Algorithm for finding particular solutions

This replaces the 9 cases listed in Farlow's Table 3.2 on page 153.

Consider the following second order, linear, constant coefficients, nonhomogeneous differential equation for the unknown $y(x)$

$$ay'' + by' + cy = f(x),$$

where $f(x)$ is of the form

$$f(x) = e^{\alpha x}\Big[(a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n)\cos\beta x + (b_0 + b_1 x + b_2 x^2 + \cdots b_n x^n)\sin\beta x\Big].$$

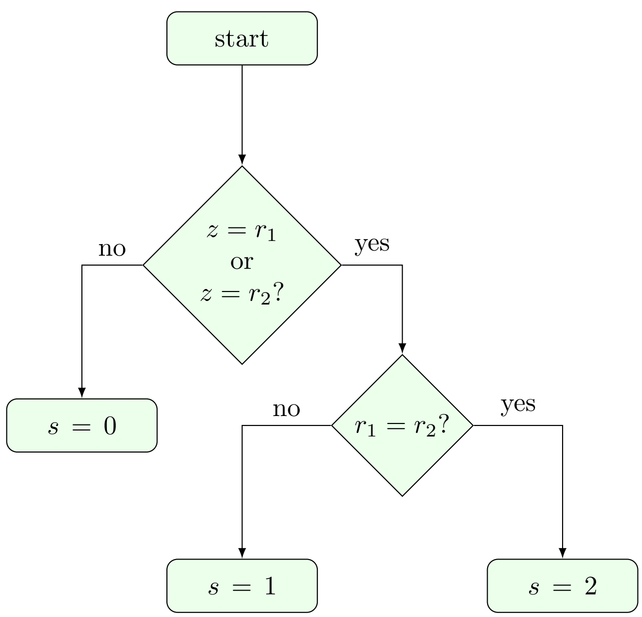
Here n is a nonnegative integer and α, β, a_k, and b_k, $k = 0, 1, \ldots, n$ are constants.

Then there exists a particular solution of the form

$$y_p(x) = x^s e^{\alpha x}\Big[(A_0 + A_1 x + A_2 x^2 + \cdots A_n x^n)\cos\beta x + (B_0 + B_1 x + B_2 x^2 + \cdots B_n x^n)\sin\beta x\Big],$$

where A_k and B_k, $k = 0, 1, \ldots, n$, are constants to be determined. The exponent s is a number from the set $\{0, 1, 2\}$, picked as follows.

Let us introduce the generally complex number $z = \alpha + i\beta$,[1] and let's write r_1 and r_2 for the roots of the characteristic equation $ar^2 + br + c = 0$.[2] Then select $s \in \{0, 1, 2\}$ according to the flowchart below.



To state the flowchart in words: If z matches neither of r_1 or r_2, then $s = 0$. If z matches only one of r_1 or r_2, then $s = 1$. If z matches both r_1 and r_2, then $s = 2$.

[1] Note that z is completely determined by $f(x)$, and that z is a real number if $\beta = 0$.

[2] The roots r_1 and r_2 may be real, complex or repeated, depending on the sign of the discriminant $b^2 - 4ac$. Regardless, we refer to the roots by r_1 and r_2 in all cases.

Example: Let us find a particular solution to $y''+2y'+y = (2x+3)e^{-x}$. Comparing the right-hand side against $f(x)$, we see that $\alpha = -1$, $\beta = 0$, $a_0 = 2$, $a_1 = 3$, $n = 1$. Then our particular solution has the form $y_p(x) = x^s e^{-x}(A_0 + A_1 x)$.

To determine s, we note that the characteristic equation $r^2 + 2r + 1 = 0$ which factorizes as $(r+1)^2 = 0$, and therefore the roots are $r_1 = r_2 = -1$. Since $\alpha = -1$, $\beta = 0$, we have $z = -1$. Referring to the flowchart, we see that $s = 2$. Thus, we seek a particular solution of the form $y_p(x) = x^2 e^{-x}(A_0 + A_1 x)$, or equivalently, $y_p(x) = e^{-x}(A_0 x^2 + A_1 x^3)$.

It remains to determine A_0 and A_1. Toward that end, we calculate

$$\begin{aligned}
y_p'(x) &= -e^{-x}(A_0x^2 + A_1x^3) + e^{-x}(2A_0x + 3A_1x^2)\\
&= e^{-x}\big[-A_1x^3 + (3A_1 - A_0)x^2 + 2A_0x\big],\\
y_p''(x) &= -e^{-x}\big[-A_1x^3 + (3A_1 - A_0)x^2 + 2A_0x\big] + e^{-x}\big[-3A_1x^2 + 2(3A_1 - A_0)x + 2A_0\big]\\
&= e^{-x}\big[A_1x^3 + (-6A_1 + A_0)x^2 + (6A_1 - 4A_0)x + 2A_0\big]
\end{aligned}$$

Substituting these into the differential equation we obtain $(6A_1x + 2A_0)e^{-x} = (2x+3)e^{-x}$, that is, $6A_1x + 2A_0 = 2x + 3$, from which it follows that $A_1 = 1/3$ and $A_0 = 3/2$, and consequently,

$$y_p(x) = x^2\Big(\frac{3}{2} + \frac{1}{3}x\Big)e^{-x}.$$

Since the solution of the homogeneous equation is $y_h(x) = c_1e^{-x} + c_2xe^{-x}$, the general solution of our differential equation is

$$y(x) = c_1e^{-x} + c_2xe^{-x} + x^2\Big(\frac{3}{2} + \frac{1}{3}x\Big)e^{-x}.$$