Partial Differential Equations

Lecture Notes for Math 404

Rouben Rostamian

Department of Mathematics and Statistics
UMBC

Fall 2020

The wave equation

as a prototype of hyperbolic equations

Hyperbolic equations in applications

The *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

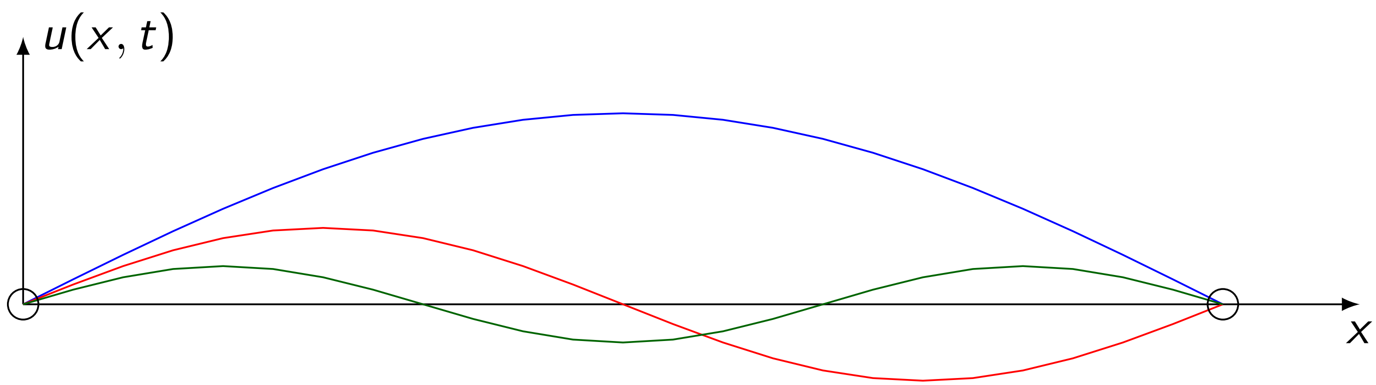
along with its many variants, is the prototype of a very large class of *hyperbolic equations* that arise in many applications such as

- vibration of solid structures (strings, beams, membranes, plates)
- propagation of seismic waves
- geological exploration, oil well detection
- aerodynamics and supersonic flight
- propagation of electromagnetic waves (radiant heat, light, radio waves, microwaves, fiber optics, antennas)

The wave equation

We wish to derive the equation of motion of a stretched string with ends fixed. (Think of a guitar string or cello string). Depending on the manner of excitation, the string may flex in many different ways. See the figure to the right. We write T for the tensile force within the string, ρ for the mass of string per unit length, $u(x, t)$ for the lateral displacement of the string, and $\theta(x, t$ for the angle between the string and the equilibrium state at the location x at time t,

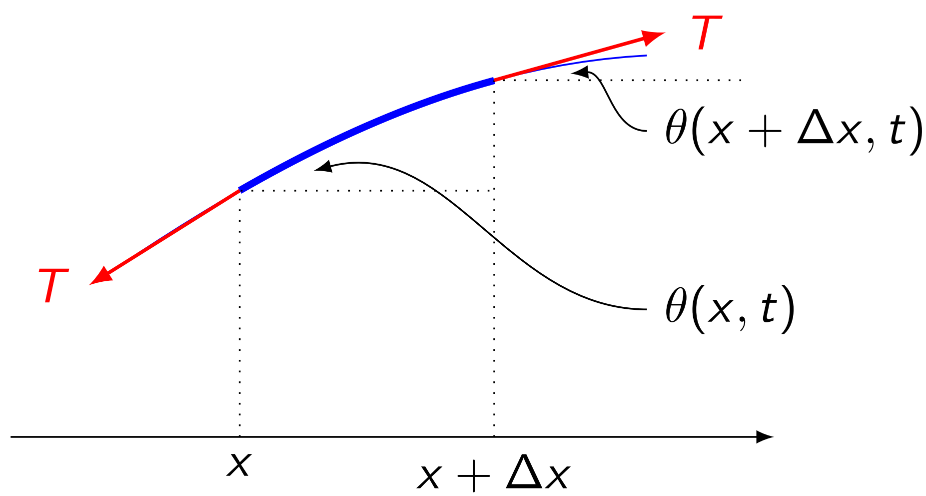
We assume that the deflection away from equilibrium is small so that we may approximate $\sin(\theta) \approx \theta$ and $\tan(\theta) \approx \theta$.



The wave equation (continued)

Let us focus on a small segment of the string between locations x and $x+\Delta x$. The mass of that segment is $\rho\Delta x$, and its vertical acceleration is $\frac{\partial^2 u}{\partial t^2}$. Therefore, according to Newton, $\rho\Delta x\frac{\partial^2 u}{\partial t^2}$ equals the resultant of vertical forces acting on the string. But in the diagram below we see that the vertical component of the acting forces is $T\sin\theta(x+\Delta x,t)-T\sin\theta(x,t)$. We conclude that

$$\rho\Delta x\frac{\partial^2 u}{\partial t^2}=T\sin\theta(x+\Delta x,t)-T\sin\theta(x,t).$$



The wave equation (continued)

We divide through by Δx

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\sin\theta(x+\Delta x, t) - \sin\theta(x,t)}{\Delta x}$$

and pass to the limit as $\Delta x \to 0$:

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial}{\partial x}(\sin\theta).$$

However, by our smallness assumption of θ we have $\frac{\partial u}{\partial x} \stackrel{\text{slope}}{=} \tan\theta \approx \sin\theta$ and therefore

$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = T \frac{\partial^2 u}{\partial x^2}.$$

We let $T/\rho = c^2$ and cast the equation above into the standard form of the *wave equation*. It expresses *Newton's law of motion* applied to a stretched string:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

The vibrating string

Consider the stretched string depicted in Slide 4. We have seen that its motion $u(x,t)$ is a solution of the wave equation. We supply that equation with initial and boundary conditions to obtain a well-posed *initial boundary value problem*:

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < L, \quad t > 0, \tag{2a}$$

$$u(0,t) = 0 \qquad t > 0, \tag{2b}$$

$$u(L,t) = 0 \qquad t > 0, \tag{2c}$$

$$u(x,0) = f(x) \qquad 0 < x < L, \tag{2d}$$

$$u_t(x,0) = g(x) \qquad 0 < x < L. \tag{2e}$$

Note the specification of the initial condition. The condition (2d) specifies the string's *deflection* at $t = 0$. The condition (2e) specifies the string's *velocity* at $t = 0$.

