Partial Differential Equations

Lecture Notes for Math 404

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Introduction to PDEs

You are already familiar with *Ordinary Differential Equations* (ODEs). Here are a few representative samples:

$$p'(r) = -kp(r), \qquad u''(x) + \omega^2 u(x) = 0, \qquad my''(t) + cy'(t) + ky(t) = f(t).$$

In these equations the unknowns p, u, y (also known as the *dependent variables*) are functions of the single variables, r, x and t (called the *independent variables*).

In *Partial Differential Equations* (PDEs), unknowns are functions of more than one independent variable. Here are a few representative samples:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$$ advection: $u(x, t)$ in one space dimension

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$ diffusion: $u(x, t)$ in one space dimension

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$ wave propagation: $u(x, y, z, t)$ in three space dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$ static gravitational field: $u(x, y, z)$ in three space dimensions

Equations of parabolic, hyperbolic, and elliptic types

Read the textbook's Lesson 1 on an extensive discussion of classifications of PDEs. In this course we will focus on linear equations of the type

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad \textit{parabolic}\text{, in analogy with } y = x^2 + c$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad \textit{hyperbolic}\text{, in analogy with } y^2 = x^2 + c$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad \textit{elliptic}\text{, in analogy with } x^2 + y^2 = c$$

Occasionally we will take side tours to look at other, closely related equations, but the above will be the bulk of this course's material.

The heat equation

as a prototype of parabolic equations

Parabolic equations in applications

The *heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

along with its many variants, is the prototype of a very large class of *parabolic equations* that arise in many applications such as

- heat conduction within solids, liquids, and gasses
- seepage in porous media
- diffusion of chemicals
- smoothing of supersonic shock waves (for numerical computation)
- stochastic processes in probability
- image analysis, edge detection, blurring and sharpening
- the Black-Scholes model of financial mathematics (Nobel prize in economics, 1997)

Heat conduction across a refrigerator wall

Heat flows from hot to cold

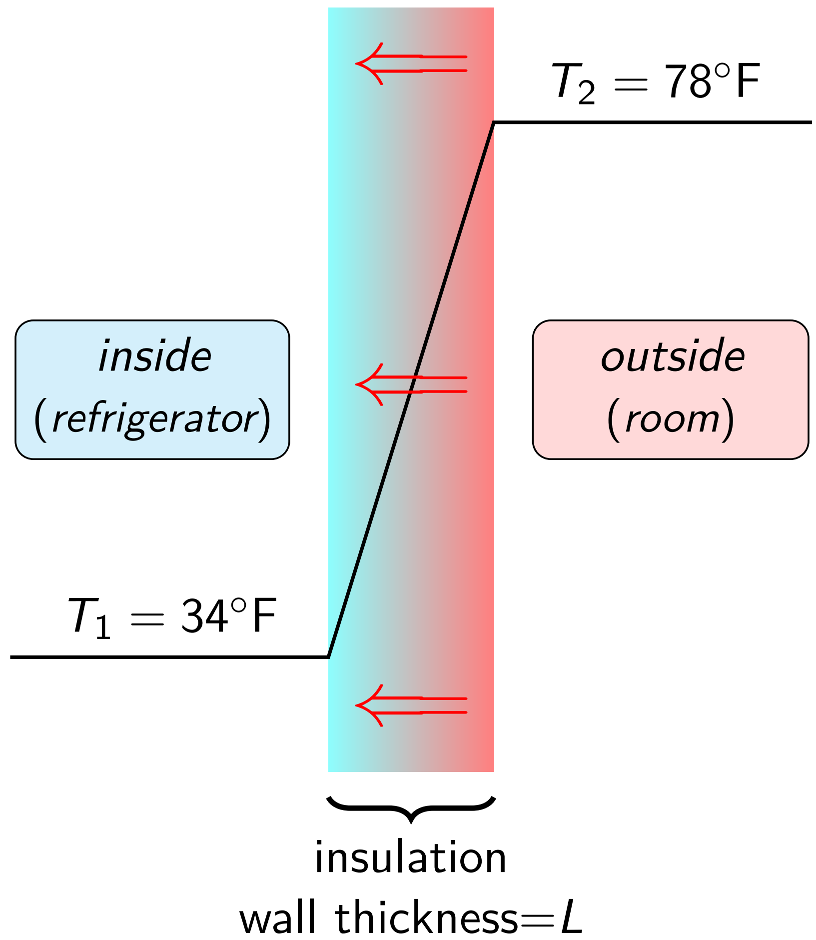
Rate of flow $\propto \frac{T_2 - T_1}{L}$

Fourier's *Law of Heat Conduction*

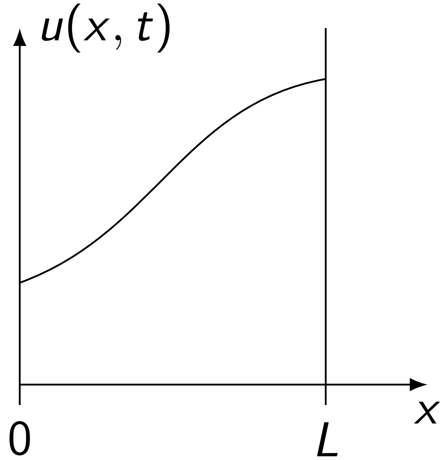
$$q = -k\frac{dT}{dx}$$

$q =$ *heat flux*
$=$ thermal energy passing through per unit area per unit time

$k =$ *thermal conductivity*



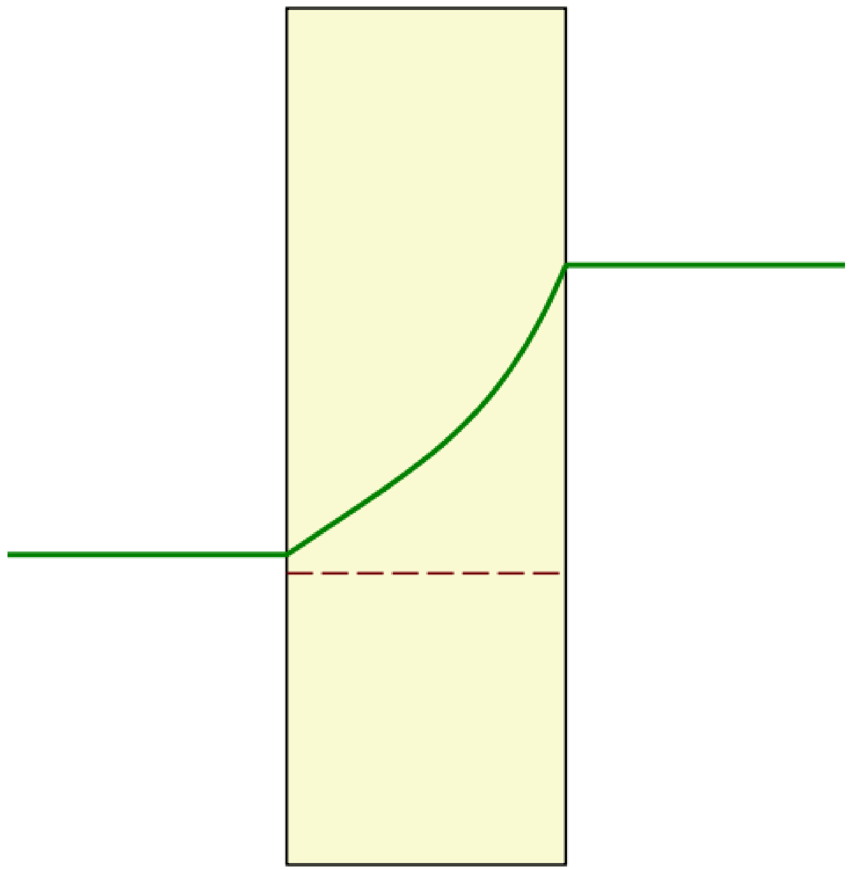
Heat conduction movie



The heat equation

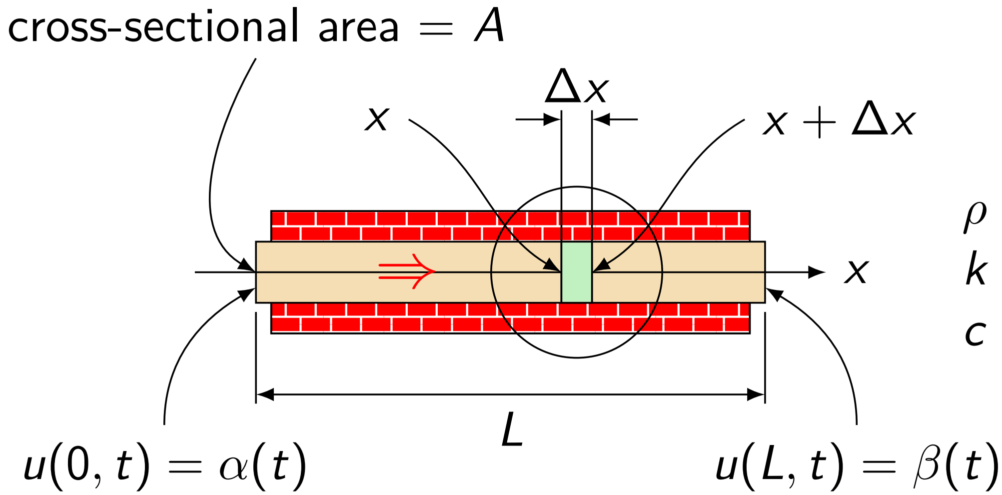
$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

Expresses conservation of thermal energy.



