$ is orthogonal to $\boldsymbol{N}$ and therefore it lies within the tangent plane T_p. The triplet $\{\boldsymbol{t}, \boldsymbol{u}, \boldsymbol{N}\}$ forms a right-handed orthonormal basis for vectors in 3D. Let us express the vector $\boldsymbol{t}'$ in that basis:

$$\boldsymbol{t}' = \kappa_g \boldsymbol{u} + \eta \boldsymbol{N}.$$

There is no $\boldsymbol{t}$ component since the derivative of the unit vector $\boldsymbol{t}$ is perpendicular to $\boldsymbol{t}$. The coefficient κ_g is called C's *geodesic curvature* at p. It measure the deviation of the curve C from a geodesic at the point p.

Considering that κ_g is the component of $\boldsymbol{t}'$ along $\boldsymbol{u}$, we have $\kappa_g = \boldsymbol{u} \cdot \boldsymbol{t}'$. Also considering that $\boldsymbol{u} = \boldsymbol{N} \times \boldsymbol{t}$, we see that

$$\kappa_g = (\boldsymbol{N} \times \boldsymbol{t}) \cdot \boldsymbol{t}' = [\boldsymbol{N}, \boldsymbol{t}, \boldsymbol{t}'] = [\boldsymbol{t}, \boldsymbol{t}', \boldsymbol{N}] = (\boldsymbol{t} \times \boldsymbol{t}') \cdot \boldsymbol{N}, \tag{1}$$

where $[\cdot, \cdot, \cdot]$ is the usual triple-scalar-product of vectors.

Let us observe that $\boldsymbol{x}_u \times \boldsymbol{x}_v$ is orthogonal to the surface, and therefore $\boldsymbol{N} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}$. We have[1]

$$\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2 = (\boldsymbol{x}_u \times \boldsymbol{x}_v) \cdot (\boldsymbol{x}_u \times \boldsymbol{x}_v) = (\boldsymbol{x}_u \cdot \boldsymbol{x}_u)(\boldsymbol{x}_v \cdot \boldsymbol{x}_v) - (\boldsymbol{x}_u \cdot \boldsymbol{x}_v)^2 = EG - F^2,$$

where E, F and G are the metric coefficients. We conclude that

$$\boldsymbol{N} = \frac{1}{\sqrt{EG - F^2}} (\boldsymbol{x}_u \times \boldsymbol{x}_v). \tag{2}$$

We now proceed to evaluate $\boldsymbol{t} \times \boldsymbol{t}'$. We have

$$\boldsymbol{t} = \boldsymbol{\alpha}' = \boldsymbol{x}_u u' + \boldsymbol{x}_v v', \tag{3}$$

where a prime indicate the derivative with respect to the arclength s. We also have

$$\begin{aligned} \boldsymbol{t}' &= \big(\boldsymbol{x}_{uu} u' + \boldsymbol{x}_{uv} v'\big) u' + \boldsymbol{x}_u u'' + \big(\boldsymbol{x}_{vu} u' + \boldsymbol{x}_{vv} v'\big) v' + \boldsymbol{x}_v v'' \\ &= \boldsymbol{x}_{uu} u'^2 + 2\boldsymbol{x}_{uv} u' v' + \boldsymbol{x}_{vv} v'^2 + \boldsymbol{x}_u u'' + \boldsymbol{x}_v v''. \end{aligned}$$

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[1]Here and elsewhere we make use of Lagrange's identity

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}).$$

Then we compute

$$\begin{aligned}
\boldsymbol{t}\times\boldsymbol{t}' &= \left(\boldsymbol{x}_u u' + \boldsymbol{x}_v v'\right) \times \left(\boldsymbol{x}_{uu} u'^2 + 2\boldsymbol{x}_{uv} u'v' + \boldsymbol{x}_{vv} v'^2 + \boldsymbol{x}_u u'' + \boldsymbol{x}_v v''\right) \\
&= (\boldsymbol{x}_u\times\boldsymbol{x}_{uu})u'^3 + 2(\boldsymbol{x}_u\times\boldsymbol{x}_{uv})u'^2v' + (\boldsymbol{x}_u\times\boldsymbol{x}_{vv})u'v'^2 \\
&\quad + (\boldsymbol{x}_v\times\boldsymbol{x}_{uu})u'^2v' + 2(\boldsymbol{x}_v\times\boldsymbol{x}_{uv})u'v'^2 + (\boldsymbol{x}_v\times\boldsymbol{x}_{vv})v'^3 \\
&\quad + (\boldsymbol{x}_u\times\boldsymbol{x}_v)(u'v'' - u''v'),
\end{aligned}$$