In the slides that follow, we will calculate the solution of this initial boundary value problem through. . . what else? Separation of variables!

The wave equation — separation of variables

We look for solutions to (2) in the form $u(x,t) = X(x)T(t)$. Plugging this into (2a) we see that $X(x)T''(t) = c^2 X''(x)T(t)$, whence

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2,$$

where we have, based on our previous experiences with such matters, picked $-\lambda^2$ (a negative number) for the separation constant. Thus, we obtain

$$T''(t) + c^2\lambda^2 T(t) = 0, \quad X''(x) + \lambda^2 X(x) = 0, \quad X(0) = 0, \quad X(L) = 0. \tag{3}$$

The last two equations are the consequences of (2b) and (2c).

The general solution of the X equation is $X(x) = A\cos\lambda x + B\sin\lambda x$. Applying the condition $X(0) = 0$ implies that $A = 0$, and thus we are left with $X(x) = B\sin\lambda x$. Applying the condition $X(L) = 0$ implies that $\sin\lambda L = 0$, whence $\lambda = n\pi/L$ for all positive integers n. We write these as $\lambda_n = n\pi/L$.

The wave equation — separation of variables 2

Having determined the values of the separation constant, we write $X_n(x) = \sin \lambda_n x$ for the corresponding solutions. Moreover, in view of the T equation in (3), we see that $T_n(t) = A \cos \lambda_n ct + B \sin \lambda_n ct$. We conclude that $X_n(x)T_n(t) = (A_n \cos \lambda ct + B_n \sin \lambda_n ct) \sin \lambda_n x$ is a solution of the equations (2a), (2b), and (2c) for any positive integer n, and therefore the following infinite linear combination is also a solution:

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \lambda_n ct + B_n \sin \lambda_n ct) \sin \lambda_n x. \tag{4}$$

In remains to pick the A's and Bs in order to satisfy the initial conditions (2d) and (2e). Let's observe that the velocity of the string at (x,t) is obtained by differentiating the displacement $u(x,t)$ with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} (-A_n \lambda_n c \sin \lambda_n ct + B_n \lambda_n c \cos \lambda_n ct) \sin \lambda_n x.$$

We set $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ and continue into the next slide.

The wave equation — separation of variables 3

We see that

$$\sum_{n=1}^{\infty} A_n \sin \lambda_n x = f(x), \qquad \sum_{n=1}^{\infty} B_n \lambda_n c \sin \lambda_n x = g(x).$$

Then A_n and B_n may be calculated from our old formulas for the Fourier sine series:

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \lambda_n x \, dx, \qquad B_n = \frac{2}{\lambda_n c L}\int_0^L g(x) \sin \lambda_n x \, dx. \tag{5}$$

This completes our analysis and solution of the vibrating string problem. The string's motion is given in (4), where the coefficients A_n and B_n are calculated according to (5).

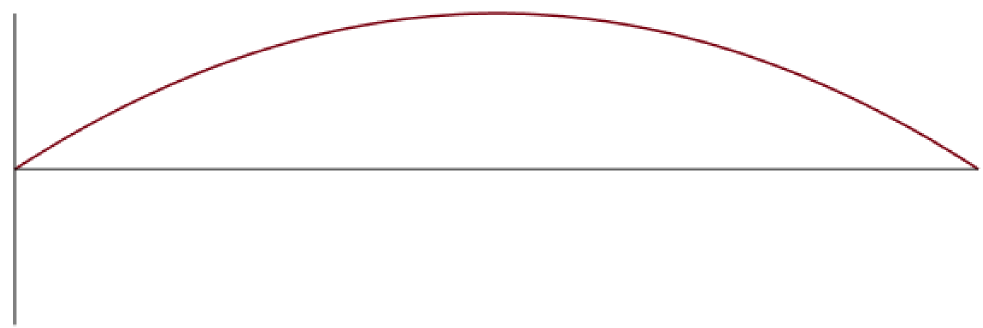
Vibrating string – An example

Suppose that we deflect the string into the shape of a parabola $f(x) = x(1 - \frac{x}{L})$ and release it without imparting any initial velocity, i.e., $g(x){=}0$. The motion is given in (4), with the As and Bs as in (5). Since $g(x) = 0$, we have all Bs equal zero, and the solution is

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \lambda_n ct \sin \lambda_n x,$$

where

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \lambda_n x \, dx = \frac{2}{L}\int_0^L x(1 - \frac{x}{L}) \sin \lambda_n x \, dx = \frac{4L}{\pi^3} \cdot \frac{1-(-1)^n}{n^3}.$$



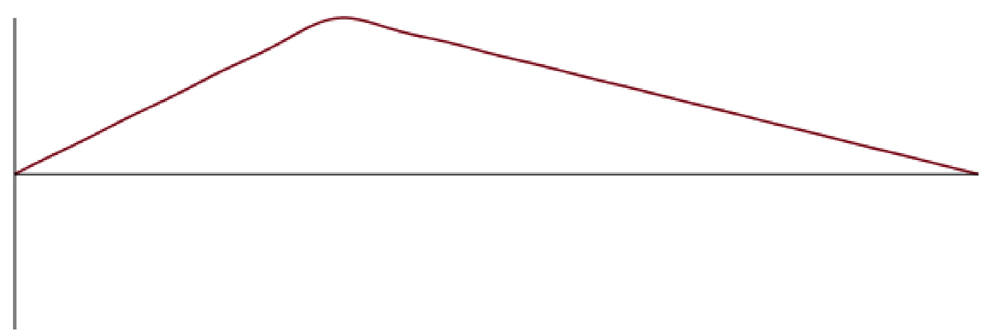
An animation with $L = 1$, $c = 1$ and infinity set to 20.

Vibrating string – Another example

We simulate the plucking of the string by setting $f(x) = \begin{cases} x & \text{if } x < L/3, \\ \frac{1}{2}(L-x) & \text{if } x > L/3. \end{cases}$

and $g(x)=0$. Then $u(x,t) = \sum_{n=1}^{\infty} A_n \cos \lambda_n ct \sin \lambda_n x$, where

$$A_n = \frac{2}{L}\int_0^L f(x) \sin \lambda_n x \, dx = \frac{3L}{\pi^2} \cdot \frac{\sin \frac{n\pi}{3}}{n^2}.$$



An animation with $L = 1$, $c = 1$ and infinity set to 20.