Temperature variations across a refrigerator wall

Where does the heat equation come from?



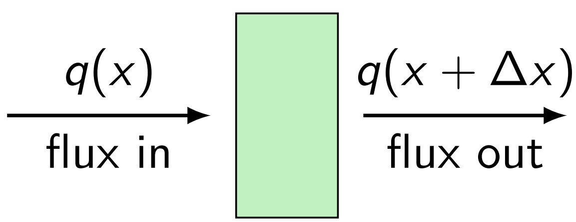
ρ = density = mass / unit volume

k = thermal conductivity

c = specific heat capacity

Specific heat capacity

thermal energy required to raise the temperature of unit mass by one degree



volume $= A\Delta x$

mass $= \rho A\Delta x$

energy content $= (\rho A\Delta x)(cu)$

$$\frac{\partial}{\partial t}\Big((\rho A\Delta x)(cu)\Big) = Aq(x) - Aq(x+\Delta x)$$

Conservation of thermal energy

Conservation of energy: *The rate of change of the thermal energy content within the green slice equals the rate of energy flowing in minus the rate of energy flowing out*

$$\frac{\partial}{\partial t}\Big((\rho A \Delta x)(cu)\Big) = Aq(x) - Aq(x+\Delta x)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{c\rho}\frac{q(x+\Delta x)-q(x)}{\Delta x}$$

Taking the limit as $\Delta x \to 0$ we arrive at a partial differential equation that expresses *conservation of energy*:

$$\frac{\partial u}{\partial t} = -\frac{1}{c\rho}\frac{\partial q}{\partial x} \tag{1a}$$

Together with *Fourier's Law of Heat Conduction*

$$q = -k\frac{\partial u}{\partial x} \tag{1b}$$

we have a system of two first order PDEs in the two unknowns u and q.

The heat equation

Eliminating q between equations (1a) and (1b), we obtain a second order PDE for the unknown temperature u;

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho}\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)$$

That's the heat equation!

The coefficients c, ρ, and k may vary with the position x, but if they are constants, then we obtain the classic heat equation:

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} \qquad \left(\text{where } \kappa = \frac{k}{c\rho}\right)$$

κ is called the heat equation's *diffusion coefficient*

Remarks

The formulation of the heat conduction as a system of first order PDEs

$$\frac{\partial u}{\partial t} = -\frac{1}{c\rho}\frac{\partial q}{\partial x}, \qquad q = -k\frac{\partial u}{\partial x} \tag{2}$$

seems to be equivalent to the single second order PDE

$$\frac{\partial u}{\partial t} = \frac{1}{c\rho}\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right) \tag{3}$$

but there are subtle and significant differences.

In (3) the diffusion coefficient k is under a differentiation sign while in (2) it is not. If k is a constant or a smoothly varying function, that's not a big deal, but what if k is discontinuous?

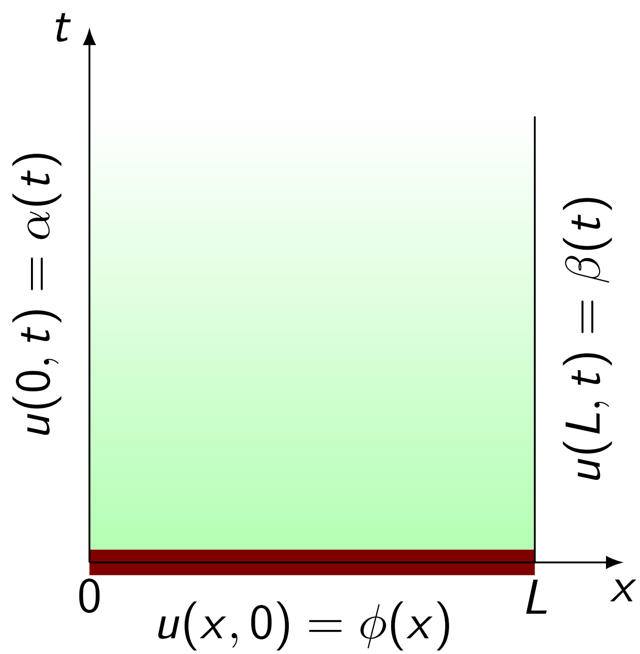
Recall the example of heat conduction through a refrigerator wall. The wall consists of a metal layer on the outside, a plastic layer on the inside, and styrofoam filling in between. The conductivities of these materials are drastically different, therefore k varies discontinuously as we move through the wall.

Remarks (continued)

There are various ways of handling discontinuous k at theoretical and computational levels.

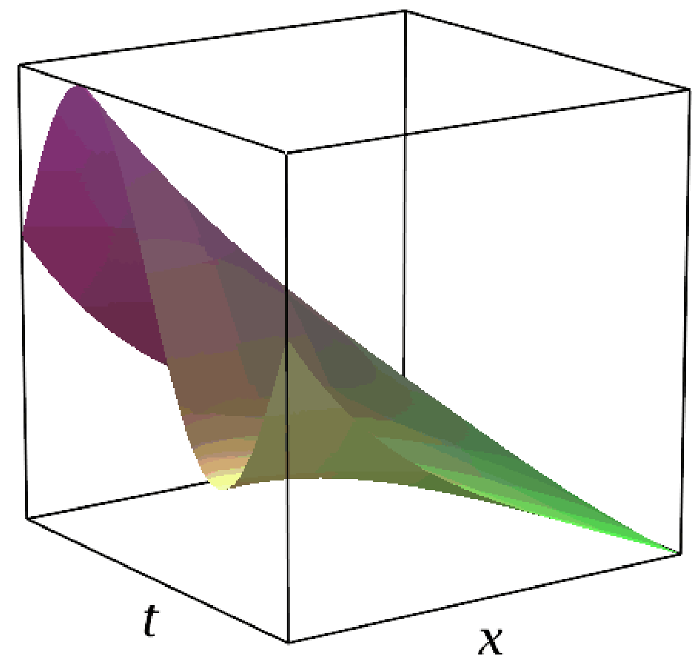
- [Theoretical] Generalize the classical definitions of functions and their derivative to non-smooth functions. This leads to the theory of generalized functions and distributions. Dirac's *delta function* falls in that category.
- [Theoretical] Formulate differentiation as an operator in a function space. This leads to Sobolev spaces and *weak formulations of PDEs*.
- [Computational] In the weak formulation of a PDE, replace the infinite-dimensional Sobolev space with an appropriate finite-dimensional approximation. This leads to Galerkin's formulation and the *method of finite elements*.
- [Computational] Approximate the derivatives in (2) through difference quotients. This leads to a *finite difference formulation* of the problem.
- [Computational] Approximate the derivatives in (3) through difference quotients. A naive implementation will produce junk since it will attempt to differentiate k. Special-purpose finite difference schemes are available for producing correct results.
- [Computational] Apply (3) separately within each layer where k is differentiable. Connect the layers through equations that enforce the conservation of energy.

The domain of $u(x, t)$



Domain of solution:

$0 < x < L, \ T > 0$



The graph of temperature $u(x, t)$ within the refrigerator's wall, as a function of x and t.

Initial/boundary value problems for the heat equation

Prescribed boundary temperature:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2} + f(x,t) & 0 < x < L, \quad t > 0 \\ u(0,t) = \alpha(t) & t > 0 \\ u(L,t) = \beta(t) & t > 0 \\ u(x,0) = \phi(x) & 0 < x < L \end{cases}$$

Prescribed boundary flux at one end:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2} + f(x,t) & 0 < x < L, \quad t > 0 \\ u(0,t) = \alpha(t) & t > 0 \\ -k \dfrac{\partial u}{\partial x}\bigg|_{x=L} = \gamma(t) & t > 0 \\ u(x,0) = \phi(x) & 0 < x < L \end{cases}$$

Separation of variables

for homogeneous equations

The separation of variables trick

The simplest initial/boundary value problem:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, \quad t > 0 \tag{4a}$$

$$u(0, t) = 0 \qquad t > 0 \tag{4b}$$

$$u(L, t) = 0 \qquad t > 0 \tag{4c}$$

$$u(x, 0) = \phi(x) \qquad 0 < x < L \tag{4d}$$

Try for a solution of the form $u(x, t) = X(x)T(t)$:

$$X(x)T'(t) = \kappa X''(x)T(t) \qquad \Rightarrow \frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} \tag{5a}$$

$$X(0)T(t) = 0 \qquad \Rightarrow X(0) = 0 \tag{5b}$$

$$X(L)T(t) = 0 \qquad \Rightarrow X(L) = 0 \tag{5c}$$

$$X(x)T(0) = \phi(x) \qquad \Rightarrow \text{? (will worry about this one later)} \tag{5d}$$

The separation of variables trick – part 2

Equation (5a) implies that

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = \text{some constant, say } \eta \tag{6}$$

The constant η may be positive, zero, or negative

Spoiler! Turns out that only $\eta < 0$ leads to anything interesting.

