Math 423, Spring 2024 **Exam #0 on prerequisites**

Please make an effort to *write neatly*, and *insert a few words* here and there to get your ideas across. It's difficult to understand (and evaluate) mathematics in the absence of guiding words.

You may write your solutions by hand, but a it would be terrific if you could do it in LaTeX. In any case, upload your solutions to the course's site on Blackboard by the midnight of Wednesday February 7.

1. [3 pts] Let $f(x) = x^3 + 1$. Find the inverse function f^{-1}.

   Solution: Solving the equation $y = x^3 + 1$ for x we obtain $x = (y-1)^{1/3}$. We conclude that the inverse function of f is $f^{-1}(y) = (y-1)^{1/3}$.

2. [3 pts] Let $\boldsymbol{a} = \langle 1, 1, 1\rangle$ and $\boldsymbol{b} = \langle 1, 2, 3\rangle$. Evaluate the cross product $\boldsymbol{a} \times \boldsymbol{b}$.

   Solution: Let $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be the usual set of orthonormal vectors, and let's write

$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \quad \boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k}.$$

   Then

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} = (3-2)\,\boldsymbol{i} - (3-1)\,\boldsymbol{j} + (2-1)\,\boldsymbol{k} = \boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}.$$

   For the sake of consistency with the original notation, it's better to express the result as $\boldsymbol{a} \times \boldsymbol{b} = \langle 1, -2, 1\rangle$.

3. [3 pts] Find the equation of the tangent line to the parabola $y = x^2$ at $x = 2$.

   Solution: The y coordinate of the point of tangency is $y = x^2\big|_{x=2} = 2^2 = 4$. The slope there is $y' = 2x\big|_{x=2} = 4$. Therefore, the equation of the tangent line is $y - 4 = 4(x-2)$ which simplifies to $y = 4x - 4$.

4. [3 pts] S is the sphere of radius 1 centered at the origin of the xyz Cartesian coordinate system. Verify that the point P with coordinates $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{1}{2}\right)$ lies on the sphere. Find the equation of the tangent plane to the sphere at P.

   Solution: We have

$$\left(\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{16} + \frac{9}{16} + \frac{1}{4} = 1,$$

   which verifies that P lies on the sphere.

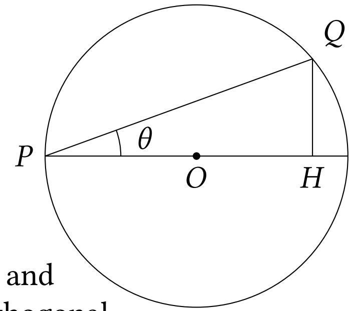
   The vector $\vec{OP} = \langle \frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{1}{2}\rangle$ from the origin O to the point P is orthogonal to the sphere, and therefore to the tangent plane. We conclude that the equation of the tangent plane is

$$\frac{\sqrt{3}}{4}\left(x - \frac{\sqrt{3}}{4}\right) + \frac{3}{4}\left(y - \frac{3}{4}\right) + \frac{1}{2}\left(z - \frac{1}{2}\right) = 0,$$

   which simplifies to

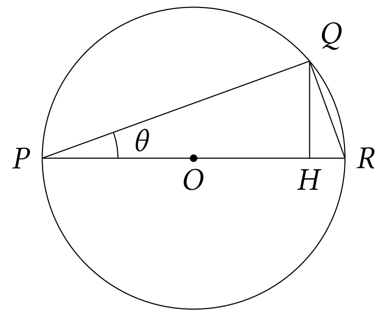
$$\frac{\sqrt{3}}{4}x + \frac{3}{4}y + \frac{1}{2}z = 1.$$

5. [3 pts] In the adjacent diagram the radius of the circle is a. What is the length of PH?

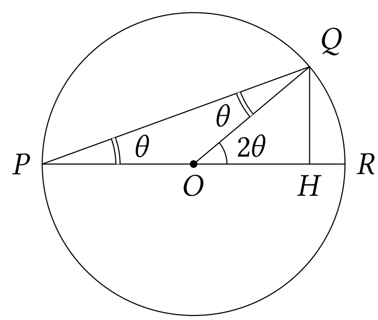


Solution #1: Let R be the diametrically opposite point of P. Referring to the figure below, the angle PQR is a right angle since it subtends the circle's diagonal.