The piano wire – an exercise

What sets a piano wire into motion is not an *initial deflection*, but an *initial velocity*, imparted to it by the hammer. In a piano wire of length L, let's take the striking region to be $1/16$th of the wire's length at either side of the wire's center. Then the wire's initial displacement is zero while the initial velocity is

$$g(x) = \begin{cases} 1 & \text{if } x > \frac{L}{2} - \frac{L}{16} \text{ and } x < \frac{L}{2} + \frac{L}{16}, \\ 0 & \text{otherwise.} \end{cases}$$

Find the wire's displacement $u(x,t)$. Here is what it looks like:

An animation with $L = 1$, $c = 1$ and infinity set to 100 (large!) in order to adequately resolve the discontinuous initial velocity.

The method of characteristics

as a means of analyzing the wave equation

The characteristic lines

Let $u(x,t)$ be a solution to the one-dimensional wave equation

$$u_{tt} = c^2 u_{xx}. \tag{6}$$

Let us observe:

$$(u_t + cu_x)_t = u_{tt} + cu_{xt},$$
$$(u_t + cu_x)_x = u_{tx} + cu_{xx}.$$

Multiply the second equation by $-c$ and add it to the first. We get

$$(u_t + cu_x)_t - c(u_t + cu_x)_x = u_{tt} - c^2 u_{xx} = 0 \quad \text{(by (6))}.$$

Letting $v = u_t + cu_x$, this becomes

$$v_t - cv_x = 0. \tag{7}$$

That's nice! We have gotten a first order PDE out of the second order PDE (6)
But that's not all...

The characteristic lines (continued)

Similarly, we calculate

$$(u_t - cu_x)_t = u_{tt} - cu_{xt},$$
$$(u_t - cu_x)_x = u_{tx} - cu_{xx}.$$

Multiply the second equation by c and add it to the first. We get

$$(u_t - cu_x)_t + c(u_t - cu_x)_x = u_{tt} - c^2 u_{xx} = 0 \quad \text{(by (6))}.$$

Letting $w = u_t - cu_x$, this becomes

$$w_t + cw_x = 0. \tag{8}$$

That's a second 1st order PDE emerging from the wave equation (6).

The characteristic lines – Summary

Summary: The 2nd order wave equation $u_{tt} = c^2 u_{xx}$ *is equivalent to the system of two 1st order equations*

$$v_t - cv_x = 0, \quad w_t + cw_x = 0, \tag{9}$$

where

$$v \overset{def}{=} u_t + cu_x, \quad w \overset{def}{=} u_t - cu_x. \tag{10}$$

Note that $u_t = \frac{1}{2}(v + w)$ and $u_x = \frac{1}{2c}(v - w)$, so once we find v and w, we can calculate u.

Terminology: Either of the equations (9) is called *a one-dimensional first order wave equation.*

The first order wave equation

Let us look at the first order wave equation for w in (9):

$$w_t + cw_x = 0. \tag{11}$$

Its solution, $w(x,t)$, expresses the value of w at the position x at time t.

Suppose that we have an observer that moves along the x axis according to some arbitrary motion $x(t)$. Then the value of w that the observer sees at time t is $w\big(x(t),t\big)$. The rate of change of w, as seen by the observer, is obtained by the chain rule

$$\frac{d}{dpt} w\big(x(t),t\big) = w_x\big(x(t),t\big)x'(t) + w_t\big(x(t),t\big), \tag{12}$$

where $x'(t) = \frac{d}{dt}x(t)$ is the observer's velocity.

What happens if the observer moves at the constant velocity c, where c is the coefficient in (11)? Then we would have $x(t) = ct + x_0$, and (12) would reduce to

$$\frac{d}{dt} w\big(x(t),t\big) = w_x\big(x(t),t\big)c + w_t\big(x(t),t\big) = 0 \qquad \text{(by (11))}.$$

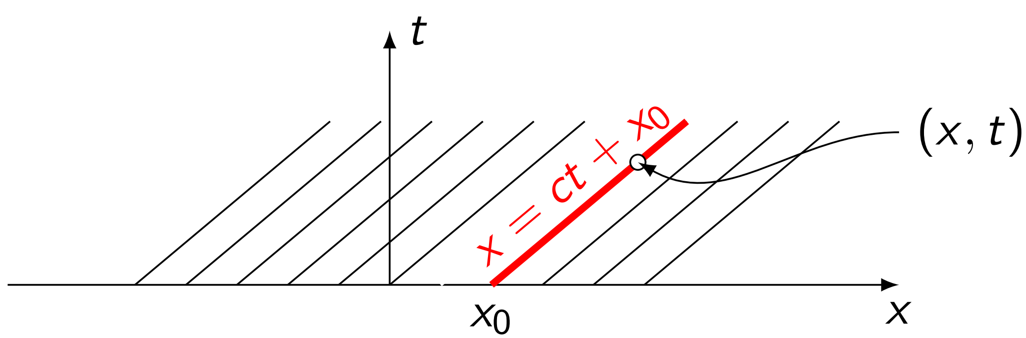
This says that the observer moving with velocity c sees no changes at all in w!

A summary of the previous slide

We have seen that if $w(x,t)$ is the solution of $w_t + cw_x = 0$, then an observer moving with velocity c will perceive no changes in the value of w. The position of

an observer moving with the constant velocity c is given by $x(t) = ct + x_0$, where x_0 is the observer's location at time $t = 0$.

The lines $x = ct + x_0$ in the x-t plane are called *the characteristic lines*, or just *the characteristics* for short, of the equation $w_t + cw_x = 0$.



We have seen that the solution w is constant along the characteristics. Thus, referring to the picture above, $w(x,t) = w(x_0, 0)$.

Solving a first order wave equation via characteristics

Let's say the value of w along the x axis is prescribed, that is, the initial condition is $w(x,0) = \phi(x)$ for some given ϕ. Then the value of of w at the point (x,t) is the same as the value of w at the point x_0 where the characteristic through (x,t) intersects the x axis. Thus

$$w(x,t) = w(x_0) = \phi(x_0).$$

But the equation of the characteristic is $x = ct + x_0$, and therefore $x_0 = x - ct$. We conclude that

$$w(x,t) = \phi(x - ct). \tag{13}$$

Important conclusion: Equation (13) expresses the solution $w(x,t)$ of the PDE $w_t + cw_x = 0$ in terms of its initial condition $\phi(x)$.