Case $\eta = \lambda^2 > 0$: From (6) we get:

$$T'(t) = \kappa\lambda^2 T(t), \qquad X''(x) = \lambda^2 X(x)$$

From the second equation above we get $X(x) = A \sinh \lambda x + B \cosh \lambda x$, and therefore $X(0) = B$. Then from (5b) we get $B = 0$. Thus, we are left with $X(x) = A \sinh \lambda x$, and therefore $X(L) = A \sinh \lambda L$. Then from (5c) we get $A \sinh \lambda L = 0$. Since $\lambda L \neq 0$, we must have $A = 0$, and therefore the solution is $X(x) = 0$ for all x. Not interesting.

Case $\eta = 0$: You do it. (conclusion: Not interesting)

The separation of variables trick – part 3

Case $\eta = -\lambda^2 < 0$: From (6) we get:

$$T'(t) + \kappa\lambda^2 T(t) = 0, \qquad X''(x) + \lambda^2 X(x) = 0 \tag{7}$$

From the second equation above we get $X(x) = A\sin\lambda x + B\cos\lambda x$, and therefore $X(0) = B$. Then from (5b) we get $B = 0$. Thus, we are left with $X(x) = A\sin\lambda x$, and therefore $X(L) = A\sin\lambda L$. Then from (5c) we get $A\sin\lambda L = 0$. We don't want A to be zero (not interesting) so we get $\sin\lambda L = 0$ and therefore $\lambda L = n\pi$, for any integer n, will do. We let

$$\lambda_n = \frac{n\pi}{L}, \qquad n = 1, 2, \dots \tag{8}$$

and thus, $X(x) = A\sin\lambda_n x$.

Furthermore, from the first equation in (7) we get $T(t) = Ce^{-\kappa\lambda_n^2 t}$, and therefore we arrive at $u(x,t) = ACe^{-\kappa\lambda_n^2 t}\sin\lambda_n x$ as a solution that satisfies the equations (5a), (5b), and (5c). and consequently, equations (4a), (4b), and (4c).

We have not yet accounted for equation (5d) (or equivalently, equation (4d)). We turn to that issue now.

The separation of variables trick – part 4

Equations (4a)–(4c) are *linear* and *homogeneous*, which is the technical way of saying that if $u_1(x,t)$ and $u_2(x,t)$ satisfy those equations, then any linear combination $c_1 u_1(x,t) + c_2 u_2(x,t)$ with constant coefficients c_1 and c_2, also satisfy those equations. (Verify this for yourself; it's not hard!)

In the previous slide (slide 18) we saw that $u(x,t) = e^{-\kappa\lambda_n^2 t}\sin\lambda_n x$ satisfies the equations (4a)–(4c) for any integer n. Therefore, so does the (infinite) linear combination

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\kappa\lambda_n^2 t}\sin\lambda_n x \tag{9}$$

where the choice of the (constant) coefficients a_n is at our disposal. We are going to choose those coefficients so that $u(x,t)$, expressed as (9), satisfies the one last remaining requirement, that is, the equation (4d).

From (9) we have $u(x,0) = \sum_{n=1}^{\infty} a_n \sin\lambda_n x$, and therefore from (4d) we get

$$\sum_{n=1}^{\infty} a_n \sin\lambda_n x = \phi(x). \tag{10}$$

The separation of variables trick – part 5

Question: *Can any function ϕ be expressed as the infinite sum in* (10)*?*

The answer is *yes!* provided that ϕ satisfies certain regularity conditions such as sufficient continuity and integrability. (We won't get into those conditions in this course, but for practical purposes it is safe to assume that those are satisfied.) If so, we multiply (10) by $\sin \lambda_m x$ and integrate over the interval $(0, L)$:

$$\sum_{n=1}^{\infty} a_n \int_0^L \sin \lambda_m x \sin \lambda_n x \, dx = \int_0^L \phi(x) \sin \lambda_m x \, dx \tag{11}$$

It is left to you as an exercise to show that for λs defined as in (8), and any two integers m and n:

$$\int_0^L \sin \lambda_m x \sin \lambda_n x \, dx = \begin{cases} 0 & \text{if } m \neq n \\ L/2 & \text{if } m = n \end{cases}$$

and therefore in the infinite sum in (11) only one term survives and we arrive at

$$\frac{L}{2} a_m = \int_0^L \phi(x) \sin \lambda_m x \, dx.$$

This tells us the value of a_m for all m, since the initial condition ϕ is known.

Summary of the two preceding slides

A function ϕ defined in the interval $(0, L)$ may be expressed as the infinite sum

$$\phi(x) = \sum_{n=1}^{\infty} a_n \sin \lambda_n x, \tag{12}$$

where

$$a_n = \frac{2}{L} \int_0^L \phi(x) \sin \lambda_n x \, dx. \tag{13}$$

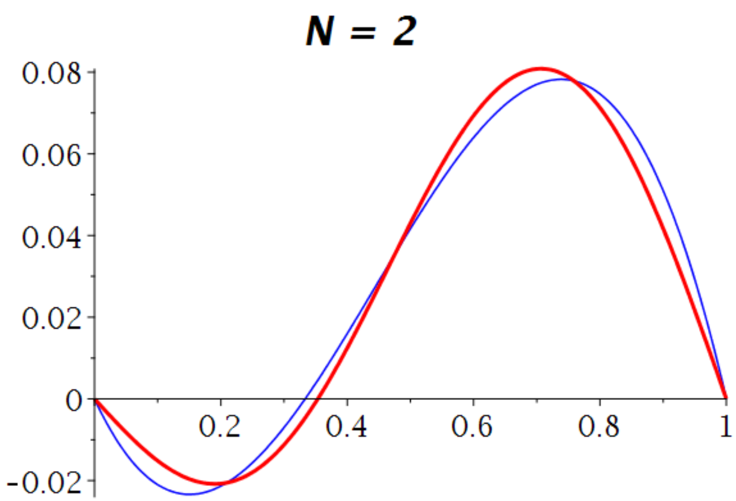
and where

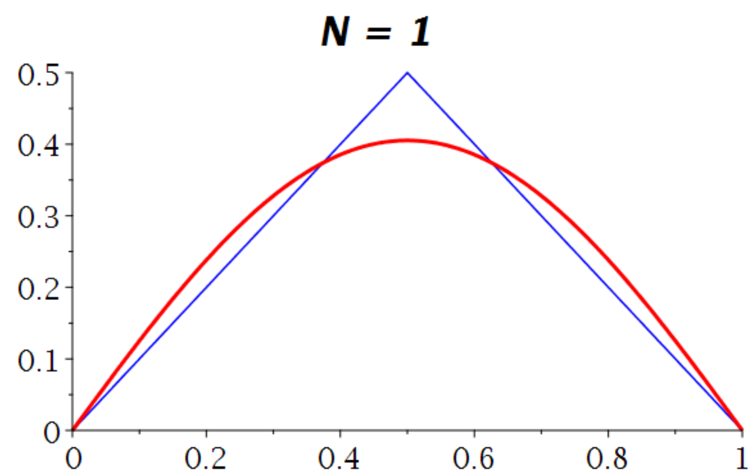
$$\lambda_n = \frac{n\pi}{L}, \qquad n = 1, 2, \dots \tag{14}$$

The expression on the right-hand side of (12) is called the *Fourier sine series representation* of the function ϕ. The coefficients a_n are called the corresponding *Fourier coefficients* (named after the French mathematician Joseph Fourier, 1767–1830).

How good is the Fourier series?