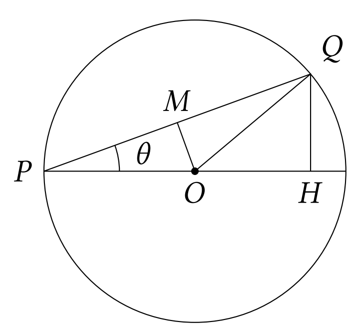
In the right-angled triangle PQR, the length of the hypotenuse is $2a$, and therefore the length of PQ is $2a\cos\theta$. The line segment PH is the orthogonal projection of PQ over the diagonal, and therefore its length is $\cos\theta$ times the length of PQ. We conclude that the length of PH is $2a\cos^2\theta$.



Solution #2: Referring to the diagram below, the angle QOR is an external angle of the isosceles triangle POQ, therefore it measures 2θ. Then in the right triangle QOH, the length of the leg OH is $a\cos 2\theta$, and therefore the length of PH is $a + a\cos 2\theta = a(1+\cos 2\theta) = 2a\cos^2\theta$.



Solution #3: Referring to the diagram below, drop a perpendicular OM from O onto PQ. In the right triangle PMO the length of the hypotenuse is a, and therefore the length of PM is $a\cos\theta$. Since POQ is an isosceles triangle, the length of PQ is twice that of PM, that is, $2a\cos\theta$. Then from the geometry of the right triangle PHQ we conclude that length of PH is $2a\cos^2\theta$.



6. [6 pts] Find the length of the parabolic arc $y = x^2$, $0 \le x \le 1$.

Solution: The length of any curve $y = f(x)$, $a \le x \le b$ is given by

$$L = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

In the present case we have $f(x) = x^2$, and therefore $f'(x) = 2x$ and the length is

$$L = \int_0^1 \sqrt{1 + 4x^2}\, dx$$

To evaluate the integral, we let $2x = \sinh u$. Then $2\,dx = \cosh u\, du$, that is, $dx = \frac{1}{2} \cosh u\, du$, and therefore

$$\begin{aligned}
\int \sqrt{1+4x^2}\, dx &= \frac{1}{2} \int \sqrt{1+\sinh^2 u}\, \cosh u\, du = \frac{1}{2} \int \cosh^2 u\, du = \frac{1}{4} \int (1 + \cosh 2u)\, du \\
&= \frac{1}{4}\left[u + \frac{1}{2} \sinh 2u\right] = \frac{1}{4}\left[u + \sinh u \cosh u\right] = \frac{1}{4}\left[u + \sinh u \sqrt{1+\sinh^2 u}\right] \\
&= \frac{1}{4}\left[\operatorname{arcsinh} 2x + (2x)\sqrt{1+(2x)^2}\right] = \frac{1}{4} \operatorname{arcsinh} 2x + \frac{1}{2} x \sqrt{1+4x^2} \\
&= \frac{1}{4} \ln\left[2x + \sqrt{1+4x^2}\right] + \frac{1}{2} x \sqrt{1+4x^2}.
\end{aligned}$$

We conclude that

$$L = \int_0^1 \sqrt{1+4x^2}\, dx = \frac{1}{4} \ln[2 + \sqrt{5}] + \frac{1}{2}\sqrt{5} \approx 1.4789.$$

A note on hyperbolic functions

The hyperbolic cosine and sine functions are defined as

$$\cosh u = \frac{1}{2}\left(e^u + e^{-u}\right), \quad \sinh u = \frac{1}{2}\left(e^u - e^{-u}\right).$$

From these definitions it follows that

$$\frac{d}{du} \cosh u = \sinh u, \tag{1a}$$

$$\frac{d}{du} \sinh u = \cosh u, \tag{1b}$$

$$\cosh^2 u - \sinh^2 u = 1, \tag{1c}$$

$$\cosh 2u = \cosh^2 u + \sinh^2 u \tag{1d}$$

$$\sinh 2u = 2 \sinh u \cosh u. \tag{1e}$$

(Be sure to verify these yourself. It’s not hard at all.)

