NOTES OF DIFFERENTIAL GEOMETRY

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1. The shape operator [sections 2.2 and 2.3 of the textbook]

We write M for a surface and $\boldsymbol{N}$ for a generic unit normal to M. We also write T_p for the tangent plane to the surface at the point $p \in M$.

Let $\boldsymbol{v}$ be a vector in T_p and let $\nabla_{\boldsymbol{v}}\boldsymbol{N}$ be the derivative of $\boldsymbol{N}$ in the direction $\boldsymbol{v}$. Since $\boldsymbol{N}$ is a unit vector, the derivative is orthogonal to it, and therefore it lies within T_p. The linear mapping $\mathscr{S}_p : \boldsymbol{v} \mapsto -\nabla_{\boldsymbol{v}}\boldsymbol{N}$ of T_p into itself is called the surface's *shape operator* at p.

Theorem 1. *The shape operator is symmetric.*

Proof. Let $\boldsymbol{x}(u, v)$ be a surface patch covering the point p. The vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are in T_p, so $\mathscr{S}_p(\boldsymbol{x}_u)$ and $\mathscr{S}_p(\boldsymbol{x}_v)$ are well-defined. Considering that $\boldsymbol{x}_u$ is orthogonal to $\boldsymbol{N}$, we have $\boldsymbol{x}_u \cdot \boldsymbol{N} = 0$. Differentiating this in the direction $\boldsymbol{x}_v$ we get

$$0 = \frac{\partial \boldsymbol{x}_u}{\partial v} \cdot \boldsymbol{N} + \boldsymbol{x}_u \cdot \frac{\partial \boldsymbol{N}}{\partial v} = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} + \boldsymbol{x}_u \cdot \nabla_{\boldsymbol{x}_v}\boldsymbol{N} = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} - \boldsymbol{x}_u \cdot \mathscr{S}_p(\boldsymbol{x}_v).$$

We conclude that

$$\boldsymbol{x}_u \cdot \mathscr{S}_p(\boldsymbol{x}_v) = \boldsymbol{x}_{uv} \cdot \boldsymbol{N}.$$

For the same reason we have

$$\boldsymbol{x}_v \cdot \mathscr{S}_p(\boldsymbol{x}_u) = \boldsymbol{x}_{vu} \cdot \boldsymbol{N}.$$

But $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$. Therefore

$$\boldsymbol{x}_u \cdot \mathscr{S}_p(\boldsymbol{x}_v) = \boldsymbol{x}_v \cdot \mathscr{S}_p(\boldsymbol{x}_u).$$

This *almost* shows that $\mathscr{S}_p$ is symmetric. To complete the proof, we need to show that

$$\boldsymbol{a} \cdot \mathscr{S}_p(\boldsymbol{b}) = \boldsymbol{b} \cdot \mathscr{S}_p(\boldsymbol{a})$$

foe all vectors $\boldsymbol{a}, \boldsymbol{b} \in T_p$. But that's easy since any vector in T_p may be written as a linear combination of the basis vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$. □

We have seen that the eigenvalues of a symmetric linear operator are real. We conclude that:

Corollary 1. *The eigenvalues of the shape operator $\mathscr{S}_p$ are real.*

Exercise 1. Show that

$$\boldsymbol{x}_u \cdot \mathscr{S}_p(\boldsymbol{x}_u) = \boldsymbol{x}_{uu} \cdot \boldsymbol{N}, \tag{1a}$$

$$\boldsymbol{x}_v \cdot \mathscr{S}_p(\boldsymbol{x}_v) = \boldsymbol{x}_{vv} \cdot \boldsymbol{N}, \tag{1b}$$

$$\boldsymbol{x}_u \cdot \mathscr{S}_p(\boldsymbol{x}_v) = \boldsymbol{x}_v \cdot \mathscr{S}_p(\boldsymbol{x}_u) = \boldsymbol{x}_{uv} \cdot \boldsymbol{N}. \tag{1c}$$

Hint: The last of these equations was proved in the proof of Theorem 1. Apply a similar technique to prove the other two. We will need these formulas later for calculating the Gaussian curvature.

Date: April 5, 2024.

2. The normal curvature [sections 2.4 of the textbook]

Lemma 1. *Let $\boldsymbol{\alpha}(t)$ be the parametric representation of a curve that lies on the surface M. Then at every t we have*

$$\boldsymbol{\alpha}'' \cdot \boldsymbol{N} = \boldsymbol{\alpha}' \cdot \mathscr{S}(\boldsymbol{\alpha}'),$$

where $\boldsymbol{N}$ and $\mathscr{S}$ are the unit normal and the shape operator at $\boldsymbol{\alpha}(t)$.