and the dot-multiply the result by $\boldsymbol{N}$. In view of (1) we conclude that

$$\begin{aligned}
\kappa_g &= (\boldsymbol{x}_u\times\boldsymbol{x}_{uu})\cdot\boldsymbol{N}u'^3 + 2(\boldsymbol{x}_u\times\boldsymbol{x}_{uv})\cdot\boldsymbol{N}u'^2v' + (\boldsymbol{x}_u\times\boldsymbol{x}_{vv})\cdot\boldsymbol{N}u'v'^2 \\
&\quad + (\boldsymbol{x}_v\times\boldsymbol{x}_{uu})\cdot\boldsymbol{N}u'^2v' + 2(\boldsymbol{x}_v\times\boldsymbol{x}_{uv})\cdot\boldsymbol{N}u'v'^2 + (\boldsymbol{x}_v\times\boldsymbol{x}_{vv})\cdot\boldsymbol{N}v'^3 \\
&\quad + (\boldsymbol{x}_u\times\boldsymbol{x}_v)\cdot\boldsymbol{N}(u'v'' - u''v'),
\end{aligned} \tag{4}$$

Now we examine the individual terms. We have:

$$\begin{aligned}
(\boldsymbol{x}_u\times\boldsymbol{x}_{uu})\cdot\boldsymbol{N} &= \frac{1}{\sqrt{EG-F^2}}\,(\boldsymbol{x}_u\times\boldsymbol{x}_{uu})\cdot(\boldsymbol{x}_u\times\boldsymbol{x}_v) \\
&= \frac{1}{\sqrt{EG-F^2}}\Big((\boldsymbol{x}_u\cdot\boldsymbol{x}_u)(\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_v) - (\boldsymbol{x}_u\cdot\boldsymbol{x}_v)(\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_u)\Big) \\
&= \frac{1}{\sqrt{EG-F^2}}\,\Big((\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_v)E - (\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_u)F\Big)
\end{aligned}$$

To further simplify this, let us recall that

$$\boldsymbol{x}_{uu} = \Gamma^1_{11}\boldsymbol{x}_u + \Gamma^2_{11}\boldsymbol{x}_v + e\boldsymbol{N},$$

and therefore

$$\begin{aligned}
\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_u &= (\Gamma^1_{11}\boldsymbol{x}_u + \Gamma^2_{11}\boldsymbol{x}_v + e\boldsymbol{N})\cdot\boldsymbol{x}_u = \Gamma^1_{11}E + \Gamma^2_{11}F, \\
\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_v &= (\Gamma^1_{11}\boldsymbol{x}_u + \Gamma^2_{11}\boldsymbol{x}_v + e\boldsymbol{N})\cdot\boldsymbol{x}_v = \Gamma^1_{11}F + \Gamma^2_{11}G.
\end{aligned}$$

We see that

$$(\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_v)E - (\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_u)F = (\Gamma^1_{11}F + \Gamma^2_{11}G)E - (\Gamma^1_{11}E + \Gamma^2_{11}F)F = \Gamma^2_{11}(EG-F^2),$$

and therefore

$$(\boldsymbol{x}_u\times\boldsymbol{x}_{uu})\cdot\boldsymbol{N} = \Gamma^2_{11}\sqrt{EG-F^2}. \tag{5a}$$