In these demos, the original function ϕ is plotted in blue, while the approximations by the first N terms of the Fourier series are plotted in red.





$$\phi(x) = x(x - 1/3)(1 - x) = \frac{4}{3\pi^3} \sum_{n=1}^{\infty} \frac{(5(-1)^n + 4) \sin n\pi x}{n^3}$$

$$\phi(x) = 1/2 - |x - 1/2| = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2} \sin n\pi x}{n^2}$$

The separation of variables trick – part 6 and conclusion

Summary:
In the previous slides we have developed the bits and pieces needed for calculating the solution $u(x,t)$ of the initial/boundary value problem (4). In (9) we saw that

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\kappa\lambda_n^2 t} \sin \lambda_n x \tag{15a}$$

and we learned that the coefficients a_n are obtained from (13)

$$a_n = \frac{2}{L}\int_0^L \phi(x) \sin \lambda_n x \, dx, \tag{15b}$$

where

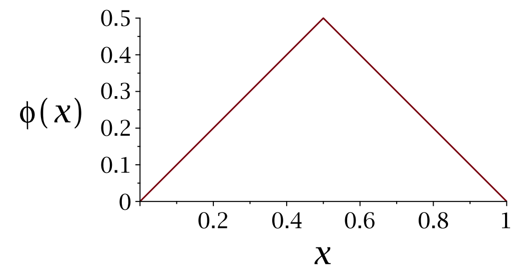
$$\lambda_n = \frac{n\pi}{L}. \tag{15c}$$

A fully worked-out example

Equations (15) on the previous slide present the solution $u(x,t)$ of the initial/boundary value problem (4) (on slide 17) for an arbitrary initial condition $u(x,0) = \phi(x)$.

Calculating the solution for a specific ϕ is a matter of carrying out the integration in (15b). Here is a sketch of the calculations.

$$\phi(x) = \frac{L}{2} - \left|x - \frac{L}{2}\right| = \begin{cases} x & \text{if } x < L/2 \\ L - x & \text{if } x > L/2 \end{cases}$$



The graph of $\phi(x)$ with $L = 1$

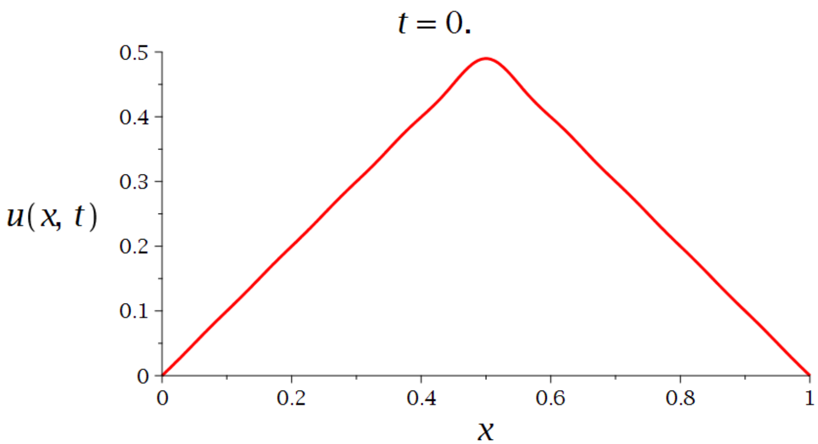
$$a_n = \frac{2}{L}\int_0^L \phi(x)\sin\lambda_n x\,dx = \frac{2}{\lambda_n^2 L}\left[2\sin\frac{\lambda_n L}{2} - \sin\lambda_n L\right] \qquad \text{(from Quiz \#1)}$$

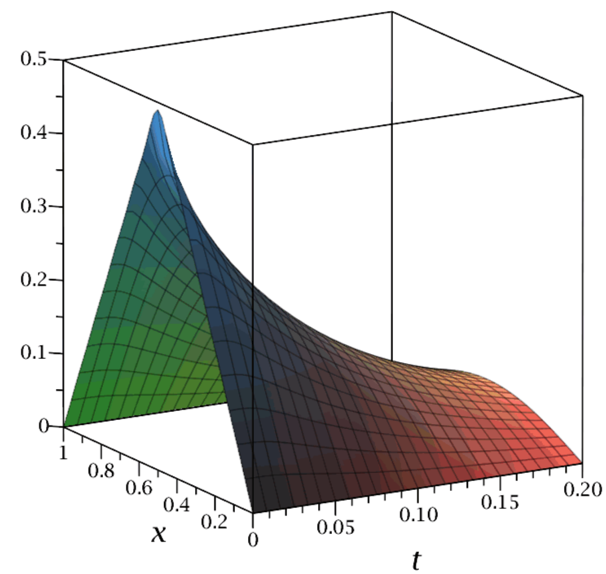
$$= \frac{2L}{n^2\pi^2}\left[2\sin\frac{n\pi}{2} - \sin n\pi\right] = \frac{4L}{n^2\pi^2}\sin\frac{n\pi}{2}. \qquad \text{(from (15c))}$$

The solution

$$u(x,t) = \frac{4L}{\pi^2}\sum_{n=1}^{\infty} e^{-\kappa\lambda_n^2 t}\,\frac{\sin\frac{n\pi}{2}\sin\lambda_n x}{n^2}$$

$$= \frac{4L}{\pi^2}\left[e^{-\kappa(\pi/L)^2 t}\sin\frac{\pi x}{L} - \frac{1}{3^2}e^{-\kappa(3\pi/L)^2 t}\sin\frac{3\pi x}{L} + \frac{1}{5^2}e^{-\kappa(5\pi/L)^2 t}\sin\frac{5\pi x}{L} - \cdots\right]$$



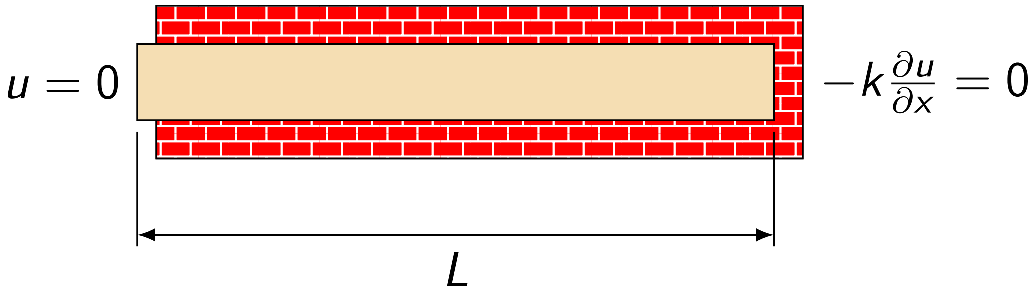


The solution $u(x,t)$ evaluated with $L = 1$, $\kappa = 1$ and truncated as $\sum_{n=1}^{19}$ (ten terms)

Insulated boundary

$$-k\frac{\partial u}{\partial x}\bigg|_{x=L} = 0$$

Insulated boundary at $x = L$



$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{16a}$$

$$u(0,t) = 0 \tag{16b}$$

$$-k \frac{\partial u}{\partial x}\bigg|_{x=L} = 0 \tag{16c}$$

$$u(x,0) = \phi(x) \tag{16d}$$

Separate the variables: $u(x,t) = X(x)T(t)$. Then $X(x)T'(t) = \kappa X''(x)T(t)$ and therefore

$$\frac{1}{\kappa}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2, \qquad X(0) = 0, \qquad X'(L) = 0$$

Separation of variables

$$T'(t) = -\kappa\lambda^2 T(t),$$

$$X''(x) + \lambda^2 X(x) = 0, \qquad X(0) = 0, \qquad X'(L) = 0$$

The general solution of the X equation is $X(x) = A \sin \lambda x + B \cos \lambda x$. Applying the boundary condition $X(0) = 0$, we get $B = 0$. Therefore $X(x) = A \sin \lambda x$.

Then $X'(x) = \lambda A \cos \lambda x$. Therefore applying the boundary condition $X'(L) = 0$ we get $\cos \lambda L = 0$. We conclude that λL is an odd multiple of $\pi/2$, that is $\lambda_n L = (2n-1)\frac{\pi}{2}$, and therefore

$$\lambda_n = \frac{(2n-1)\pi}{2L}, \quad X_n(x) = \sin \lambda_n x, \quad T_n(t) = e^{-\kappa\lambda^2 t} \quad n = 1, 2, \ldots \qquad (17)$$

and

$$u(x,t) = \sum_{n=1}^{\infty} a_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} a_n e^{-\kappa\lambda_n^2 t} \sin \lambda_n x$$

$$= \sum_{n=1}^{\infty} a_n e^{-\kappa\left[(2n-1)\pi/(2L)\right]^2 t} \sin \frac{(2n-1)\pi}{2L} x.$$

Separation of variables continued

The coefficients a_n are determined by applying the initial condition $u(x,0) = \phi(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} a_n X_n(x) = \phi(x)$$

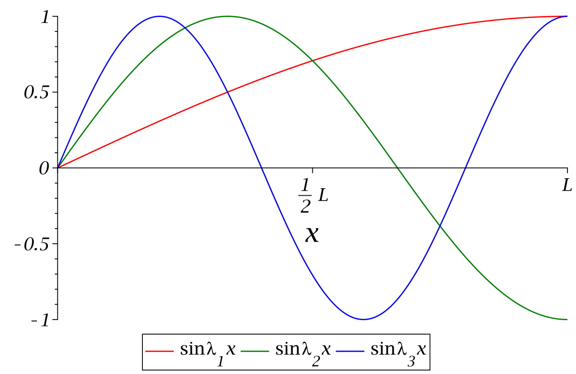
Exercise: Show that for any integer m and n, and λ_n defined as in (17), we have:

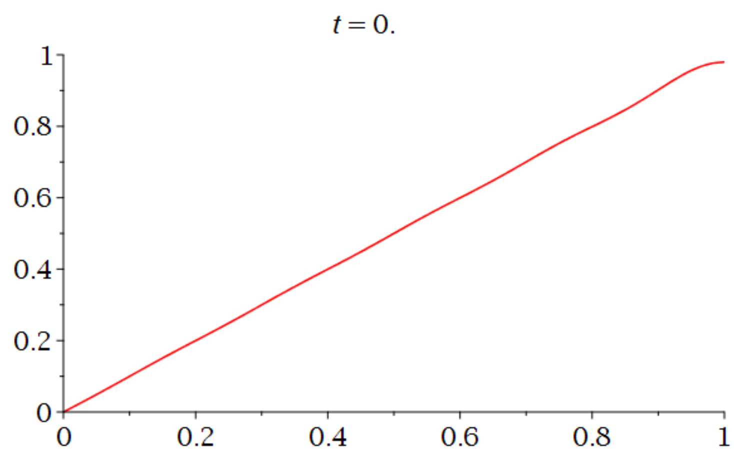
$$\int_0^L X_m(x) X_n(x)\,dx = \int_0^L \sin\lambda_m x \sin\lambda_n x\,dx = \begin{cases} 0 & \text{if } m \neq n \\ L/2 & \text{if } m = n \end{cases}$$

Therefore

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L \phi(x) X_n(x)\,dx = \frac{2}{L}\int_0^L \phi(x) \sin\lambda_n x\,dx \\ &= \frac{2}{L}\int_0^L \phi(x) \sin\frac{(2n-1)\pi x}{2L}\,dx \end{aligned}$$

The modal shapes and an animation





The solution $u(x, t)$ with the initial condition $\phi(x) = x/L$, evaluated with $L = 1$, $\kappa = 1$ and truncated as $\sum_{n=1}^{10}$ (ten terms)

Equations with heat source

...but zero boundary conditions

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \kappa \frac{\partial^2 u}{\partial x^2} + f(x,t) \\
u(0,t) &= 0 \\
u(L,t) &= 0 \\
u(x,0) &= \phi(x)
\end{aligned}
$$

Eigenfunction expansion

We are going to solve the initial/boundary value problem

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad 0 < x < L, \quad t > 0 \tag{18a}$$

$$u(0,t) = 0 \qquad t > 0 \tag{18b}$$

$$u(L,t) = 0 \qquad t > 0 \tag{18c}$$

$$u(x,0) = \phi(x) \qquad 0 < x < L \tag{18d}$$

On slide 21 we saw that any function of x defined in the interval $0 < x < L$ may be expanded into a Fourier sine series. We let

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \lambda_n x, \quad f(x,t) = \sum_{n=1}^{\infty} \bar{f}_n(t) \sin \lambda_n x, \quad \phi(x) = \sum_{n=1}^{\infty} \bar{\phi}_n \sin \lambda_n x,$$

where the coefficients $a_n(t)$ are unknown, but $\bar{f}_n(t)$ and $\bar{\phi}_n$ may be calculated from:

$$\bar{f}_n(t) = \frac{2}{L} \int_0^L f(x,t) \sin \lambda_n x \, dx, \quad \bar{\phi}_n = \frac{2}{L} \int_0^L \phi(x) \sin \lambda_n x \, dx.$$

Reducing the PDE into a set of infinitely many ODEs

Substitute the expansions into equations (18a) and (18d):

$$\sum_{n=1}^{\infty} a_n'(t) \sin \lambda_n x = \kappa \sum_{n=1}^{\infty} (-\lambda_n^2) a_n(t) \sin \lambda_n x + \sum_{n=1}^{\infty} \bar{f}_n(t) \sin \lambda_n x,$$

$$\sum_{n=1}^{\infty} a_n(0) \sin \lambda_n x = \sum_{n=1}^{\infty} \bar{\phi}_n \sin \lambda_n x,$$

and groups the summands

$$\sum_{n=1}^{\infty} \left(a_n'(t) + \kappa \lambda_n^2 a_n(t) - \bar{f}_n(t) \right) \sin \lambda_n x = 0,$$

$$\sum_{n=1}^{\infty} \left(a_n(0) - \bar{\phi}_n \right) \sin \lambda_n x = 0.$$

Since $\{\sin \lambda_n x\}_{n=1}^{\infty}$ is a basis, it follows that

$$a_n'(t) + \kappa \lambda_n^2 a_n(t) = \bar{f}_n(t), \quad a_n(0) = \bar{\phi}_n, \quad n = 1, 2, \ldots \tag{19}$$

Calculating the coefficients $a_n(t)$

Equations (19) express a set of infinitely many initial value problems for ODEs in the unknowns $a_n(t)$. which may be solved with the *integrating factor* method learned in a course in ODEs.

So we multiply through by the integrating factor $e^{\kappa\lambda_n^2 t}$ and combine terms:

$$\left(e^{\kappa\lambda_n^2 t}a_n(t)\right)' = e^{\kappa\lambda_n^2 t}\bar{f}_n(t),$$

and integrate:

$$\left(e^{\kappa\lambda_n^2 s}a_n(s)\right)\Big|_{s=0}^{s=t} = \int_0^t e^{\kappa\lambda_n^2 s}\bar{f}_n(s)\,ds.$$

but

$$\left(e^{\kappa\lambda_n^2 s}a_n(s)\right)\Big|_{s=0}^{s=t} = e^{\kappa\lambda_n^2 t}a_n(t) - a_n(0) = e^{\kappa\lambda_n^2 t}a_n(t) - \bar{\phi}_n,$$

therefore

$$e^{\kappa\lambda_n^2 t}a_n(t) - \bar{\phi}_n = \int_0^t e^{\kappa\lambda_n^2 s}\bar{f}_n(s)\,ds.$$

Calculation of $a_n(t)$: Conclusion

From the previous slide:

$$e^{\kappa\lambda_n^2 t} a_n(t) - \bar{\phi}_n = \int_0^t e^{\kappa\lambda_n^2 s} \bar{f}_n(s)\, ds.$$

therefore

$$a_n(t) = e^{-\kappa\lambda_n^2 t}\bar{\phi}_n + \int_0^t e^{-\kappa\lambda_n^2 (t-s)} \bar{f}_n(s)\, ds.$$

We conclude that the solution $u(x, t)$ of the initial/boundary value problem (18) is

$$u(x,t) = \sum_{n=1}^{\infty} \left(e^{-\kappa\lambda_n^2 t}\bar{\phi}_n + \int_0^t e^{-\kappa\lambda_n^2 (t-s)} \bar{f}_n(s)\, ds \right) \sin \lambda_n x.$$

A worked-out example

Let's solve the initial/boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2} + \sigma \sin \omega t, & 0 < x < L, \quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u(L,t) = 0 & t > 0 \\ u(x,0) = 0 & 0 < x < L \end{cases} \tag{20}$$

This corresponds to $f(x,t) = \sigma \sin \omega t$, and therefore

$$\begin{aligned} \bar{f}_n(t) &= \frac{2}{L}\int_0^L \sigma \sin \omega t \sin \lambda_n x \, dx = \frac{2\sigma \sin \omega t}{L} \int_0^L sin \lambda_n x \, dx \\ &= \frac{2\sigma \sin \omega t}{L} \cdot \frac{L}{\pi}\left(\frac{1-(-1)^n}{n}\right) = \frac{2\sigma}{\pi}\left(\frac{1-(-1)^n}{n}\right) \sin \omega t. \end{aligned}$$

A worked-out example (continued)

Then equations (19) on slide 33 take the form

$$a_n'(t) + \kappa\lambda_n^2 a_n(t) = \frac{2\sigma}{\pi}\left(\frac{1-(-1)^n}{n}\right)\sin\omega t, \quad a_n(0) = 0, \quad n = 1, 2, \ldots$$

which may be solved with an integrating factor as before, but in this case it is quicker to express the solution as the sum of homogeneous and particular solutions, as is done in a course in ODEs.

The homogeneous equation is $a_n'(t) + \kappa\lambda_n^2 a_n(t) = 0$, whence $a_n(t) = Ce^{-\kappa\lambda_n^2 t}$.