Proof. The vector $\boldsymbol{\alpha}'$ is tangent to the surface M and therefore it is orthogonal to $\boldsymbol{N}$, that is, $\boldsymbol{\alpha}' \cdot \boldsymbol{N} = 0$ at every t. Differentiating this with respect to t we see that

$$0 = \boldsymbol{\alpha}'' \cdot \boldsymbol{N} + \boldsymbol{\alpha}' \cdot \frac{\partial \boldsymbol{N}}{\partial t} = \boldsymbol{\alpha}'' \cdot \boldsymbol{N} + \boldsymbol{\alpha}' \cdot \nabla_{\boldsymbol{\alpha}'} \boldsymbol{N} = \boldsymbol{\alpha}'' \cdot \boldsymbol{N} - \boldsymbol{\alpha}' \cdot \mathscr{S}(\boldsymbol{\alpha}').$$

□

Now suppose that the curve $\boldsymbol{\alpha}$ of the lemma above is a *unit speed curve.* Then $\boldsymbol{t} = \boldsymbol{\alpha}'$ is the unit tangent to the curve, and by Frenet's formulas, $\boldsymbol{t}' = \kappa \boldsymbol{n}$, where κ and $\boldsymbol{n}$ are the curve's curvature and unit normal. Equivalently, $\boldsymbol{\alpha}'' = \kappa \boldsymbol{n}$, and therefore the lemma's conclusion may be stated as

$$\kappa \boldsymbol{n} \cdot \boldsymbol{N} = \boldsymbol{\alpha}' \cdot \mathscr{S}(\boldsymbol{\alpha}') \qquad \text{(assuming a unit speed curve).} \tag{2}$$

That's an interesting conclusion. The left-hand side depends on the curve's curvature but right-hand side doesn't! Specifically, the right-hand side depends only on the curve's *direction*, $\boldsymbol{\alpha}'$, which would be the same for *all curves on the surface that pass through the point p* on the in that direction. In other words, the right-hand side is telling us something about a *property of the surface* in the direction $\boldsymbol{\alpha}'$, not the curve! This leads to

Definition 1. For a unit vector $\boldsymbol{u} \in T_p$, the quantity

$$k(\boldsymbol{u}) = \boldsymbol{u} \cdot \mathscr{S}_p(\boldsymbol{u})$$

is called the *normal curvature* of the surface at p.

Remark 1. According to (2), if a curve lies on the surface M and passes through the point $p \in M$, and if it has the unit tangent $\boldsymbol{u}$ at p, then

$$k(\boldsymbol{u}) = \kappa \boldsymbol{n} \cdot \boldsymbol{N}, \tag{3}$$

where $\boldsymbol{N}$ is the surface's unit normal, $\boldsymbol{n}$ is the curve's unit normal, and κ is the curve's curvature at p.

A useful way of realizing the formula (3) is to construct a plane P through p which is spanned by the vectors $\boldsymbol{u}$ and $\boldsymbol{N}$. Let $\boldsymbol{\alpha}$ be the curve of intersection of P and the surface M. Then $\boldsymbol{u}$ is the curve's unit tangent at p. The curve's normal vector $\boldsymbol{n}$ lies in P and is perpendicular to the tangent vector $\boldsymbol{u}$. Therefore $\boldsymbol{n}$ is collinear with $\boldsymbol{N}$ and consequently $\boldsymbol{n} \cdot \boldsymbol{N} = \pm 1$. We conclude that $|k(\boldsymbol{u})| = |\kappa \boldsymbol{n} \cdot \boldsymbol{N}| = |\kappa|$, that is, $k(\boldsymbol{u}) = \pm\kappa$. The sign of that expression is of no great consequence because it depends on the choice of the orientation of the surface's normal vector $\boldsymbol{N}$.

As the vector $\boldsymbol{u}$ turns in the tangent plane T_p, the plane P turns and therefore the curve of of its intersection with M changes and consequently the normal curvature $k(\boldsymbol{u})$ changes. We may view $k(\boldsymbol{u})$ as a scalar-valued function of the angular orientation θ of the vector $\boldsymbol{u}$. Evidently $k(\boldsymbol{u})$ is a π-periodic function of θ. If the surface is smooth, $k(\boldsymbol{u})$ achieves a minimum and maximum, k_1 and k_2 respectively, within the interval $[0, \pi]$. The quantities k_1 and k_2 are called the surface's *principal curvatures* at p. The directions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ in which the minimum and maximum are achieved are called the surface's *principal vectors* at p.

Theorem 2 (Euler). *The principal curvatures k_1 and k_2 are the eigenvalues of the shape operator $\mathscr{S}_p$. The principal vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the corresponding eigenvectors.*

Proof. We write $\mathscr{S}$ for $\mathscr{S}_p$, implying that all the calculations pertain to the point p of the surface. Let λ_1 and λ_2 be the eigenvalues of $\mathscr{S}$, and let $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ be an orthonormal pair of eigenvalues. In view of the definition 1 we have

$$k(\boldsymbol{u}_1) = \boldsymbol{u}_1 \cdot \mathscr{S}(\boldsymbol{u}_1) = \boldsymbol{u}_1 \cdot (\lambda_1 \boldsymbol{u}_1) = \lambda_1(\boldsymbol{u}_1 \cdot \boldsymbol{u}_1) = \lambda_1,$$

and similarly, $k(\boldsymbol{u}_2) = \lambda_2$.