Similarly, we calculate

$$(\boldsymbol{x}_u\times\boldsymbol{x}_{uv})\cdot\boldsymbol{N} = \cdots, \tag{5b}$$
$$(\boldsymbol{x}_u\times\boldsymbol{x}_{vv})\cdot\boldsymbol{N} = \cdots, \tag{5c}$$
$$(\boldsymbol{x}_v\times\boldsymbol{x}_{uu})\cdot\boldsymbol{N} = \cdots, \tag{5d}$$
$$(\boldsymbol{x}_u\times\boldsymbol{x}_{uv})\cdot\boldsymbol{N} = \cdots, \tag{5e}$$
$$(\boldsymbol{x}_v\times\boldsymbol{x}_{vv})\cdot\boldsymbol{N} = \cdots, \tag{5f}$$
$$(\boldsymbol{x}_u\times\boldsymbol{x}_v)\cdot\boldsymbol{N} = \cdots, \tag{5g}$$

and plug the results into (4) and observe that it reduces to

$$\begin{aligned}
\kappa_g = \Big[\Gamma^2_{11}u'^3 &+ (2\Gamma^2_{12} - \Gamma^1_{11})u'^2v' \\
&+ (\Gamma^2_{22} - 2\Gamma^1_{12})u'v'^2 - \Gamma^1_{22}v'^3 + u'v'' - u''v'\Big]\sqrt{EG-F^2}.
\end{aligned} \tag{6}$$

This is a significant result as it expresses κ_g solely in terms of the metric coefficients E, F, and G. It says that the geodesic curvature is an intrinsic property of the surface—it can be determined by taking measurements within the surface by the inhabitants of that two-dimensional world.

Remark 1. The geodesic curvature of the parametric line $v \equiv$ constant is obtained by setting $v' = 0$ in the equation above:

$$\kappa_g\Big|_{v\equiv\text{const}} = \Gamma^2_{11} u'^3 \sqrt{EG - F^2}.$$

This simplifies somewhat in view of (3) together with the fact that $\boldsymbol{t}$ is a unit vector and $v' = 0$. We have

$$1 = \boldsymbol{t} \cdot \boldsymbol{t} = (\boldsymbol{x}_u u') \cdot (\boldsymbol{x}_u u') = (\boldsymbol{x}_u \cdot (\boldsymbol{x}_u) u'^2 = E u'^2,$$

whence $u' = 1/\sqrt{E}$. We conclude that

$$\kappa_g\Big|_{v\equiv\text{const}} = \Gamma^2_{11} \frac{\sqrt{EG - F^2}}{E^{3/2}}. \tag{7a}$$

and similarly

$$\kappa_g\Big|_{u\equiv\text{const}} = -\Gamma^1_{22} \frac{\sqrt{EG - F^2}}{G^{3/2}}. \tag{7b}$$

Exercise 1. Show that if the the parametric curves are mutually orthogonal (that is $F = 0$) then (6) reduces to (there may be typos here, check!)

$$\kappa_g = \Big[-\frac{E_v}{G} u'^3 + (2\frac{G_u}{G} - \frac{E_u}{2E}) u'^2 v' + (\frac{G_v}{G} - 2\frac{E_v}{2E}) u' v'^2 + \frac{G_u}{E} v'^3 + u' v'' - u'' v' \Big] \sqrt{EG}.$$

2. A second look at the geodesic curvature

Figure 1 depicts a surface parametrized as $\boldsymbol{x}(u, v)$ and a closed curve C that lies on the surface. For the sake of simplifying this first exposition, we assume that:

- the part of the surface enclosed by C is homeomorphic to the unit disk, that is, it has no holes;
- the coordinate lines u = const and v = const are mutually orthogonal, that is, $F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = 0$;
- C is parameterized by the arclength, that is, its equation is $\boldsymbol{\alpha}(s) = \boldsymbol{x}\big(u(s), v(s)\big)$, its tangent vector is $\boldsymbol{t}(s) = \boldsymbol{\alpha}'(s)$, and $\|\boldsymbol{t}(s)\| = 1$ at all s.

We define the unit normal $\boldsymbol{N}$ to the surface through

$$\boldsymbol{N} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|},$$

and therefore rotating $\boldsymbol{x}_u$ through a 90 angle about $\boldsymbol{N}$ produced the vector $\boldsymbol{x}_v$. (Recall that we are assuming that the u and v coordinate lines are mutually orthogonal.) The curve C inherits an orientation through that rotation. That orientation is marked by an arrowhead on C is Figure 1.