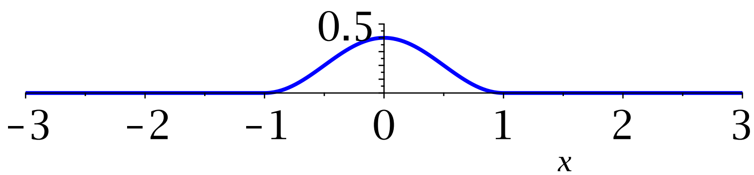
Riding on the characteristics

Let's solve the initial value problem for the function $w(x,t)$:

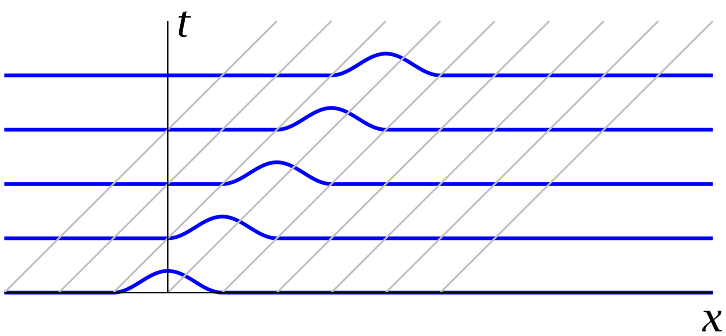
$$w_t + cw_x = 0,$$
$$w(x,0) = \phi(x),$$

where ϕ is a blip:

$$\phi(x) = \begin{cases} \frac{1}{5}(1+\cos \pi x) & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$$



We know that the solution is $w(x,t) = \phi(x-ct)$. But what does it look like?

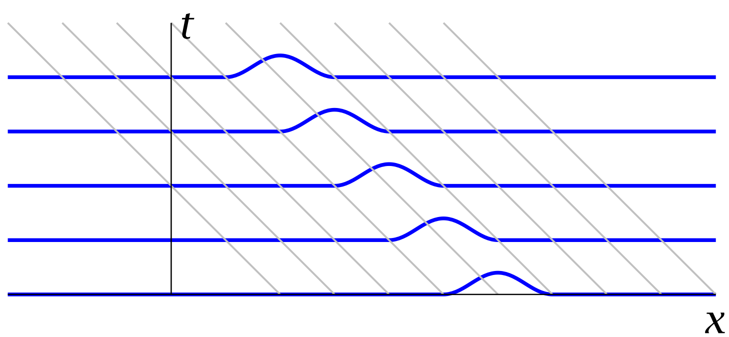


It's a traveling wave!

The traveling blip

Traveling in the opposite direction

Returning to Slide 17, recall that we split the 2nd order wave equation $u_{tt} = c^2 u_{xx}$ into a pair of two 1st order PDEs $v_t - cv_x = 0$ and $w_t + cw_x = 0$. We have completely analyzed the w equation. The v equation is pretty much the same except for the wave speed $+c$ has been changed to $-c$. Everything that has been said about w carries over to v, but the waves travel in the opposite direction.



The solution of the initial value problem

$$v_t - cv_x = 0,$$
$$v(x,0) = \psi(x),$$

is $v(x,t) = \psi(x + ct)$.

d'Alembert's solution to the second order wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Solving the second order wave equation

We have completely analyzed the 1st order initial value problems

$$w_t + cw_x = 0, \qquad v_t - cv_x = 0,$$
$$w(x,0) = \phi(x), \qquad v(x,0) = \psi(x),$$

and have obtained their solutions $w(x,t) = \phi(x-ct)$ and $v(x,t) = \psi(x+ct)$. On Slide 17 we saw that the solution $u(x,t)$ of the 2nd order wave equation $u_{tt} = c^2 u_{xx}$ is related to v and w through

$$u_t(x,t) = \frac{1}{2}\Big[v(x,t) + w(x,t)\Big], \quad u_x(x,t) = \frac{1}{2c}\Big[v(x,t) - w(x,t))\Big].$$

With what we have learned, these become

$$u_t(x,t) = \frac{1}{2}\Big[\psi(x+ct) + \phi(x-ct)\Big], \quad u_x(x,t) = \frac{1}{2c}\Big[\psi(x+ct) - \phi(x-ct)\Big],$$

Let us introduce the the function F and G defined through their derivatives as

$$F'(x) = -\frac{1}{2c}\phi(x), \quad G'(x) = \frac{1}{2c}\psi(x). \tag{14}$$

Then

$$u_t(x,t) = cG'(x+ct) - cF'(x-ct), \quad u_x(x,t) = G'(x+ct) + F'(x-ct).$$

Solving the second order wave equation (continued)

In the previous slide we arrived at

$$u_t(x,t) = cG'(x+ct) - cF'(x-ct), \quad u_x(x,t) = G'(x+ct) + F'(x-ct).$$

Integrating the first equation with respect to t, and the second equation with respect to x we get

$$u(x,t) = G(x+ct) + F(x-ct) + A(x), \quad u(x,t) = G(x+ct) + F(x-ct) + B(t),$$

where $A(x)$ and $B(t)$ are the integration "constants". Subtracting the two equations results in $A(x) = B(t)$. This says that $A(x)$ does not depend on x (since it's equal to $B(t)$ for all x). Therefore $A(x)$ is a constant, and therefore $B(t)$ is also a constant. Let's write C for that common constant.

Thus, we arrive at $u(x,t) = G(x+ct) + F(x-ct) + C$. The presence of C there is immaterial since each of F and G are defined through their derivatives only in (14). We conclude that

Solving the second order wave equation (continued)

The reasoning in the previous slide has lead us to

$$u(x, t) = F(x - ct) + G(x + ct) \tag{15}$$

as a solution of the wave equation $u_{tt} = c^2 u_{xx}$. The functions F and G are defined in (14) in terms of the arbitrary functions ϕ and ψ, therefore they may be regarded as arbitrary functions as well.

Important! It can be shown (but not in this course) that (15) is *the general solution of the wave equation $u_{tt} = c^2 u_{xx}$*, that is, every solution of the wave equation has that form.

The functions F and G may be determined from a set of prescribed initial conditions to the wave equation. We will address that in the next slide.

d'Alembert's solution

Here we consider the initial value problem for the function $u(x,t)$:

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty, \quad t > 0, \tag{16a}$$
$$u(x,0) = f(x), \qquad -\infty < x < \infty, \tag{16b}$$
$$u_t(x,0) = g(x), \qquad -\infty < x < \infty, \tag{16c}$$

where the initial displacement, f, and the initial velocity, g, are given. The general solution to the PDE (16a) is available in (15). Our task is to determine F and G in terms of the given data f and g.

