Math 423: Differential Geometry

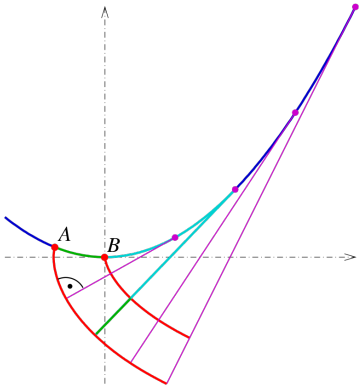
Spring 2024

Exam 1

1. [10 pts] The following figure is taken from the Wikipedia page

   `https://en.wikipedia.org/wiki/Involute`

   It shows a parabola and two involutes. Find the equations of the two involutes and produce a similar drawing in a software of your choice. It need not be exactly the same in every detail.



2. [8 pts] The definition of the involute generalizes from plane curves to space curves as follows. To construct the involute of the space curve $\boldsymbol{\alpha}(t)$, let $\boldsymbol{T}(t)$ be the unit tangent vector to the curve, and let $s(t)$ be the arclength measured from some $t = t_0$. Then the involute that starts out at the point $\boldsymbol{\alpha}(t_0)$ is defined to be the curve $\boldsymbol{\beta}(t) = \boldsymbol{\alpha}(t) - s(t)\boldsymbol{T}(t)$, just as in the plane case.

   Find the equation of the involute of the helix $\boldsymbol{\alpha}(t) = \langle a\cos t, a\sin t, bt\rangle$ that starts out at $t = 0$. [Interestingly, it turns out that the involute is a plane curve!]

3. [12 pts] The parametric curve

$$\boldsymbol{\alpha}(t) = \begin{pmatrix} \sin t + 2\sin 2t \\ \cos t - 2\cos 2t \\ -\sin 3t \end{pmatrix}$$

   is called a *trefoil knot*. You may be interested in looking it up in the Wikipedia page

   `https://en.wikipedia.org/wiki/Trefoil%20knot`

   (a) Calculate the trefoil knot's Frenet triad $\{\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b}\}$ as a function of t.

   (b) Determine the corresponding Darboux vector $\boldsymbol{\omega}$.

   (c) Plot together in one diagram the graphs of the trefoil knot's curvature $\kappa(t)$, the torsion $\tau(t)$, and $|\boldsymbol{\omega}(t)|$.

   (d) Make an animation that shows the Frenet triad together with the Darboux vector, as the group flies around the trefoil curve.

   (e) Derive the equation of a tube of radius r that surrounds the trefoil knot. Plot the tube with a radius of your choice.