We introduce unit normals $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ along $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, respectively. Observe that the length of $\boldsymbol{x}_u$ is $\sqrt{\boldsymbol{x}_u \cdot \boldsymbol{x}_u} = \sqrt{E}$. Similarly, the length of $\boldsymbol{x}_v$ is $\sqrt{G}$. Therefore, we have

$$\boldsymbol{e}_1 = \frac{\boldsymbol{x}_u}{\sqrt{E}}, \quad \boldsymbol{e}_2 = \frac{\boldsymbol{x}_v}{\sqrt{G}}. \tag{8}$$

Referring to Figure 1, we write $\theta(s)$ for the angle between $\boldsymbol{x}_u$ and $\boldsymbol{t}$ at any point along C. Then C's unit tangent vector $\boldsymbol{t}$ may be expressed in terms of components along $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ through

$$\boldsymbol{t} = \boldsymbol{e}_1 \cos\theta + \boldsymbol{e}_2 \sin\theta. \tag{9a}$$

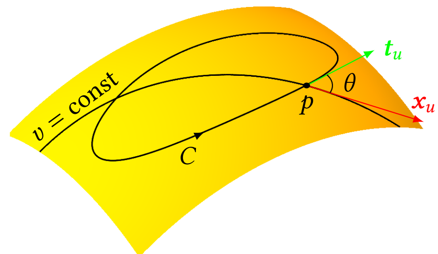

FIGURE 1. The u coordinate line intersects the curve C at p. The curve's tangent vector $\boldsymbol{t}$ at p makes an angle of θ relative to $\boldsymbol{x}_u$. If we go one complete turn around the curve, the angle θ increments by 2π radians.

We introduce the unit vector $\boldsymbol{u} = \boldsymbol{N} \times \boldsymbol{t}$ which is obtained by rotating $\boldsymbol{t}$ by 90 degrees counterclockwise within the tangent plane; see Figure 2. We have

$$\boldsymbol{u} = -\boldsymbol{e}_1 \sin\theta + \boldsymbol{e}_2 \cos\theta. \tag{9b}$$

Recall that the geodesic curvature of C at p is the projection of $\boldsymbol{t}'$ onto $\boldsymbol{u}$, that is

$$\kappa_g = \boldsymbol{u} \cdot \boldsymbol{t}',$$

where the prime indicates the derivative with respect to the arclength along C. To evaluate κ_g, we begin with calculating $\boldsymbol{t}'$:

$$\begin{aligned}
\boldsymbol{t}' &= \boldsymbol{e}_1' \cos\theta - \boldsymbol{e}_1\theta' \sin\theta + \boldsymbol{e}_2' \sin\theta + \boldsymbol{e}_2\theta' \cos\theta \\
&= (-\boldsymbol{e}_1 \sin\theta + \boldsymbol{e}_2 \cos\theta)\,\theta' + \boldsymbol{e}_1' \cos\theta + \boldsymbol{e}_2' \sin\theta \\
&= \boldsymbol{u}\,\theta' + \boldsymbol{e}_1' \cos\theta + \boldsymbol{e}_2' \sin\theta.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\kappa_g &= \boldsymbol{u} \cdot \Big[\boldsymbol{u}\,\theta' + \boldsymbol{e}_1' \cos\theta + \boldsymbol{e}_2' \sin\theta\Big] \\
&= \boldsymbol{u} \cdot \boldsymbol{u}\,\theta' + \boldsymbol{u} \cdot \Big[\boldsymbol{e}_1' \cos\theta + \boldsymbol{e}_2' \sin\theta\Big] \\
&= \theta' + (-\boldsymbol{e}_1 \sin\theta + \boldsymbol{e}_2 \cos\theta) \cdot (\boldsymbol{e}_1' \cos\theta + \boldsymbol{e}_2' \sin\theta) \\
&= \theta' - \boldsymbol{e}_1 \cdot \boldsymbol{e}_1' \sin\theta\cos\theta - \boldsymbol{e}_1 \cdot \boldsymbol{e}_2' \sin^2\theta + \boldsymbol{e}_2 \cdot \boldsymbol{e}_1' \cos^2\theta + \boldsymbol{e}_2 \cdot \boldsymbol{e}_2' \sin^2\theta.
\end{aligned}$$