We have

$$u(x,t) = F(x-ct) + G(x+ct),$$
$$u_t(x,t) = -cF'(x-ct) + cG'(x+ct).$$

Letting $t = 0$ and applying the initial conditions we get

$$F(x) + G(x) = f(x), \tag{17a}$$
$$-cF'x) + cG'(x) = g(x). \tag{17b}$$

d'Alembert's solution (continued)

Isolate $G(x)$ in (17a) and plug the result into (17b):

$$-cF'(x) + c\Big[f'(x) - F'(x)\Big] = g(x),$$

solve for $F'(x)$:

$$F'(x) = \frac{1}{2}f'(x) - \frac{1}{2c}g(x),$$

and integrate:

$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\xi)\,d\xi + K. \tag{18}$$

Note: The integration constant, K, cancels a $-K$ in the final answer in the next slide, and therefore it is of no practical significance.

Having determined $F(x)$, now we calculate $G(x)$ from (17a):

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\xi)\,d\xi - K. \tag{19}$$

d'Alembert's solution (continued)

We conclude that

$$F(x-ct) = \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(\xi)\,d\xi - K,$$
$$G(x+ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(\xi)\,d\xi + K,$$

whence the general solution

$$u(x,t) = F(x-ct) + G(x+ct) \tag{20}$$

takes the form

$$u(x,t) = \frac{1}{2}\Big[f(x-ct) + f(x+ct)\Big] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi. \tag{21}$$

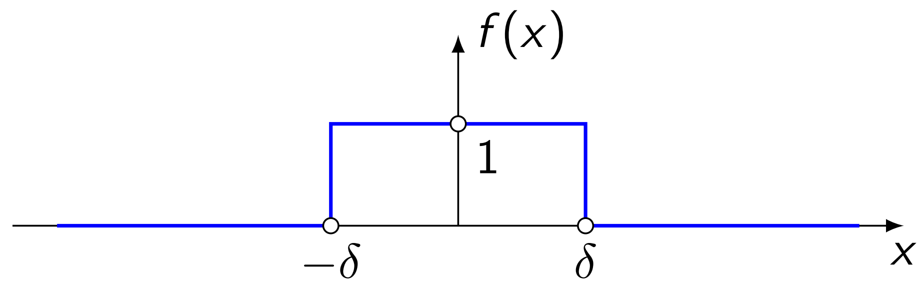
The representation (21) of the initial value problem (16) was discovered by *Jean-Baptiste le Rond d'Alembert* in 1747 and is referred to as *d'Alembert's solution*.

Note: The expression (21) is pleasing, but it's not the most convenient form for hand calculations. To calculate $u(x,t)$, it's more practical to calculate the functions F and G from (18) and (19), and then apply (20) to determine $u(x,t)$.

A worked out example

Consider a string which at the initial time is deformed into a rectangular blip, as shown below, and is released with zero initial velocity:

$$f(x) = \begin{cases} 1 & \text{if } |x| < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

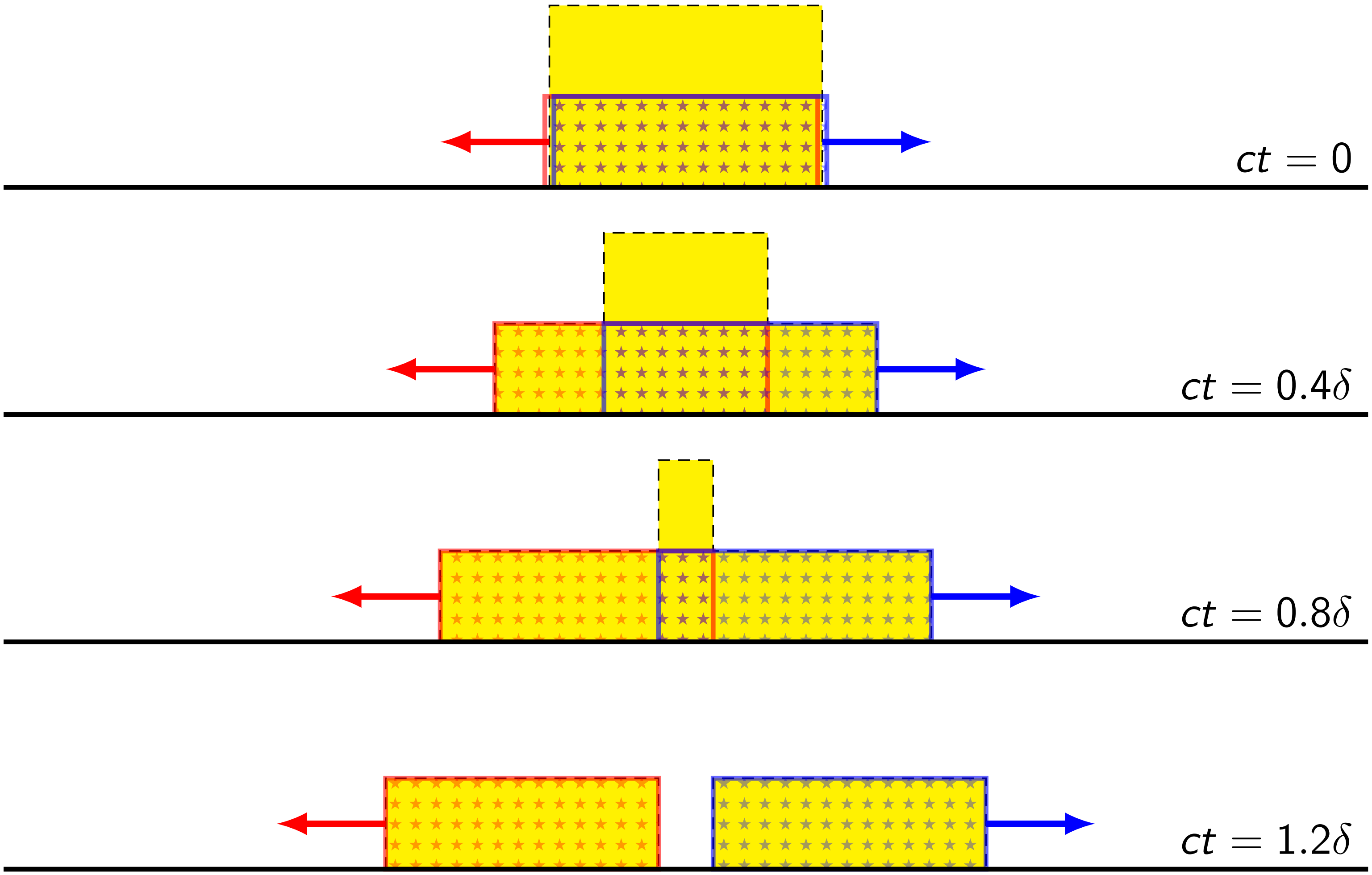


This fits the formulation of d'Alembert's problem in equations (16) with $f(x)$ as the blip given above, and $g(x) = 0$. We apply (20) to calculate the solution $u(x,t)$.