We wish to show that λ_1 and λ_2 are the principal curvatures at p. Toward that end, suppose $\lambda_1 > \lambda_2$ and let $\boldsymbol{u}(\theta) = \boldsymbol{u}_1 \cos\theta + \boldsymbol{u}_2 \sin\theta$ for some arbitrary θ. Then calculate

$$\begin{aligned}
k\big(\boldsymbol{u}(\theta)\big) &= \boldsymbol{u}(\theta) \cdot \mathscr{S}\big(\boldsymbol{u}(\theta)\big) \\
&= (\boldsymbol{u}_1 \cos\theta + \boldsymbol{u}_2 \sin\theta) \cdot \mathscr{S}\big(\boldsymbol{u}_1 \cos\theta + \boldsymbol{u}_2 \sin\theta\big) \\
&= (\boldsymbol{u}_1 \cos\theta + \boldsymbol{u}_2 \sin\theta) \cdot \big(\cos\theta \mathscr{S}\big(\boldsymbol{u}_1) + \sin\theta \mathscr{S}(\boldsymbol{u}_2)\big) \\
&= \cos^2\theta\, \boldsymbol{u}_1 \cdot \mathscr{S}(\boldsymbol{u}_1) + \cos\theta \sin\theta\, \boldsymbol{u}_1 \cdot \mathscr{S}(\boldsymbol{u}_2) + \sin\theta\cos\theta\, \boldsymbol{u}_2 \cdot \mathscr{S}(\boldsymbol{u}_1) + \sin^2\theta\, \boldsymbol{u}_2 \cdot \mathscr{S}(\boldsymbol{u}_2) \\
&= \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta \\
&= \lambda_1(1 - \sin^2\theta) + \lambda_2 \sin^2\theta \\
&= \lambda_1 - (\lambda_1 - \lambda_2)\sin^2\theta.
\end{aligned}$$

Since $\sin^2\theta$ ranges from 0 to 1, and since $\lambda_1 - \lambda_2 > 0$ we see that the right-hand side is at a maximum when $\theta = 0$ and at a minimum when $\theta = \pi/2$. We conclude that the largest and smallest values of $k\big(\boldsymbol{u}(\theta)\big)$ are λ_1 and λ_2, and they are achieved respectively at $\theta = 0$ and $\theta = \pi/2$. As $\boldsymbol{u}(0) = \boldsymbol{u}_1$ and $\boldsymbol{u}(\pi/2) = \boldsymbol{u}_2$, this completes the demonstration of the theorem. □

Remark 2. Since the principal curvatures k_1 and k_2 are defined to be the largest and smallest values of $k\big(\boldsymbol{u}(\theta)\big)$, we may calculate the curvature at any θ through

$$k\big(\boldsymbol{u}(\theta)\big) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

3. The Gaussian and mean curvatures [sections 3.1 and 3.2 of the textbook]

Having defined the principal curvatures k_1 and k_2 at a point p of a surface M, we now define the Gaussian curvature K, and the mean curvature H, at the point p through

$$K = k_1 k_2, \quad H = \frac{1}{2}(k_1 + k_2).$$

At first sight these seem to be counterproductive since each merges two meaningful quantities into one. We will see, however, that the Gaussian and mean curvatures carry a lot of information about the surface. Extracting that information calls for some effort. That is the bulk of the textbook's Chapter three.

Remark 3. We have seen that the principal curvatures k_1 and k_2 at $p \in M$ are the eigenvalues of the the shape operator $\mathscr{S}_p$, and that the principal eigenvectors form an orthonormal basis of the tangent plane T_p. Expressing $\mathscr{S}_p$ as a 2×2 matrix in that basis results in

$$[\mathscr{S}_p] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

We observe that the Gaussian and mean curvatures may be expressed in terms of the determinant and trace of $\mathscr{S}_p$:

$$K = \det \mathscr{S}_p, \quad H = \frac{1}{2} \operatorname{tr} \mathscr{S}_p.$$

A naive approach to calculating the Gaussian and mean curvatures can lead to intractable algebra. Here is a carefully designed systematic approach. We begin with introducing a notation.

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be a linearly independent pair of vectors in the tangent plane T_p of a surface M. (These need not be of unit length or orthogonal.) Thus, any vector in T_p may be expressed as a linear combination $\boldsymbol{u}$ and $\boldsymbol{v}$. If $\boldsymbol{x} = \alpha\boldsymbol{u}+\beta\boldsymbol{v}$, we associate $[\boldsymbol{x}] = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \in R^2$ with $\boldsymbol{x} \in T_p$. In particular, $\mathscr{S}_p(\boldsymbol{u})$ and $\mathscr{S}_p(\boldsymbol{v})$ may be expressed as

$$\mathscr{S}_p(\boldsymbol{u}) = a\boldsymbol{u} + b\boldsymbol{v}, \quad \mathscr{S}_p(\boldsymbol{v}) = c\boldsymbol{u} + d\boldsymbol{v},$$

and therefore

$$\big[\mathscr{S}_p(\boldsymbol{u})\big] = \begin{pmatrix}a\\ b\end{pmatrix}, \quad \big[\mathscr{S}_p(\boldsymbol{v})\big] = \begin{pmatrix}c\\ d\end{pmatrix},$$

for some numbers a, b, c, d. Now, with $\boldsymbol{x} = \alpha\boldsymbol{u} + \beta\boldsymbol{v}$ as before, we have

$$\begin{aligned}\mathscr{S}_p(\boldsymbol{x}) &= \mathscr{S}_p(\alpha\boldsymbol{u} + \beta\boldsymbol{v})\\ &= \alpha\mathscr{S}_p(\boldsymbol{u}) + \beta\mathscr{S}_p(\boldsymbol{v})\\ &= \alpha(a\boldsymbol{u} + b\boldsymbol{v}) + \beta(c\boldsymbol{u} + d\boldsymbol{v})\\ &= (a\alpha + c\beta)\boldsymbol{u} + (b\alpha + d\beta)\boldsymbol{v}.\end{aligned}$$