Look for a particular solution of the form $a_n(t) = A\cos\omega t + B\sin\omega t$.

$$\big(-A\omega\sin\omega t + B\omega\cos\omega t\big) + \kappa\lambda_n^2\big(A\cos\omega t + B\sin\omega t\big) = \frac{2\sigma}{\pi}\left(\frac{1-(-1)^n}{n}\right)\sin\omega t,$$

$$\big(-A\omega + B\kappa\lambda_n^2\big)\sin\omega t + \big(B\omega + A\kappa\lambda_n^2\big)\cos\omega t = \frac{2\sigma}{\pi}\left(\frac{1-(-1)^n}{n}\right)\sin\omega t,$$

$$\begin{cases} -A\omega + B\kappa\lambda_n^2 = \dfrac{2\sigma}{\pi}\left(\dfrac{1-(-1)^n}{n}\right) \equiv Q_n \\ A\kappa\lambda_n^2 + B\omega = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} A = -\dfrac{Q_n\omega}{\omega^2 + \kappa^2\lambda_n^4} \\ B = \dfrac{Q_n\kappa\lambda_n^2}{\omega^2 + \kappa^2\lambda_n^4} \end{cases}$$

A worked-out example (continued)

Particular solution:

$$a_n(t) = -\frac{Q_n\omega}{\omega^2+\kappa^2\lambda_n^4}\cos\omega t + \frac{Q_n\kappa\lambda_n^2}{\omega^2+\kappa^2\lambda_n^4}\sin\omega t, \qquad \text{where } Q_n = \frac{2\sigma}{\pi}\left(\frac{1-(-1)^n}{n}\right)$$

General solution:

$$a_n(t) = Ce^{-\kappa\lambda_n^2 t} - \frac{Q_n\omega}{\omega^2+\kappa^2\lambda_n^4}\cos\omega t + \frac{Q_n\kappa\lambda_n^2}{\omega^2+\kappa^2\lambda_n^4}\sin\omega t.$$

Initial condition:

$$a_n(0) = 0 \qquad \Rightarrow \qquad 0 = C - \frac{Q_n\omega}{\omega^2+\kappa^2\lambda_n^4} \qquad \Rightarrow \qquad C = \frac{Q_n\omega}{\omega^2+\kappa^2\lambda_n^4}$$

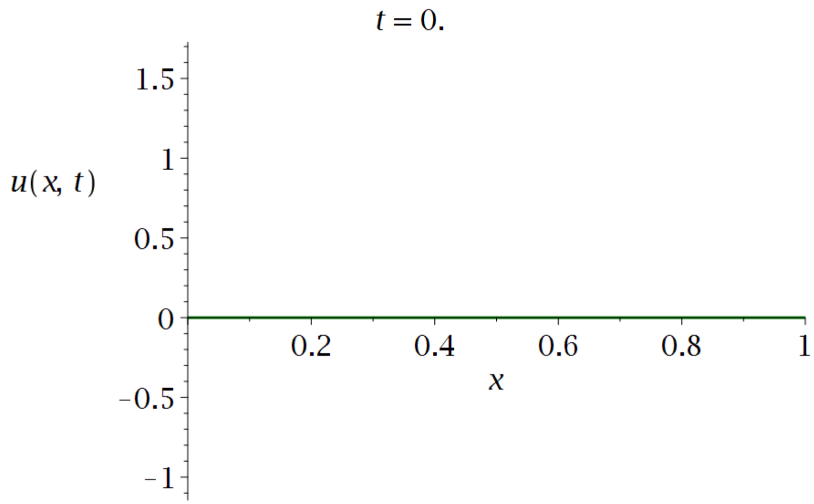
$$\begin{aligned} a_n(t) &= \frac{Q_n\omega}{\omega^2+\kappa^2\lambda_n^4}e^{-\kappa\lambda_n^2 t} - \frac{Q_n\omega}{\omega^2+\kappa^2\lambda_n^4}\cos\omega t + \frac{Q_n\kappa\lambda_n^2}{\omega^2+\kappa^2\lambda_n^4}\sin\omega t \\ &= \frac{Q_n}{\omega^2+\kappa^2\lambda_n^4}\left(\omega e^{-\kappa\lambda_n^2 t} - \omega\cos\omega t + \kappa\lambda_n^2\sin\omega t\right) \end{aligned}$$

A worked-out example (conclusion)

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} a_n(t) \sin \lambda_n x \\ &= \sum_{n=1}^{\infty} \frac{Q_n}{\omega^2 + \kappa^2\lambda_n^4}\left(\omega e^{-\kappa\lambda_n^2 t} - \omega\cos\omega t + \kappa\lambda_n^2 \sin\omega t\right)\sin\lambda_n x \\ &= \frac{2\sigma}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\left(\omega^2+\kappa^2\lambda_n^4\right)}\left(\omega e^{-\kappa\lambda_n^2 t} - \omega\cos\omega t + \kappa\lambda_n^2 \sin\omega t\right)\sin\lambda_n x \end{aligned}$$

An animation of $u(x,t)$ evaluated as $\sum_{n=1}^{10}$ (five terms)

Note the transient behavior.



Prescribed temperature at the boundary

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + f(x,t)$$
$$u(0,t) = \alpha(t)$$
$$u(L,t) = \beta(t)$$
$$u(x,0) = \phi(x)$$

Prescribed temperature at the boundary

Up to now all of our boundary conditions have been of the form $u = 0$ (zero temperature) of $\frac{\partial u}{\partial x} = 0$ (zero flux). We sought solutions in the form $u(x,t) = \sum_{n=1}^{\infty} a_n(t) X_n(t)$, where $X_n(x)$ were selected expressly to satisfy those zero boundary conditions. As a result, the sum satisfies the those zero boundary conditions and we are done.

But what if the boundary conditions are other than zero? There is no use in changing the X_ns to satisfy those boundary conditions because even if each X_n satisfies a nonzero boundary condition, it does not follow that the sum $\sum_{n=1}^{\infty} a_n(t) X_n(t)$ also satisfies that boundary condition. (This clearly shows that a zero boundary condition is something very special!)

Here is a bright idea: Split $u(x,t)$ into a sum $u(x,t) = v(x,t) + \xi(x,t)$. For $\xi(x,t)$ pick a function, any function, that satisfies the problem's boundary conditions. Since $u(x,t)$ also satisfies those boundary conditions, it follows that $v(x,t)$ satisfies the corresponding zero boundary conditions!

In the PDE, replace $u(x,t)$ by $v(x,t) + \xi(x,t)$. This will yield a PDE involving v. But v satisfies zero boundary conditions, and therefore we may calculate it through our previous techniques. Once we have v, we add ξ to it to obtain u.

Temperature prescribed at the boundaries

Heat condition in a rod with prescribed temperatures at the ends:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2} + f(x,t) & 0 < x < L, \quad t > 0 \\ u(0,t) = \alpha(t) & t > 0 \\ u(L,t) = \beta(t) & t > 0 \\ u(x,0) = \phi(x) & 0 < x < L \end{cases} \tag{21}$$

For the function $\xi(x,t)$ we pick

$$\xi(x,t) = \left(1 - \frac{x}{L}\right)\alpha(t) + \frac{x}{L}\beta(t). \tag{22}$$

and note that $\xi(0,t) = \alpha(t)$, $\xi(L,t) = \beta(t)$.

Then substitute

$$u(x,t) = v(x,t) + \left(1 - \frac{x}{L}\right)\alpha(t) + \frac{x}{L}\beta(t)$$

into (21).

Equation with homogeneous boundary conditions

The v equation:

$$\begin{cases} \dfrac{\partial v}{\partial t} + \left(1 - \dfrac{x}{L}\right)\alpha'(t) + \dfrac{x}{L}\beta'(t) = \kappa \dfrac{\partial^2 v}{\partial x^2} + f(x,t) \\ v(0,t) = 0 \\ v(L,t) = 0 \\ v(x,0) + \left(1 + \dfrac{x}{L}\right)\alpha(0) + \dfrac{x}{L}\beta(0) = \phi(x) \end{cases}$$

Rearrange:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \kappa \dfrac{\partial^2 v}{\partial x^2} + f(x,t) - \left(1 - \dfrac{x}{L}\right)\alpha'(t) - \dfrac{x}{L}\beta'(t) \\ v(0,t) = 0 \\ v(L,t) = 0 \\ v(x,0) = \phi(x) - \left(1 + \dfrac{x}{L}\right)\alpha(0) - \dfrac{x}{L}\beta(0) \end{cases} \tag{23}$$

So going from u equations in (21) to the v equations in (23) amounts to modifying the heat source function f and the initial condition ϕ.