Equations (18) and (19) indicate that $F(x) = G(x) = \frac{1}{2}f(x)$, that is, each of F and G is just like the original blip but with half the height.

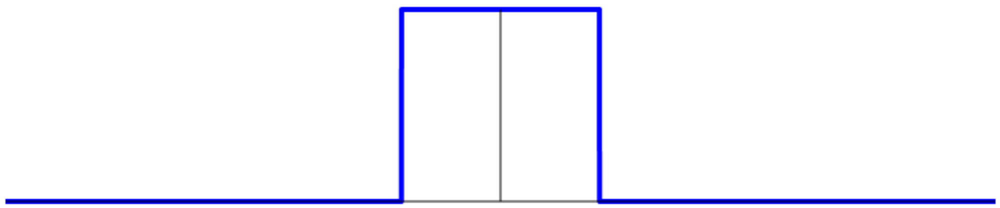
To apply (20), we need to calculate $F(x - ct)$ and $G(x + ct)$. But the graph of $F(x - ct)$ is obtained by translating the graph of $F(x)$ to the right along the x axis by the amount ct. Similarly, the graph of $G(x + ct)$ is obtained by translating the graph of $G(x)$ to the left by ct. The resulting $u(x,t)$ is shown in the next slide.

A worked out example (continued)

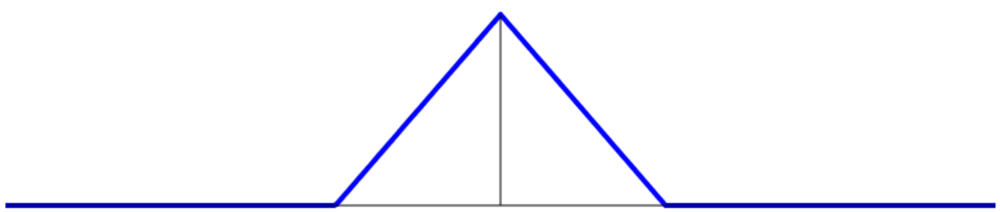


Animated traveling waves

Initial conditions: $f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$ and $g(x) = 0$



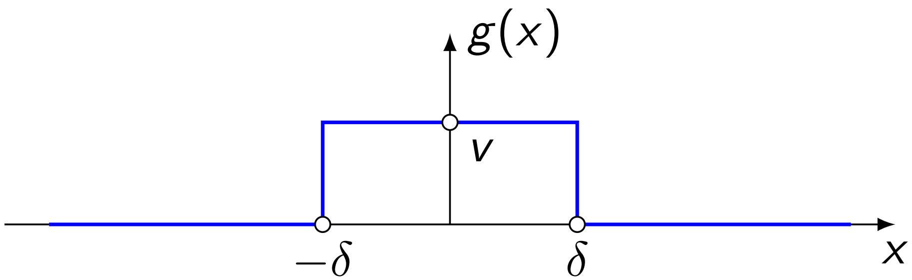
Initial conditions: $f(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$ and $g(x) = 0$



Another worked out example (hitting a piano wire)

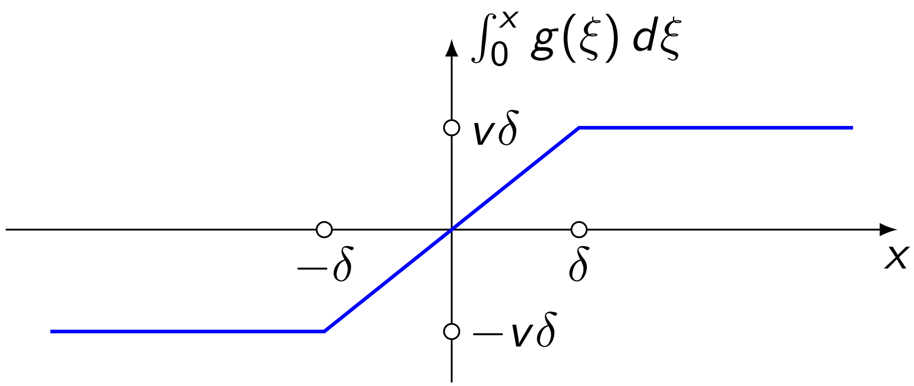
Consider a string which at $t = 0$ is in its equilibrium position (i.e. $f(x) = 0$), but it is given an initial velocity $g(x)$ in the form of a rectangular blip, as shown below:

$$g(x) = \begin{cases} v & \text{if } |x| < \delta, \\ 0 & \text{otherwise.} \end{cases}$$



Calculating the function $F(x)$ and $G(x)$ in (18) and (19), calls for finding the antiderivative of $g(x)$. We see that

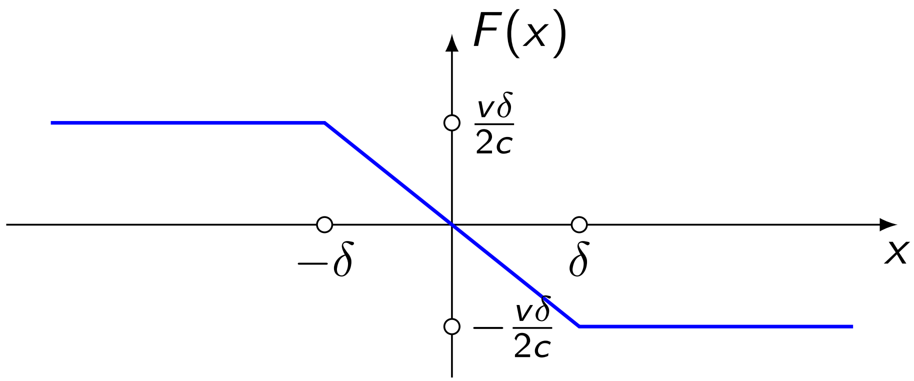
$$\int_0^x g(\xi)\, d\xi = \begin{cases} -v\delta & \text{if } x < -\delta, \\ vx & \text{if } |x| \le \delta, \\ v\delta & \text{if } x > \delta, \end{cases}$$