But $\boldsymbol{e}_1 \cdot \boldsymbol{e}_1' = 0$ and $\boldsymbol{e}_2 \cdot \boldsymbol{e}_2' = 0$ because $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors. Furthermore, since $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are mutually orthogonal, we have $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = 0$ and therefore $\boldsymbol{e}_1' \cdot \boldsymbol{e}_2 + \boldsymbol{e}_1 \cdot \boldsymbol{e}_2' = 0$, that is, $-\boldsymbol{e}_1 \cdot \boldsymbol{e}_2' = \boldsymbol{e}_1' \cdot \boldsymbol{e}_2$, and therefore the expression above simplifies to

$$\kappa_g = \theta' + \boldsymbol{e}_1' \cdot \boldsymbol{e}_2.$$

To proceed further, let us calculate

$$\boldsymbol{e}_1' = \frac{d\boldsymbol{e}_1}{ds} = \frac{\partial \boldsymbol{e}_1}{\partial u}\frac{du}{ds} + \frac{\partial \boldsymbol{e}_1}{\partial v}\frac{dv}{ds} = \Big(\frac{\boldsymbol{x}_u}{\sqrt{E}}\Big)_u u' + \Big(\frac{\boldsymbol{x}_u}{\sqrt{E}}\Big)_v v',$$

where we have replaced $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ by their definitions in (8). We calculate each of the terms of the right-hand side separately and then add up. We have

$$\Big(\frac{\boldsymbol{x}_u}{\sqrt{E}}\Big)_u = \frac{\boldsymbol{x}_{uu}E^{1/2} - \boldsymbol{x}_u\big(E^{1/2}\big)_u}{E} = \frac{\boldsymbol{x}_{uu}E^{1/2} - \frac{1}{2}\boldsymbol{x}_u E^{-1/2}E_u}{E}$$

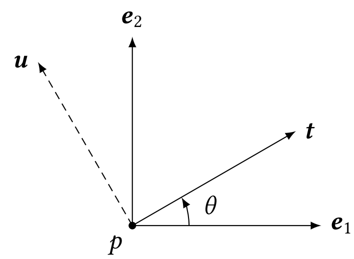

FIGURE 2. This is a "bird's eye view" of the tangent plane T_p at p. The unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ point along the directions $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, and the unit vector $\boldsymbol{t}$ is the unit tangent to the curve (not shown) that passes through p. The unit vector $\boldsymbol{u}$ is obtained by rotating $\boldsymbol{t}$ counterclockwise by 90 degrees.

But $E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u$ and therefore $E_u = 2\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u$. We conclude that

$$\left(\frac{\boldsymbol{x}_u}{\sqrt{E}}\right)_u = \frac{\boldsymbol{x}_{uu}E^{1/2} - \boldsymbol{x}_u E^{-1/2}(\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u)}{E}.$$

Similarly, we calculate

$$\left(\frac{\boldsymbol{x}_u}{\sqrt{E}}\right)_v = \frac{\boldsymbol{x}_{uv}E^{1/2} - \boldsymbol{x}_u\left(E^{1/2}\right)_v}{E} = \frac{\boldsymbol{x}_{uv}E^{1/2} - \frac{1}{2}\boldsymbol{x}_u E^{-1/2}E_v}{E}$$

But $E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u$ and therefore $E_v = 2\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u$. We conclude that

$$\left(\frac{\boldsymbol{x}_u}{\sqrt{E}}\right)_v = \frac{\boldsymbol{x}_{uv}E^{1/2} - \boldsymbol{x}_u E^{-1/2}(\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u)}{E}.$$

Resuming the interrupted calculation, we have

$$\boldsymbol{e}_1' = \frac{\boldsymbol{x}_{uu}E^{1/2} - \boldsymbol{x}_u E^{-1/2}(\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u)}{E}u' + \frac{\boldsymbol{x}_{uv}E^{1/2} - \boldsymbol{x}_u E^{-1/2}(\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u)}{E}v'.$$