We conclude that

$$\big[\mathscr{S}_p(\boldsymbol{x})\big] = \big[(a\alpha + c\beta)\boldsymbol{u} + (b\alpha + d\beta)\boldsymbol{v}\big] = \begin{pmatrix}a\alpha + c\beta\\ b\alpha + d\beta\end{pmatrix} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}a & c\\ b & d\end{pmatrix}[\boldsymbol{x}]$$

and therefore $\begin{pmatrix}a & c\\ b & d\end{pmatrix}$ is the matrix representation of $\mathscr{S}_p$. In particular,

$$\det \mathscr{S}_p = ad - bc, \quad \operatorname{tr} \mathscr{S}_p = a + d.$$

Now let us calculate

$$\begin{aligned}\mathscr{S}_p(\boldsymbol{u}) \times \mathscr{S}_p(\boldsymbol{v}) &= (a\boldsymbol{u} + b\boldsymbol{v}) \times (c\boldsymbol{u} + d\boldsymbol{v})\\ &= ac\, \boldsymbol{u} \times \boldsymbol{u} + ad\, \boldsymbol{u} \times \boldsymbol{v} + bc\, \boldsymbol{v} \times \boldsymbol{u} + bd\, \boldsymbol{v} \times \boldsymbol{v}\\ &= (ad - bc)\, \boldsymbol{u} \times \boldsymbol{v}\\ &= (\det \mathscr{S}_p)\, \boldsymbol{u} \times \boldsymbol{v}\\ &= K\, \boldsymbol{u} \times \boldsymbol{v}.\end{aligned}$$

Finally, we dot-multiply this on both sides by $\boldsymbol{u} \times \boldsymbol{v}$

$$\big(\mathscr{S}_p(\boldsymbol{u}) \times \mathscr{S}_p(\boldsymbol{v})\big) \cdot (\boldsymbol{u} \times \boldsymbol{v}) = K(\boldsymbol{u} \times \boldsymbol{v}) \cdot (\boldsymbol{u} \times \boldsymbol{v})$$

and expand the products through applying Lagrange's identity[1]

$$\big(\boldsymbol{u} \cdot \mathscr{S}_p(\boldsymbol{u})\big)\,\big(\boldsymbol{v} \cdot \mathscr{S}_p(\boldsymbol{v})\big) - \big(\boldsymbol{v} \cdot \mathscr{S}_p(\boldsymbol{u})\big)\,\big(\boldsymbol{u} \cdot \mathscr{S}_p(\boldsymbol{v})\big) = K\Big[(\boldsymbol{u} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})(\boldsymbol{v} \cdot \boldsymbol{u})\Big],$$

[1]Lagrange's identity holds for all vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ in the three-dimensional space:

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}).$$

and then solve for the Gaussian curvature K:

$$K = \frac{\big(\boldsymbol{u}\cdot\mathscr{S}_p(\boldsymbol{u})\big)\,\big(\boldsymbol{v}\cdot\mathscr{S}_p(\boldsymbol{v})\big) - \big(\boldsymbol{v}\cdot\mathscr{S}_p(\boldsymbol{u})\big)\,\big(\boldsymbol{u}\cdot\mathscr{S}_p(\boldsymbol{v})\big)}{(\boldsymbol{u}\cdot\boldsymbol{u})\,(\boldsymbol{v}\cdot\boldsymbol{v}) - (\boldsymbol{u}\cdot\boldsymbol{v})\,(\boldsymbol{v}\cdot\boldsymbol{u})}. \tag{4}$$

Remark 4. The right-hand side of (4) is expressed in terms of the arbitrary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in T_p. We know that the Gaussian curvature K is a well-defined quantity and should not depend on such arbitrariness. It follows that despite the appearances, the right-hand side of (4) is independent of $\boldsymbol{u}$ and $\boldsymbol{v}$.

Exercise 2. Following a line of calculation similar to the one above, show that

$$\mathscr{S}_p(\boldsymbol{u})\times\boldsymbol{v} + \boldsymbol{u}\times\mathscr{S}_p(\boldsymbol{v}) = 2H\,\boldsymbol{u}\times\boldsymbol{v}. \tag{5}$$

Exercise 3. Dot-multiply (5) by $\boldsymbol{u}\times\boldsymbol{v}$ and simplify the result to conclude that

$$H = \frac{\big(\mathscr{S}_p(\boldsymbol{u})\cdot\boldsymbol{u}\big)\,(\boldsymbol{v}\cdot\boldsymbol{v}) - \big(\mathscr{S}_p(\boldsymbol{u})\cdot\boldsymbol{v}\big)\,(\boldsymbol{v}\cdot\boldsymbol{u}) + \big(\mathscr{S}_p(\boldsymbol{v})\cdot\boldsymbol{v}\big)\,(\boldsymbol{u}\cdot\boldsymbol{u}) - \big(\mathscr{S}_p(\boldsymbol{v})\cdot\boldsymbol{u}\big)\,(\boldsymbol{u}\cdot\boldsymbol{v})}{2\big[(\boldsymbol{u}\cdot\boldsymbol{u})\,(\boldsymbol{v}\cdot\boldsymbol{v}) - (\boldsymbol{u}\cdot\boldsymbol{v})\,(\boldsymbol{v}\cdot\boldsymbol{u})\big]}. \tag{6}$$