The identities (1c) and (1d) may be combined to produce a few other useful identities. Look, for instance, at

$$\cosh 2u = \cosh^2 u + \sinh^2 u \overset{\text{by (1c)}}{=} \cosh^2 u + (\cosh^2 u - 1) = 2\cosh^2 u - 1.$$

and therefore

$$\cosh^2 u = \frac{1}{2}\big(1 + \cosh 2u\big). \tag{1f}$$

Yet another point: The functions arcsinh and arccosh may be expressed in terms of elementary functions. For instance, let $y = \operatorname{arcsinh} u$. Then $u = \sinh y = \frac{1}{2}\big(e^y - e^{-y}\big)$, and therefore $e^y - e^{-y} = 2u$. We multiply through by e^y and rearrange that into $(e^y)^2 - 2u(e^y) - 1 = 0$. This is a quadratic in e^y whose solution is $e^y = \frac{2u \pm \sqrt{4u^2+4}}{2} = u \pm \sqrt{1+u^2}$. The minus sign in the $\pm$ is not acceptable because e^y cannot be negative. We conclude that $e^y = u + \sqrt{1+u^2}$, and therefore $y = \ln\big[u + \sqrt{1+u^2}\big]$. Recalling that $y = \operatorname{arcsinh} u$, we conclude that

$$\operatorname{arcsinh} u = \ln\Big[u + \sqrt{1+u^2}\Big]. \tag{1g}$$

Do you see where these identities are used in evaluating the integral?

Challenge yourself: Derive the counterparts of (1f) and (1g) for $\sinh^2 u$ and $\operatorname{arccosh} u$.

7. [6 pts] Find the length of the parametrically defined curve

$$\boldsymbol{r}(t) = \langle t - \sin t, 1 - \cos t\rangle, \quad 0 \le t \le 2\pi.$$

Solution: The length of any parametric curve $\boldsymbol{r}(t) = \langle x(t), y(t)\rangle$, $a \le t \le b$, is given by

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dx.$$

In our case we have $x(t) = t - \sin t$, $y(t) = 1 - \cos t$, and therefore $x'(t) = 1 - \cos t$ and $y'(t) = \sin t$. Then we calculate

$$x'(t)^2 + y'(t)^2 = (1-\cos t)^2 + (\sin t)^2 = 1 - 2\cos t + \cos^2 t + \sin^2 t = 2(1-\cos t) = 4\sin^2\frac{t}{2}.$$

It follows that

$$\sqrt{x'(t)^2 + y'(t)^2} = 2\left|\sin\frac{t}{2}\right|.$$

We are going to integrate this over $0 \le t \le 2\pi$. Then we have $\le \frac{t}{2} \le \pi$ and therefore $\sin\frac{t}{2} \ge 0$ and the absolute value signs are not required. That is, in this range we have

$$\sqrt{x'(t)^2 + y'(t)^2} = 2\sin\frac{t}{2}.$$

Then the curve's length is

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_0^{2\pi} 2\sin\frac{t}{2}\, dt = -4\cos\frac{t}{2}\Big|_0^{2\pi} = -4\big[\cos\pi - \cos 0\big] = 8.$$

8. [6 pts] Consider the graph of $z = x^2 - y^2 + 5$ in the Cartesian coordinates. Find the surface area of the part of that graph that lies above the disk $x^2 + y^2 \le 1$.

Solution: The area of the graph of any function $z = f(x,y)$ over a region D is the value of the double integral

$$\iint_D \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA,$$

where f_x and f_y denote the partial derivatives of f with respect to x and y. In the current problem we have $f(x, y) = x^2 + y^2$ and therefore the integrand is $\sqrt{1 + 4x^2 + 4y^2}$.

Considering that our domain D is a disk, it makes sense to do the calculations in polar coordinates (r, θ), where

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Then the integrand simplifies to $\sqrt{1 + 4r^2}$. The area element dA, as we have seen in multivariable calculus, is $dA = r\,dr\,d\theta$. We now calculate

$$\text{area} = \int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2}\, r\,dr\,d\theta = 2\pi \int_0^1 \sqrt{1 + 4r^2}\, r\,dr.$$

To continue, we let $u = 1 + 4r^2$, whence $du = 8r\,dr$, that is, $r\,dr = \frac{1}{8}du$. The limits of integration change to 1 and 5. We conclude that

$$\text{area} = 2\pi \int_1^5 u^{1/2}\,\frac{du}{8} = 2\pi \times \frac{1}{8} \times \frac{2}{3}u^{3/2}\Big|_1^5 = \frac{\pi}{6}\left[5\sqrt{5} - 1\right] \approx 5.33.$$