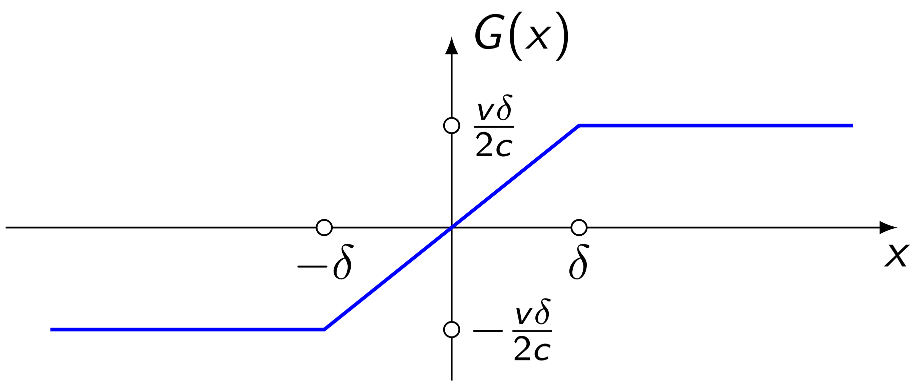
Piano wire (continued)

From (18) and (19):

$$F(x) = \begin{cases} \frac{v\delta}{2c} & \text{if } x < -\delta, \\ -\frac{v}{2c}x & \text{if } |x| \le \delta, \\ -\frac{v\delta}{2c} & \text{if } x > \delta, \end{cases}$$



$$G(x) = \begin{cases} -\frac{v\delta}{2c} & \text{if } x < -\delta, \\ \frac{v}{2c}x & \text{if } |x| \le \delta, \\ \frac{v\delta}{2c} & \text{if } x > \delta, \end{cases}$$

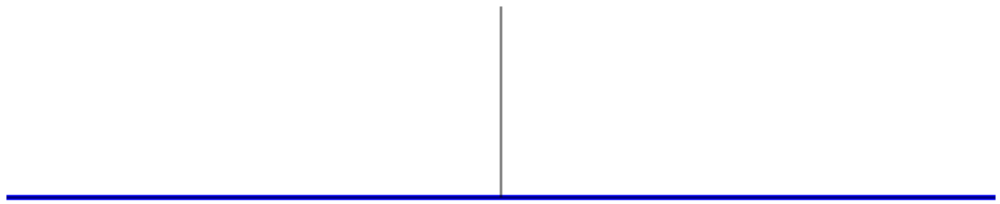


Then from (20):

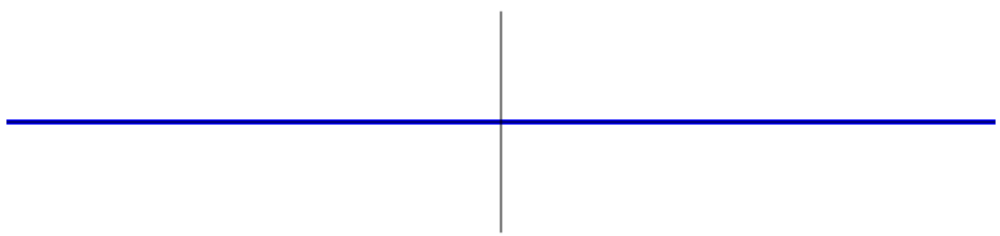
$$u(x,t) = F(x - ct) + G(x + ct).$$

Animations

Initial conditions: $f(x) = 0$ and $g(x) = \begin{cases} v & \text{if } |x| < \delta, \\ 0 & \text{otherwise.} \end{cases}$



Initial conditions: $f(x) = 0$ and $g(x) = \begin{cases} \sin \pi x & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases}$



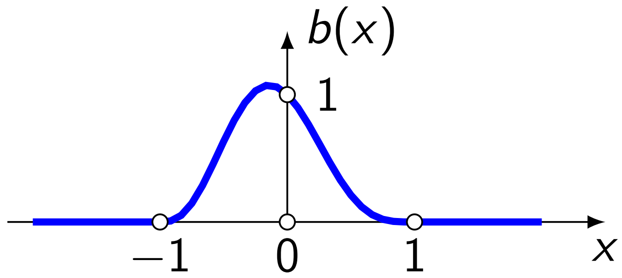
Waves in semi-infinite domains and reflections from the boundary

The *Method of Images*

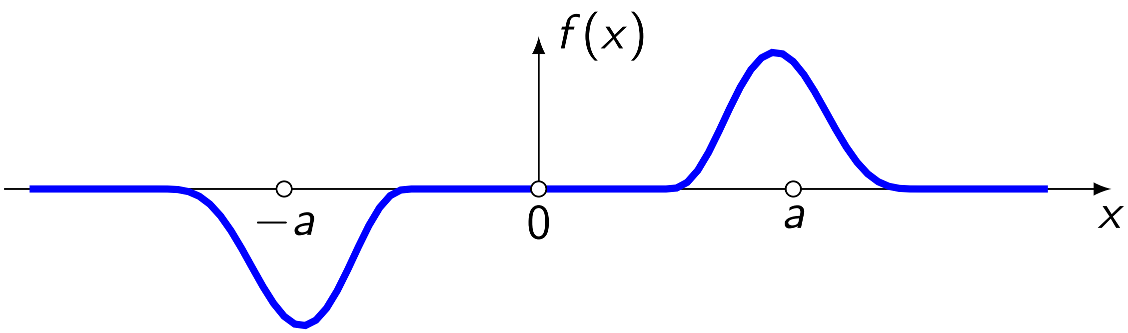
An introduction to the *Method of Images*

Traveling wave in an infinite string with an *odd function* for the initial condition.

A blip: $b(x) = \begin{cases} (x-1)^4(x+1)^3 & \text{if } |x| < 1, \\ 0 & \text{otherwise.} \end{cases}$



Initial displacement: $f(x) = b(x-a) - b(-x-a)$.

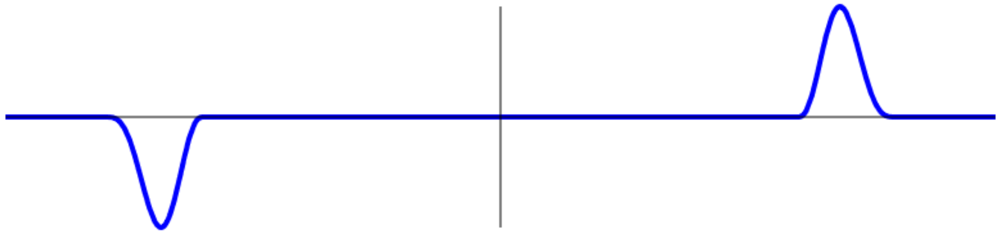


Note that f is odd: $f(-x) = -f(x)$.