The heat equation with oscillating temperature at the boundary

Oscillatory temperature imposed at the right-hand end:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2} & 0 < x < L, \quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u(L,t) = \sigma \sin \omega t & t > 0 \\ u(x,0) = 0 & 0 < x < L \end{cases} \tag{24}$$

This is a special case of the problem (21) on slide 42. The ξ function in (22) is $\xi(x,t) = \frac{x}{L}\sigma \sin \omega t$, and therefore $u(x,t) = v(x,t) + \frac{x}{L}\sigma \sin \omega t$ and then problem (23) takes the form

$$\begin{cases} \dfrac{\partial v}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2} - \dfrac{x}{L}\sigma\omega \cos \omega t & 0 < x < L, \quad t > 0 \\ v(0,t) = 0 & t > 0 \\ v(L,t) = 0 & t > 0 \\ v(x,0) = 0 & 0 < x < L \end{cases} \tag{25}$$

Solution continued

The initial/boundary value problem (25) is quite similar to the system (20) on slide 36. Solving it is left to you as homework. When you work out the details, you will find that:

$$v(x,t) = \frac{2\sigma\omega}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n(\omega^2 + \kappa^2\lambda_n^4)} \left[-\kappa\lambda_n^2 e^{-\kappa\lambda_n^2 t} + \kappa\lambda_n^2 \cos\omega t + \omega \sin\omega t \right] \sin\lambda_n x.$$

and therefore

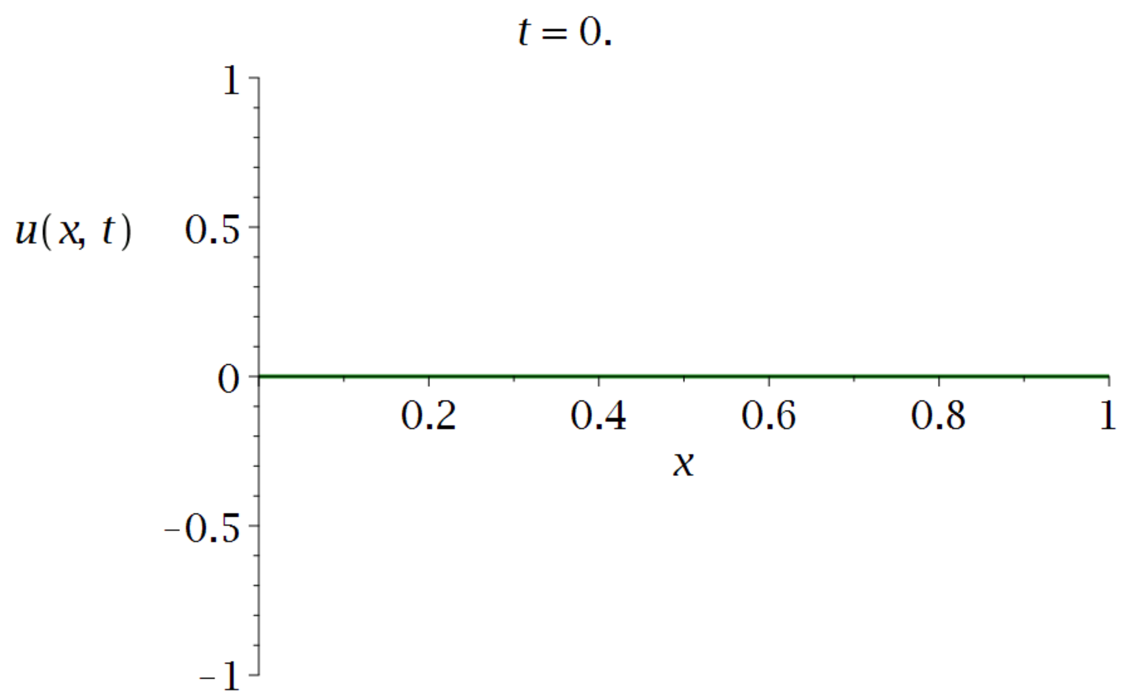
$$\begin{aligned} u(x,t) &= \frac{x}{L}\sigma\sin\omega t \\ &\quad + \frac{2\sigma\omega}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n(\omega^2 + \kappa^2\lambda_n^4)} \left[-\kappa\lambda_n^2 e^{-\kappa\lambda_n^2 t} + \kappa\lambda_n^2 \cos\omega t + \omega \sin\omega t \right] \sin\lambda_n x. \end{aligned}$$

Animation of the solution

We animate the solution with the parameter values

$$L = 1, \quad \omega = 1, \quad \sigma = 1, \quad \kappa = 0.02,$$

and truncate the series at the tenth term.



A compact notation for partial derivatives

$$u_t = \frac{\partial u}{\partial t} \qquad u_x = \frac{\partial u}{\partial x} \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

$$u_x(L,t) = \left.\frac{\partial u}{\partial x}\right|_{x=L}$$

A compact notation for partial derivatives

Initial/boundary value problem in the expanded notation:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \kappa \dfrac{\partial^2 u}{\partial x^2} + f(x,t) & 0 < x < L, \quad t > 0 \\ u(0,t) = \alpha(t) & t > 0 \\ -k \dfrac{\partial u}{\partial x}\Big|_{x=L} = \gamma(t) & t > 0 \\ u(x,0) = \phi(x) & 0 < x < L \end{cases}$$

The same problem in compact notation:

$$\begin{cases} u_t = \kappa u_{xx} + f(x,t) & 0 < x < L, \quad t > 0 \\ u(0,t) = \alpha(t) & t > 0 \\ -k u_x(L,t) = \gamma(t) & t > 0 \\ u(x,0) = \phi(x) & 0 < x < L \end{cases}$$

Handling inhomogeneous boundary conditions

$$u_t = \kappa u_{xx} + f(x,t)$$
$$\alpha_1(t)u(0,t) + \alpha_2(t)u_x(0,t) = \alpha(t)$$
$$\beta_1(t)u(L,t) + \beta_2(t)u_x(L,t) = \beta(t)$$
$$u(x,0) = \phi(x)$$

Handling inhomogeneous boundary conditions

Initial/boundary value problem with inhomogeneous boundary conditions:

$$u_t = \kappa u_{xx} + f(x,t) \tag{26a}$$

$$\alpha_1(t)u(0,t) + \alpha_2(t)u_x(0,t) = \alpha(t) \tag{26b}$$

$$\beta_1(t)u(L,t) + \beta_2(t)u_x(L,t) = \beta(t) \tag{26c}$$

$$u(x,0) = \phi(x) \tag{26d}$$

Introduce a new unknown $v(x,t)$ through

$$u(x,t) = v(x,t) + c_1(t) + c_2(t)x \tag{27}$$

and eliminate u in favor of v in the problem. Then, pick $c_1(t)$ and $c_2(t)$ so as to eliminate the inhomogeneous terms $\alpha(t)$ and $\beta(t)$ in (26b) and (26c).

Eliminating the inhomogeneous terms

Substituting $u(x,t)$ from (27) into (26b) and (26c) we get

$$\alpha_1\big(v(0,t)+c_1\big)+\alpha_2\big(v_x(0,t)+c_2\big)=\alpha,$$
$$\beta_1\big(v(L,t)+c_1+c_2L\big)+\beta_2\big(v_x(L,t)+c_2\big)\beta_2=\beta.$$

whence

$$\alpha_1 v(0,t)+\alpha_2 v_x(0,t)=\alpha-\alpha_1c_1-\alpha_2c_2 \tag{28a}$$
$$\beta_1 v(L,t)+\beta_2 v_x(L,t)=\beta-\beta_1c_1-(\beta_1L+\beta_2)c_2 \tag{28b}$$

To get homogeneous boundary conditions on v, set the right-hand sides to zero:

$$\alpha_1c_1+\alpha_2c_2=\alpha, \tag{29a}$$
$$\beta_1c_1+(\beta_1L+\beta_2)c_2=\beta \tag{29b}$$

and solve the system for the unknowns c_1 and c_2.

Eliminating the inhomogeneous terms – continued

$$c_1 = \frac{(\beta_1 L + \beta_2)\alpha - \alpha_2\beta}{\alpha_1(\beta_1 L + \beta_2) - \alpha_2\beta_1}, \quad c_2 = \frac{\alpha_1\beta - \beta_1\alpha}{\alpha_1(\beta_1 L + \beta_2) - \alpha_2\beta_1}. \tag{30}$$

Observation: Since α, α_1, α_2, β, β_1, β_2 are generally functions of time, c_1 and c_2 calculated above are also functions of time. Occasionally we will write $c_1(t)$ and $c_2(t)$ to stress that.

In view of (29), the boundary conditions (28) on v reduce to

$$\alpha_1 v(0,t) + \alpha_2 v_x(0,t) = 0, \tag{31a}$$

$$\beta_1 v(L,t) + \beta_2 v_x(L,t) = 0 \tag{31b}$$

which are homogeneous by design.