In practice, the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in (4) and (6) are picked to be the vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ of a surface patch $\boldsymbol{x}(u,v)$ that includes the point p. As we need to calculate the products $\boldsymbol{u}\cdot\boldsymbol{u}$, etc., we introduce the notation

$$E = \boldsymbol{x}_u\cdot\boldsymbol{x}_u, \quad F = \boldsymbol{x}_u\cdot\boldsymbol{x}_v, \quad G = \boldsymbol{x}_v\cdot\boldsymbol{x}_v, \tag{7a}$$

$$e = \boldsymbol{x}_u\cdot\mathscr{S}_p(\boldsymbol{x}_u), \quad f = \boldsymbol{x}_u\cdot\mathscr{S}_p(\boldsymbol{x}_v) = \boldsymbol{x}_v\cdot\mathscr{S}_p(\boldsymbol{x}_u), \quad g = \boldsymbol{x}_v\cdot\mathscr{S}_p(\boldsymbol{x}_v), \tag{7b}$$

whereby (4) and (6) take the forms

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{eG + gE - 2fF}{2(EG - F^2)}. \tag{8}$$

The expressions E, F, and G defined in (7a) are straightforward to calculate. The expressions e, f, and g defined in (7b) involve the shape operator and their calculation may not be immediately obvious. Referring to Exercise 1, however, we see that we have practical formulas for them. Specifically

$$e = \boldsymbol{x}_{uu}\cdot\boldsymbol{N}, \quad f = \boldsymbol{x}_{uv}\cdot\boldsymbol{N}, \quad g = \boldsymbol{x}_{vv}\cdot\boldsymbol{N}. \tag{9}$$

These, together with

$$\boldsymbol{N} = \frac{\boldsymbol{x}_u\times\boldsymbol{x}_v}{\|\boldsymbol{x}_u\times\boldsymbol{x}_v\|} \tag{10}$$

provide a practical way of computing the Gaussian and mean curvatures.

4. The Christoffel symbols [section 3.4 of the textbook]

The symbols E, F, G defined in (7a) are called the surface patch's *metric coefficients*. These can be computed solely by performing measurements *within the surface*.

In contrast, the symbols e, f, g defined in (7b) and further refined in (9) are expressed in terms of the surface normal $\boldsymbol{N}$. Their evaluation necessitates looking *outside the surface*. Therefore, at first sight, it appears that the curvatures K and H, given in (8), cannot be determined by taking measurements within the surface alone; one needs to look at the surface from the outside. It turns out, however, due to fortuitous cancellations, the Gaussian curvature K (but not the mean curvature H) may be expressed only in terms of the metric coefficients E, F, and G. That is, K can be determined by taking measurements within the surface alone; it's not necessary to look at the surface from the outside! This is the content of Gauss's *Theorema Egregium* (Latin for *The Remarkable Theorem*).

To establish that theorem we note that the triple $\{\boldsymbol{x}_u, \boldsymbol{x}_v, \boldsymbol{N}\}$ is a basis for the three-dimensional space, and therefore we may express any vector as a linear combination of those. In particular, the vectors $\boldsymbol{x}_{uu}$, $\boldsymbol{x}_{uv}$, $\boldsymbol{x}_{vv}$ that appear in (9) may be expressed as such linear combinations. Referring to the equations (9), we see that the components of these vectors along the normal direction $\boldsymbol{N}$ are e, f, and g, respectively. It remains to determine their components along the vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$. We denote these by Γ^i_{jk}:

$$\boldsymbol{x}_{uu} = \Gamma^1_{11}\boldsymbol{x}_u + \Gamma^2_{11}\boldsymbol{x}_v + e\boldsymbol{N}, \tag{11a}$$

$$\boldsymbol{x}_{uv} = \Gamma^1_{12}\boldsymbol{x}_u + \Gamma^2_{12}\boldsymbol{x}_v + f\boldsymbol{N}, \tag{11b}$$

$$\boldsymbol{x}_{vu} = \Gamma^1_{21}\boldsymbol{x}_u + \Gamma^2_{21}\boldsymbol{x}_v + f\boldsymbol{N},$$

$$\boldsymbol{x}_{vv} = \Gamma^1_{22}\boldsymbol{x}_u + \Gamma^2_{22}\boldsymbol{x}_v + g\boldsymbol{N}. \tag{11c}$$

The coefficients Γ^i_{jk} are called the *Christoffel symbols* of the parametrization $\boldsymbol{x}(u, v)$. The third of these equations is redundant since $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$ and that's why it's not numbered. That also tells us that $\Gamma^1_{12} = \Gamma^1_{21}$ and $\Gamma^2_{12} = \Gamma^2_{21}$. Therefore there are altogether six Christoffel symbols that need to be determined.