Recalling that $\boldsymbol{e}_2 = \boldsymbol{x}_v/\sqrt{G}$, we arrive at

$$\boldsymbol{e}_1' \cdot \boldsymbol{e}_2 = \frac{\boldsymbol{x}_{uu}E^{1/2} - \boldsymbol{x}_u E^{-1/2}(\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u)}{E\sqrt{G}}u' \cdot \boldsymbol{x}_v + \frac{\boldsymbol{x}_{uv}E^{1/2} - \boldsymbol{x}_u E^{-1/2}(\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u)}{E\sqrt{G}}v' \cdot \boldsymbol{x}_v.$$

Due to our assumption that $F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = 0$, the expression above simplifies to

$$\boldsymbol{e}_1' \cdot \boldsymbol{e}_2 = \frac{1}{\sqrt{EG}}\left[\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v\, u' + \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v\, v'\right]. \tag{10}$$

To further simplify, we recall that

$$\boldsymbol{x}_{uu} = \Gamma^1_{11}\boldsymbol{x}_u + \Gamma^2_{11}\boldsymbol{x}_v + e\boldsymbol{N}, \quad \boldsymbol{x}_{uv} = \Gamma^1_{12}\boldsymbol{x}_u + \Gamma^2_{12}\boldsymbol{x}_v + g\boldsymbol{N}.$$

When $F = 0$, the Christoffel symbols reduce to

$$\Gamma^1_{11} = \frac{E_u}{2E}, \quad \Gamma^1_{12} = \frac{E_v}{2E}, \quad \Gamma^1_{22} = -\frac{G_u}{2E},$$
$$\Gamma^2_{11} = -\frac{E_v}{2G}, \quad \Gamma^2_{12} = \frac{G_u}{2G}, \quad \Gamma^2_{22} = \frac{G_v}{2G},$$

and therefore

$$\boldsymbol{x}_{uu} = \frac{E_u}{2E}\boldsymbol{x}_u - \frac{E_v}{2G}\boldsymbol{x}_v, \quad \boldsymbol{x}_{uv} = \frac{E_v}{2E}\boldsymbol{x}_u + \frac{G_u}{2G}\boldsymbol{x}_v.$$

and

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v = -\frac{E_v}{2G}\boldsymbol{x}_v \cdot \boldsymbol{x}_v = -\frac{1}{2}E_v, \quad \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v = \frac{G_u}{2G}\boldsymbol{x}_v \cdot \boldsymbol{x}_v = \frac{1}{2}G_u.$$

We conclude that

$$\boldsymbol{e}_1' \cdot \boldsymbol{e}_2 = \frac{1}{2\sqrt{EG}}\Big[G_u\, v' - E_v\, u'\Big],$$

and then, referring to (10)

$$\kappa_g = \theta' + \frac{1}{2\sqrt{EG}}\Big[G_u\, v' - E_v\, u'\Big]. \tag{11}$$

Remark 2. As Tristan Needham has pointed out, Gauss's choice of E and G for the metric coefficient is rather unfortunate, as letting $E = A^2$ and $G = B^2$ avoids all those square roots in the calculations and leads to much cleaner formulas in terms of A and B. As a case in point, the expression (11) would take the form

$$\kappa_g = \theta' + \frac{B_u}{B}u' - \frac{A_u}{A}v'.$$

Returning to (11), let us integrate its two sides over the entire boundary of C. If C is a smooth, in the sense it has no corners (e.g., it's *not* a triangle) then θ' integrates to 2π, and considering that $u' = du/ds$ and $v' = dv/ds$, we get

$$\int_C \kappa_g\, ds = 2\pi + \int_C \frac{1}{2\sqrt{EG}}\Big[G_u\, v' - E_v\, u'\Big]\, ds = 2\pi + \int_{\hat{C}} \left[\frac{G_u}{2\sqrt{EG}}\, dv - \frac{E_v}{2\sqrt{EG}}\, du\right].$$

Note the change of the domain of integration from C to $\hat{C}$ at the last step. That's because in the integration over C, the variable of integration is the arc length s, therefore $\int_C \cdots\, ds$ is tantamount to $\int_0^L \cdots\, ds$, where L is the length of the closed curve C. In the last step, we have changed the variables of the line integral to u and v. The integration now takes place over the curve $\hat{C}$ which is the pre-image of the curve C in the *parameter space uv*.