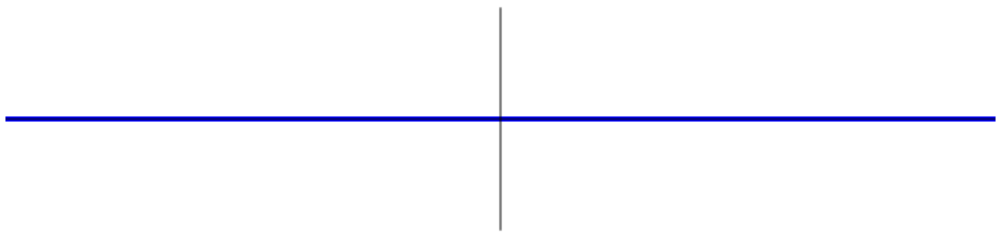
Take the initial velocity $g(x) = 0$. What does the solution look like? Let's see...

An introduction to the *Method of Images* (continued)

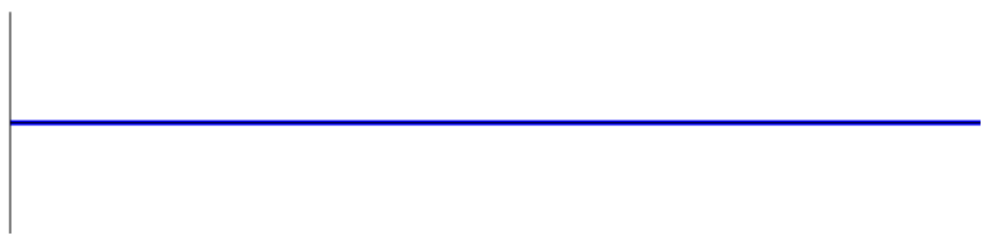
Here is what the wave looks like:



Here is the same animation, cropped from the left and right:



And here is the same animation, with the $x < 0$ hidden:



Waves in semi-infinite domains

The animations in the previous slide inspire the following "trick".

Consider the motion of a *semi-infinite* string, $0 < x < \infty$, which is tided down (cannot move) at $x = 0$.

We give it an initial displacement $f(x)$ and, for the sake of simplicity, start off with zero initial velocity, $g(x)$. Here is the mathematical statement of the corresponding initial boundary value problem:

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < \infty, \quad t > 0, \tag{22a}$$

$$u(0,t) = 0 \qquad t > 0, \tag{22b}$$

$$u(x,0) = f(x) \qquad 0 < x < \infty, \tag{22c}$$

$$u_t(x,0) = 0, \qquad 0 < x < \infty. \tag{22d}$$

To solve this, we extend $f(x)$ as an odd function to the negative x axis. That is, let $\tilde{f}(x) = \begin{cases} f(x) & \text{if } x > 0 \\ -f(-x) & \text{if } x < 0 \end{cases}$. Then, we solve the wave equation on the entire x axis, with the initial displacement $\tilde{f}(x)$. [*continued on the next slide*]

Waves in semi-infinite domains (continued)

The extended initial boundary value problem is

$$u_{tt} = c^2 u_{xx} \qquad -\infty < x < \infty, \quad t > 0, \tag{23a}$$

$$u(x,0) = \tilde{f}(x) \qquad -\infty < x < \infty, \tag{23b}$$

$$u_t(x,0) = 0, \qquad -\infty < x < \infty. \tag{23c}$$

Note that the boundary constraint (22b) has been removed.

The solution of the system of equations (23) is give by (see (21))

$$u(x,t) = \frac{1}{2}\Big[\tilde{f}(x-ct) + \tilde{f}(x+ct)\Big]. \tag{24}$$

Waves in semi-infinite domains (continued)

Now here's a nifty argument:

- The PDEs (23a) and (22a) are identical on the $x > 0$ domain. Since $u(x,t)$ given in (24) satisfies (23a) for all $-\infty < x < \infty$, it also satisfies (22a) on $x > 0$.
- Plugging $t = 0$ in (24) we see that $u(x,0) = \tilde{f}(x)$, which is not a surprise, since that is required in (23b). But the definition of $\tilde{f}$ says that $\tilde{f}$ and f coincide on $x > 0$, therefore $u(x,t)$ constructed in (24) also satisfies the initial condition (22c).
- The velocity corresponding to (24) is $u_t(x,t) = \frac{1}{2}\Big[-c\tilde{f}'(x-ct) + +c\tilde{f}'(x+ct)\Big]$ and therefore $u_t(x,0) = 0$ for $-\infty < x\infty$, and in particular, for $0 < x < \infty$. It follows that $u(x,t)$ satisfies (22d).
- Let $x = 0$ in (24). We get $u(0,t) = \frac{1}{2}\Big[\tilde{f}(-ct) + \tilde{f}(ct)\Big] = 0$ since $\tilde{f}$ is an odd.

Conclusion: The restriction of the function $u(x,t)$ given in (24) satisfies all four equations in (22) and therefore it is the desired solution.

Recipe: The Method of Images

Consider the initial boundary value problem for the wave equation on a semi-infinite domain:

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < \infty, \quad t > 0, \tag{25a}$$

$$u(0,t) = 0 \qquad t > 0, \tag{25b}$$

$$u(x,0) = f(x) \qquad 0 < x < \infty, \tag{25c}$$

$$u_t(x,0) = g(x) \qquad 0 < x < \infty. \tag{25d}$$

To solve this, extend f and g as odd functions $\tilde{f}$ and $\tilde{g}$ to the entire x axis and solve, let's say via d'Alembert's formula (21), the initial value problem

$$u_{tt} = c^2 u_{xx} \qquad 0 < x < \infty, \quad t > 0, \tag{26a}$$

$$u(x,0) = \tilde{f}(x) \qquad 0 < x < \infty, \tag{26b}$$

$$u_t(x,0) = \tilde{g}(x) \qquad 0 < x < \infty. \tag{26c}$$

Then the restriction of $u(x,t)$ to $x > 0$ is the solution of (25).
We have seen why this is true when $g = 0$. Showing that this remains true when g is nonzero is left as a homework problem.

The Method of Images

Why is this called "The Method of Images"? It's because the extensions $\tilde{f}$ and $\tilde{g}$ look like inverted mirror images of f and g.

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x > 0 \\ -f(-x) & \text{if } x \leq 0 \end{cases} \qquad \tilde{g}(x) = \begin{cases} g(x) & \text{if } x > 0 \\ -g(-x) & \text{if } x \leq 0 \end{cases}$$