Remark 5. Equations (11) are analogous to the Frenet formulas for space curves; they express the rates of change of the basis vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ in terms of the basis vectors.

To determine of the coefficients Γ^i_{jk}, we begin with listing the following self-evident identities:

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u = \frac{1}{2}(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)_u = \frac{1}{2}E_u, \tag{12a}$$

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v = (\boldsymbol{x}_u \cdot \boldsymbol{x}_v)_u - \boldsymbol{x}_u \cdot \boldsymbol{x}_{uv} = F_u - \frac{1}{2}E_v \tag{12b}$$

$$\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u = \frac{1}{2}(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)_v = \frac{1}{2}E_v, \tag{12c}$$

$$\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v = \frac{1}{2}(\boldsymbol{x}_v \cdot \boldsymbol{x}_v)_u = \frac{1}{2}G_u \tag{12d}$$

$$\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u = (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)_v - \boldsymbol{x}_v \cdot \boldsymbol{x}_{uv} = F_v - \frac{1}{2}G_u \tag{12e}$$

$$\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v = \frac{1}{2}(\boldsymbol{x}_v \cdot \boldsymbol{x}_v)_v = \frac{1}{2}G_v. \tag{12f}$$

Now, we take the dot products of the equation (11a) with $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u = \Gamma^1_{11}\boldsymbol{x}_u \cdot \boldsymbol{x}_u + \Gamma^2_{11}\boldsymbol{x}_v \cdot \boldsymbol{x}_u + e\boldsymbol{N} \cdot \boldsymbol{x}_u,$$
$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v = \Gamma^1_{11}\boldsymbol{x}_u \cdot \boldsymbol{x}_v + \Gamma^2_{11}\boldsymbol{x}_v \cdot \boldsymbol{x}_v + e\boldsymbol{N} \cdot \boldsymbol{x}_v,$$

which, in view of the identities (12a) and (12b), and the fact that $\boldsymbol{N}$ is orthogonal to $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, simplify to

$$\frac{1}{2}E_u = \Gamma^1_{11}E + \Gamma^2_{11}F,$$
$$F_u - \frac{1}{2}E_v = \Gamma^1_{11}F + \Gamma^2_{11}G.$$

We solve this as a linear system of two equations in the two unknowns Γ^1_{11} and Γ^2_{11} and obtain

$$\Gamma^1_{11} = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)}, \quad \Gamma^2_{11} = -\frac{EE_v - 2EF_u + FE_u}{2(EG - F^2)}.$$

Similar calculations with equations (11b) and (11c) enables us to calculate the remaining Christoffel symbols. Altogether, we end up with[2]

$$\begin{cases} \Gamma_{11}^1 = \dfrac{GE_u - 2FF_u + FE_v}{2(EG-F^2)}, & \Gamma_{11}^2 = -\dfrac{EE_v - 2EF_u + FE_u}{2(EG-F^2)}, \\ \Gamma_{12}^1 = \dfrac{GE_v - FG_u}{2(EG-F^2)}, & \Gamma_{12}^2 = \dfrac{EG_u - FE_v}{2(EG-F^2)}, \\ \Gamma_{22}^1 = -\dfrac{FG_v - 2GF_v + GG_u}{2(EG-F^2)}, & \Gamma_{22}^2 = \dfrac{EG_v - 2FF_v + FG_u}{2(EG-F^2)}. \end{cases} \tag{13}$$

Remark 6. We see that the Christoffel symbols are completely determined by the metric coefficients E, F, and G. It's not necessary to look outside of the surface to evaluate them.

Remark 7. If the surface patch is such that its coordinate curves are mutually orthogonal, that is, $F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = 0$, then the expressions for Christoffel symbols simplify significantly:

$$\begin{cases} \Gamma_{11}^1 = \dfrac{E_u}{2E}, & \Gamma_{11}^2 = -\dfrac{E_v}{2G}, \\ \Gamma_{12}^1 = \dfrac{E_v}{2E}, & \Gamma_{12}^2 = \dfrac{G_u}{2G}, \\ \Gamma_{22}^1 = -\dfrac{G_u}{2E}, & \Gamma_{22}^2 = \dfrac{G_v}{2G}. \end{cases} \tag{14}$$

The exposition of the textbook's Section 3.4 is limited to this special case.

5. Theorema Egregium [section 3.4 of the textbook]

Lemma 2. *We have*

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} = -\frac{1}{2}\big(E_{vv} - 2F_{uv} + G_{uu}\big). \tag{15}$$

Proof. We note that

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} = (\boldsymbol{x}_u \cdot \boldsymbol{x}_{vv})_u - (\boldsymbol{x}_u \cdot \boldsymbol{x}_{uv})_v.$$

Substituting for $\boldsymbol{x}_u \cdot \boldsymbol{x}_{vv}$ from (12e), and for $\boldsymbol{x}_u \cdot \boldsymbol{x}_{uv}$ from (12c), we get

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} = \Big(F_v - \frac{1}{2}G_u\Big)_u - \Big(\frac{1}{2}E_v\Big)_v,$$

which is equivalent to (15). □

Remark 8. What takes place in the lemma above is a surprise. The vectors $\boldsymbol{x}_{uu}$, $\boldsymbol{x}_{uv}$, $\boldsymbol{x}_{vv}$ generally have components along the normal $\boldsymbol{N}$ to the surface. In fact, the coefficients e, f, and g are precisely those normal components—see see equations (9) on page 5—so one would expect that the combination $\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv}$ would also depend on those components. But as we see above, the normal components magically disappear and the result is expressed solely in terms of the coefficients of metric coefficients E, F, and G. This is the crucial element that makes Theorema Egregium possible.