Similarly, let Ω be the surface patch in Figure 1 delimited by the curve C. Its pre-image in the parameter space is the region $\hat{\Omega}$ enclosed by the curve $\hat{C}$. The integral over $\hat{C}$ in the displayed equation above may be converted to an integral of the domain $\hat{\Omega}$ through Green's Theorem, whereby

$$\begin{aligned}\int_C \kappa_g\, ds &= 2\pi + \int_{\hat{\Omega}} \left[\left(\frac{G_u}{2\sqrt{EG}}\right)_u + \left(\frac{E_v}{2\sqrt{EG}}\right)_v\right] du\, dv. \\ &= 2\pi + \int_{\hat{\Omega}} \frac{1}{\sqrt{EG}} \left[\left(\frac{G_u}{2\sqrt{EG}}\right)_u + \left(\frac{E_v}{2\sqrt{EG}}\right)_v\right] \sqrt{EG}\, du\, dv.\end{aligned}$$

Referring to equation (20) of the class notes `dg.pdf`, we see that the Gaussian curvature formula (obtained under the assumption $F = 0$) is

$$K = -\frac{1}{\sqrt{EG}} \left[\left(\frac{G_u}{2\sqrt{EG}}\right)_u + \left(\frac{E_v}{2\sqrt{EG}}\right)_v\right].$$

This leads to

$$\int_C \kappa_g\, ds = 2\pi - \int_{\hat{\Omega}} K\sqrt{EG}\, du\, dv.$$

Finally, we change the integration from the $\hat{\Omega}$ domain to the surface patch Ω, and having in mind that the differential area element dA in Ω is related to the differential area element $du\, dv$ in $\hat{\Omega}$ through $dA = \sqrt{EG}\, du\, dv$, we arrive at

$$\int_C \kappa_g\, ds + \int_\Omega K\, dA = 2\pi. \tag{12}$$

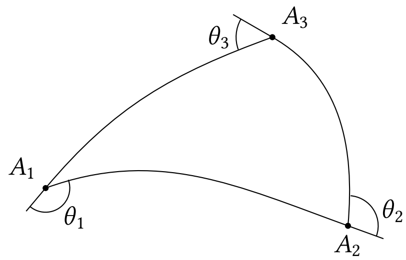

Figure 3. The "triangle" C lies on the surface $\boldsymbol{x}(u, v)$. Its edges are arbitrary curves that intersect at the three vertices. The external angles are θ_1, θ_2, θ_3.

This is the most basic form of the *Gauss–Bonnet Theorem.* The integral of the Gaussian curvature K over Ω is called the *total curvature of* Ω.

2.1. **Application to a triangular patch.** There are several variants of the Gauss–Bonnet Theorem. For instance, we have assumed that the domains of integration Ω and $\hat{\Omega}$ are homeomorphic to the unit disk, that is, they may not have holes. That excludes annular domains. As an another instance, we have assumed that the curves C and $\hat{C}$ are smooth, that is, they may not have corners. That excludes common domains such as triangles and polygons. Modified versions of (12) accommodate for these.

Let us see how we may extend (12) to a triangular domain. Figure 3 shows the curve C as "triangle" whose edges are arbitrary curves lying on the surface $\boldsymbol{x}(u, v)$ and whose external angles are θ_1, θ_2, θ_3. A we traverse the curve in a complete cycle, the tangent $\boldsymbol{t}$ turns by 2π radians as in the smooth case. The integral $\int_C d\theta$, however, picks up only the changes in θ along the curved edges; it is oblivious to the discontinuous increments at the vertices. The correct expression is

$$\theta_1 + \theta_2 + \theta_3 + \int_C d\theta = 2\pi,$$

which is better expressed as

$$\int_C d\theta = 2\pi - \theta_1 - \theta_2 - \theta_3.$$

In our previous calculation in the context of a smooth curve C, we integrated (11) around C and replaced the integral of θ' by 2π. In the case of a triangular curve, the integration of θ' yields the modified value shown above, and therefore the Gauss–Bonnet Theorem over a triangle takes the form

$$\int_C \kappa_g \, ds + \int_\Omega K \, dA = 2\pi - \theta_1 - \theta_2 - \theta_3. \tag{13}$$

Let's write α_k for the *internal angles* of the triangle. We have $\theta_k = \pi - \alpha_k$, $k = 1, 2, 3$; see Figure 4. Substituting these into (13), we get

$$\int_C \kappa_g \, ds + \int_\Omega K \, dA = \alpha_1 + \alpha_2 + \alpha_3 - \pi. \tag{14}$$

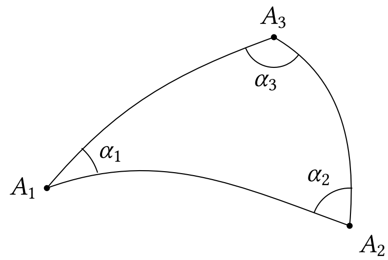

FIGURE 4. This is the a copy of Figure 3's "triangle" where θ_k marked the triangle's *external angles*. In the current figure α_k mark the triangle's *internal angles*. These are *supplementary* to the external angles, that is, $\alpha_k + \theta_k = \pi$, $k = 1, 2, 3$.

In particular, if the triangle's edges are geodesics, then $\kappa_g = 0$ on the edges, and we arrive at the interesting conclusion that

$$\int_\Omega K\,dA = \alpha_1 + \alpha_2 + \alpha_3 - \pi. \tag{15}$$

The right-hand side of that expression is called the triangle's *angular excess*. The result is expressed in words as:

> *The total curvature of a geodesic triangle equals its* angular excess.

In that regard, Gauss writes:

> "Hoc theorema, quod ni fallimur, ad elegantissima in theoria superficierum curvarum referendum esse videtur." [2]

Another way of looking at (15) is to divide it by the area of Ω:

$$\frac{1}{\text{area of }\Omega}\int_\Omega K\,dA = \frac{\alpha_1 + \alpha_2 + \alpha_3 - \pi}{\text{area of }\Omega},$$

and then pass to the limit as Ω shrinks to a point p. The left-hand side, being the average of K over Ω, converges to the curvature of the surface at p, and we arrive at a very appealing characterization of the Gaussian curvature:

$$K = \lim_{\Omega\to p}\frac{\text{angular excess of }\Omega}{\text{area of }\Omega}.$$

2.2. **Total curvature as a topological invariant.** Consider a closed surface Ω which is topologically equivalent to a sphere, that is, Ω is obtained by stretching and deforming a sphere without tearing and or introducing self-intersections. In the same way that the sphere's equator is the boundary between two hemispheres, the image of the equator under that deformation is the boundary C between two disjoint surfaces Ω_1 and Ω_2. For the sake of simplicity, assume that C is a smooth curve. Each of Ω_1 and Ω_2 is homeomorphic

[2] "This theorem, if we are not mistaken, ought to be counted among the most elegant in the theory of curved surfaces."

to a disk. and therefore the Gauss-Bonnet Theorem applies:

$$\int_C \kappa_g \, ds + \int_{\Omega_1} K \, dA = 2\pi,$$
$$-\int_C \kappa_g \, ds + \int_{\Omega_2} K \, dA = 2\pi.$$

The minus sign is to account for the fact that the orientation of C reverses when viewed as the boundary of Ω_1 versus the boundary of Ω_2. Adding the two equations above we see that

$$\int_\Omega K \, dA = 4\pi. \tag{16}$$

Stated in words:

> *The total curvature of a surface which is topologically equivalent to a sphere is 4π.*

This is an impressive result. The particularities of the surface Ω such as its shape and size do not affect affect the result. As long as the surface looks like a distorted sphere, its total curvature is 4π!.