Theorem 3 (Theorema Egregium)**.** *The Gaussian curvature K may be expressed solely in terms of the metric coefficients E, F, G:*[3]

$$\begin{aligned}(EG - F^2)\,K = -\frac{1}{2}\big(E_{vv} - 2F_{uv} + G_{uu}\big) - \big(\Gamma_{11}^1\Gamma_{22}^1 - (\Gamma_{12}^1)^2\big)\,E \\ - \big(\Gamma_{11}^1\Gamma_{22}^2 - 2\Gamma_{12}^1\Gamma_{12}^2 + \Gamma_{11}^2\Gamma_{22}^1\big)F - \big(\Gamma_{11}^2\Gamma_{22}^2 - (\Gamma_{12}^2)^2\big)\,G.\end{aligned} \tag{16}$$

[2]Beware of the minus signs in front of the fractions corresponding to Γ_{22}^1 and Γ_{11}^2.

[3]Recall that the Christoffel symbols are expressed as functions of E, F, G in (13).

Proof. Let us introduce compact versions of the three labeled equation in (11) as

$$\boldsymbol{x}_{uu} = \boldsymbol{a} + e\boldsymbol{N}, \quad \boldsymbol{x}_{uv} = \boldsymbol{b} + f\boldsymbol{N}, \quad \boldsymbol{x}_{vv} = \boldsymbol{c} + g\boldsymbol{N},$$

where

$$\boldsymbol{a} = \Gamma^1_{11}\boldsymbol{x}_u + \Gamma^2_{11}\boldsymbol{x}_v, \quad \boldsymbol{b} = \Gamma^1_{12}\boldsymbol{x}_u + \Gamma^2_{12}\boldsymbol{x}_v, \quad \boldsymbol{c} = \Gamma^1_{22}\boldsymbol{x}_u + \Gamma^2_{22}\boldsymbol{x}_v. \tag{17}$$

Observe that each of the vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ is in the surface's tangent plane, and therefore $\boldsymbol{a}\cdot\boldsymbol{N} = \boldsymbol{b}\cdot\boldsymbol{N} = \boldsymbol{c}\cdot\boldsymbol{N} = 0$. Let us calculate

$$\begin{aligned}\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_{vv} - \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} &= (\boldsymbol{a} + e\boldsymbol{N})\cdot(\boldsymbol{c} + g\boldsymbol{N}) - (\boldsymbol{b} + f\boldsymbol{N})\cdot(\boldsymbol{b} + f\boldsymbol{N}) \\ &= (\boldsymbol{a}\cdot\boldsymbol{c} + eg) - (\boldsymbol{b}\cdot\boldsymbol{b} - f^2),\end{aligned}$$

that is

$$eg - f^2 = (\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_{vv} - \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv}) - (\boldsymbol{a}\cdot\boldsymbol{c} - \boldsymbol{b}\cdot\boldsymbol{b}).$$

For the first parenthesized term we substitute from (15). As to the second parenthesized term, we calculate:

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{c} &= (\Gamma^1_{11}\boldsymbol{x}_u + \Gamma^2_{11}\boldsymbol{x}_v)\cdot(\Gamma^1_{22}\boldsymbol{x}_u + \Gamma^2_{22}\boldsymbol{x}_v) \\ &= \Gamma^1_{11}\Gamma^1_{22}\boldsymbol{x}_u\cdot\boldsymbol{x}_u + (\Gamma^1_{11}\Gamma^2_{22} + \Gamma^2_{11}\Gamma^1_{22})\boldsymbol{x}_u\cdot\boldsymbol{x}_v + \Gamma^2_{11}\Gamma^2_{22}\boldsymbol{x}_v\cdot\boldsymbol{x}_v \\ &= \Gamma^1_{11}\Gamma^1_{22}\,E + (\Gamma^1_{11}\Gamma^2_{22} + \Gamma^2_{11}\Gamma^1_{22})\,F + \Gamma^2_{11}\Gamma^2_{22}\,G.\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{b}\cdot\boldsymbol{b} &= (\Gamma^1_{12}\boldsymbol{x}_u + \Gamma^2_{12}\boldsymbol{x}_v)\cdot(\Gamma^1_{12}\boldsymbol{x}_u + \Gamma^2_{12}\boldsymbol{x}_v) \\ &= (\Gamma^1_{12})^2\boldsymbol{x}_u\cdot\boldsymbol{x}_u + 2\Gamma^1_{12}\Gamma^2_{12}\boldsymbol{x}_u\cdot\boldsymbol{x}_v + (\Gamma^2_{12})^2\boldsymbol{x}_v\cdot\boldsymbol{x}_v \\ &= (\Gamma^1_{12})^2\,E + 2\Gamma^1_{12}\Gamma^2_{12}\,F + (\Gamma^2_{12})^2\,G.\end{aligned}$$

Thus, we have arrived at

$$\begin{aligned}eg - f^2 = -\frac{1}{2}\big(E_{vv} - 2F_{uv} + G_{uu}\big) - \big(\Gamma^1_{11}\Gamma^1_{22} - (\Gamma^1_{12})^2\big)\,E \\ - \big(\Gamma^1_{11}\Gamma^2_{22} - 2\Gamma^1_{12}\Gamma^2_{12} + \Gamma^2_{11}\Gamma^1_{22}\big)\,F - \big(\Gamma^2_{11}\Gamma^2_{22} - (\Gamma^2_{12})^2\big)\,G,\end{aligned} \tag{18}$$

which, via (8), leads to (16). □

Remark 9. Equation (18) says that the combination $eg - f^2$ may be determined *without referencing the normal vector* $\boldsymbol{N}$, although as we see in (9), e, f, and g individually do depend on $\boldsymbol{N}$.

Remark 10. Equation (16) is sometimes referred to as *Gauss's equation.*

Remark 11. In a parametrization where the coordinate curves are orthogonal, that is, $F = 0$, equation (16) simplifies considerably. Let's begin with the coefficient of E in (16). In view of (14) we have

$$\Gamma^1_{11}\Gamma^1_{22} - (\Gamma^1_{12})^2 = \Big(\frac{E_u}{2E}\Big)\Big(-\frac{G_u}{2E}\Big) - \Big(\frac{E_v}{2E}\Big)^2 = -\frac{1}{4E^2}(E_uG_u + E_v^2).$$

Similarly, calculate the coefficient of G:

$$\Gamma^2_{11}\Gamma^2_{22} - (\Gamma^2_{12})^2 = \Big(-\frac{E_v}{2G}\Big)\Big(\frac{G_v}{2G}\Big) - \Big(\frac{G_u}{2G}\Big)^2 = -\frac{1}{4G^2}(E_vG_v + G_u^2).$$

Thus, equation (16) reduces to

$$EGK = -\frac{1}{2}\big(E_{vv} + G_{uu}\big) + \frac{1}{4E}(E_uG_u + E_v^2) + \frac{1}{4G}(E_vG_v + G_u^2). \tag{19}$$

An elementary but tedious calculation shows that this is equivalent to

$$K = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]. \tag{20}$$

Here are the details of that calculation. We pick the following three terms from the right-hand side of (19) and simplify:

$$\begin{aligned}
-\frac{1}{2}E_{vv} + \frac{1}{4E}E_v^2 + \frac{1}{4G}E_vG_v &= -\frac{1}{2}E_v\left[\frac{E_{vv}}{E_v} - \frac{1}{2}\frac{E_vG + EG_v}{EG}\right] \\
&= -\frac{1}{2}E_v\left[\frac{E_{vv}}{E_v} - \frac{1}{2}\frac{(EG)_v}{EG}\right] \\
&= -\frac{1}{2}E_v\left[(\ln E_v)_v - \frac{1}{2}\Big(\ln(EG)\Big)_v\right] \\
&= -\frac{1}{2}E_v\left[(\ln E_v) - \frac{1}{2}\Big(\ln(EG)\Big)\right]_v \\
&= -\frac{1}{2}E_v\left[\ln\left(\frac{E_v}{\sqrt{EG}}\right)\right]_v \\
&= -\frac{1}{2}E_v\frac{\left(\frac{E_v}{\sqrt{EG}}\right)_v}{\frac{E_v}{\sqrt{EG}}} \\
&= -\frac{1}{2}\sqrt{EG}\left(\frac{E_v}{\sqrt{EG}}\right)_v.
\end{aligned}$$

We repeat the calculations with the remaining three terms of the right-hand side of (19), and obtain

$$-\frac{1}{2}G_{uu} + \frac{1}{4G}G_u^2 + \frac{1}{4E}E_uG_u = -\frac{1}{2}\sqrt{EG}\left(\frac{G_u}{\sqrt{EG}}\right)_u.$$

Plugging these back into (19) we get

$$EGK = -\frac{1}{2}\sqrt{EG}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right].$$

Then we divide through by EG to arrive at (20).

Exercise 4. Needham[4] (page 36) criticizes Gauss's choice of the notation E, F, G for the metric coefficients (see equations (7a)), arguing that it would have been more natural to write A^2 and B^2 for E and G, that is,

$$A^2 = \boldsymbol{x}_u \cdot \boldsymbol{x}_u, \quad B^2 = \boldsymbol{x}_v \cdot \boldsymbol{x}_v.$$

He writes:

> "In the subsequent centuries, almost every research paper and textbook on Differential Geometry has slavishly perpetuated [Gauss's] E, F and G notation. …The consequence has been that a literature has arisen that is needlessly cluttered with square roots."

[4]Tristan Needham, *Visual Differential Geometry and Forms*, Princeton University Press, 2021.

He goes on to state (without proof) that in his notation, the curvature formula (20) (corresponding to $F = 0$) takes the "beautiful" (his word) symmetric form

$$K = -\frac{1}{AB}\left[\left(\frac{A_v}{B}\right)_v + \left(\frac{B_u}{A}\right)_u\right].$$

Verify his claim