To obtain a PDE on v, substitute $u(x,t)$ from (27) into (26a) and we get $v_t + c_1'(t) + c_2'(t)x = \kappa v_{xx} + f(x,t)$, that is,

$$v_t = \kappa v_{xx} + f(x,t) - c_1'(t) - c_2'(t)x \tag{32}$$

Eliminating the inhomogeneous terms – continued

To obtain the initial condition on v, substitute $u(x,t)$ from (27) into (26d). We get $v(x,0) + c_1(0) + c_2(0)x = \phi(x)$, that is

$$v(x,0) = \phi(x) - c_1(0) - c_2(0)x. \tag{33}$$

In summary, the change of variables (27) with c_1 and c_2 selected as in (30), converts the inhomogeneous boundary conditions in (26) into homogeneous boundary conditions in the modified equation:

$$v_t = \kappa v_{xx} + f(x,t) - c_1'(t) - c_2'(t)x \tag{34a}$$

$$\alpha_1(t)v(0,t) + \alpha_2(t)v_x(0,t) = 0 \tag{34b}$$

$$\beta_1(t)v(L,t) + \beta_2(t)v_x(L,t) = 0 \tag{34c}$$

$$v(x,0) = \phi(x) - c_1(0) - c_2(0)x. \tag{34d}$$

Observation: Going from (26) to (34) amounts to (a) zeroing the inhomogeneous parts of the boundary conditions; (b) replacing $f(x,t)$ by $f(x,t) - c_1'(t) - c_2'(t)x$; and (c) replacing $\phi(x)$ by $\phi(x) - c_1(0) - c_2(0)x$.

Special case: Dirichlet boundary conditions

The initial/boundary value problem

$$u_t = \kappa u_{xx} + f(x,t) \tag{35a}$$

$$u(0,t) = \alpha(t) \tag{35b}$$

$$u(L,t) = \beta(t) \tag{35c}$$

$$u(x,0) = \phi(x) \tag{35d}$$

is a special case of (26) with $\alpha_1(t) = 1, \alpha_2(t) = 0, \beta_1(t) = 1, \beta_2(t) = 0$. From (30) we get $c_1 = \alpha(t), c_2 = (\beta(t) - \alpha(t))/L$ and then (27) and (34) reduce to

$$u(x,t) = v(x,t) + \alpha(t) + \frac{\beta(t) - \alpha(t)}{L}x \tag{36}$$

and

$$v_t = \kappa v_{xx} + f(x,t) - \alpha'(t) - \frac{\beta'(t) - \alpha'(t)}{L}x \tag{37a}$$

$$v(0,t) = 0 \tag{37b}$$

$$v(L,t) = 0 \tag{37c}$$

$$v(x,0) = \phi(x) - \alpha(0) - \frac{\beta(0) - \alpha(0)}{L}x \tag{37d}$$

Special case: Dirichlet and Neumann boundary conditions

The initial/boundary value problem

$$\begin{aligned} u_t &= \kappa u_{xx} + f(x,t) \\ u(0,t) &= \alpha(t) \\ u_x(L,t) &= \beta(t) \\ u(x,0) &= \phi(x) \end{aligned}$$

is a special case of (26) with $\alpha_1(t) = 1, \alpha_2(t) = 0, \beta_1(t) = 0, \beta_2(t) = 1.$ From (30) we get $c_1 = \alpha(t), c_2 = \beta(t)$ and then (27) and (34) reduce to

$$u(x,t) = v(x,t) + \alpha(t) + \beta(t)x$$

and

$$\begin{aligned} v_t &= \kappa v_{xx} + f(x,t) - \alpha'(t) - \beta'(t)x \\ v(0,t) &= 0 \\ v(L,t) &= 0 \\ v(x,0) &= \phi(x) - \alpha(0) - \beta(0)x \end{aligned}$$

Special case: Neumann and Robin boundary conditions

The initial/boundary value problem

$$u_t = \kappa u_{xx} + f(x,t)$$
$$u_x(0,t) = \alpha(t)$$
$$\beta_1(t)u(L,t) + \beta_2(t)u_x(L,t) = \beta(t)$$
$$u(x,0) = \phi(x)$$

is a special case of (26) with $\alpha_1(t) = 0, \alpha_2(t) = 1$. From (30) we get $c_1(t) = \frac{\beta(t) - \big(\beta_1(t)L + \beta_2(t)\big)\alpha(t)}{\beta_1(t)}, c_2(t) = \alpha(t)$ and then (27) and (34) reduce to

$$u(x,t) = v(x,t) + c_1(t) + c_2(t)x$$

and

$$v_t = \kappa v_{xx} + f(x,t) - c_1'(t) - c_2'(t)x$$
$$v(0,t) = 0$$
$$v(L,t) = 0$$
$$v(x,0) = \phi(x) - c_1(0) - c_2(0)x$$

Exceptional cases

The change in (27) from $u(x,t)$ to the $v(x,t)$ works for reducing inhomogeneous boundary conditions to homogeneous ones in most cases, but not always. That's because the equations in (30) fail to provide values for c_1 and c_2 when their denominators vanish. Once such instance occurs when Neumann boundary conditions are specified at both ends:

$$u_x(0,t) = \alpha(t), \quad u_x(L,t) = \beta(t). \tag{38}$$

That's a special case of (26b) and (26c) with

$$\alpha_1(t) = 0, \quad \alpha_2(t) = 1, \quad \beta_1(t) = 0, \quad \beta_2(t) = 1.$$

Calculating c_1 and c_2 in this case fails since the denominators in (30) vanish.

A little experimentation shows that we can make things work by replacing the change of variables (27) by

$$u(x,t) = v(x,t) + c_1(t)x + c_2(t)x^2. \tag{39}$$

Determining the proper choices for these $c_1(t)$ and $c_2(t)$ is left as a homework problem.

Newton's Law of cooling

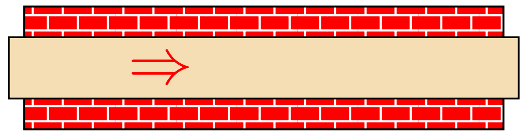
$$-ku_x(L,x) = \gamma\Big(u(L,t) - u_\infty\Big)$$

or equivalently

$$\gamma u(L,t) + ku_x(L,x) = \gamma u_\infty$$

Newton's Law of cooling – Example 1

Rod with prescribed temperature at the left, Newton's cooling on the right.

$u(0,t) = \alpha(t)$  $-ku_x(L,t) = \gamma\big(u(L,t) - u_\infty\big)$

$$\begin{cases} u_t = \kappa u_{xx} + f(x,t) & 0 < x < L, \quad t > 0 \\ u(0,t) = \alpha(t) & t > 0 \\ \gamma u(L,t) + k u_x(L,x) = \gamma u_\infty & t > 0 \\ u(x,0) = \phi(x) & 0 < x < L \end{cases} \tag{40}$$

The initial/boundary value problem (40) matches (26) on slide 50 with $\alpha_1 = 1, \alpha_2 = 0, \beta_1 = \gamma, \beta_2 = k, \beta = \gamma u_\infty$. Thus, from (30) we obtain

$$c_1 = \alpha(t), \quad c_2 = \frac{\gamma\big(u_\infty - \alpha(t)\big)}{\gamma L + k}$$

and therefore (27) takes the form

$$u(x,t) = v(x,t) + \alpha(t) + \frac{\gamma\big(u_\infty - \alpha(t)\big)}{\gamma L + k} x. \tag{41}$$

Newton's Law of cooling – Example 1 (continued)

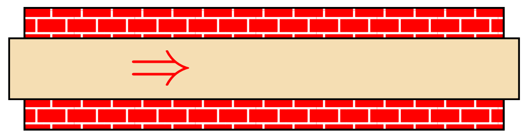
Plugging (41) into (40), and having the *Observation* on slide 53 in mind, we arrive at

$$\begin{cases} v_t = \kappa v_{xx} + f(x,t) - \dfrac{\gamma(L-x)+k}{\gamma L + k}\alpha'(t) & 0 < x < L, \quad t > 0 \\ v(0,t) = 0 & t > 0 \\ \gamma v(L,t) + k v_x(L,t) = 0 & t > 0 \\ v(x,0) = \phi(x) - \left[\alpha(0) + \dfrac{\gamma\big(u_\infty - \alpha(0)\big)}{\gamma L + k}x\right] & 0 < x < L \end{cases} \tag{42}$$

Now that we have homogeneous boundary conditions, we may solve for v through eigenfunction expansion as usual, and then obtain u from (41).

Newton's Law of cooling – Example 2

Heat conduction in a rod with forced flux at the left, Newton's cooling on the right.

$-ku_x(0,t) = \alpha(t)$  $-ku_x(L,t) = \gamma\big(u(L,t) - u_\infty\big)$

$$\begin{cases} u_t = \kappa u_{xx} + f(x,t) & 0 < x < L, \quad t > 0 \\ -ku_x(0,t) = \alpha(t) & t > 0 \\ -ku_x(L,t) = \gamma\big(u(L,t) - u_\infty\big) & t > 0 \\ u(x,0) = \phi(x) & 0 < x < L \end{cases} \tag{43}$$

Rearrange the terms in the right boundary condition as $\gamma u(L,t) + ku_x(L,t) = \gamma u_\infty$. Then (43) matches (26) on slide 50 with $\alpha_1 = 0, \alpha_2 = -k, \beta_1 = \gamma, \beta_2 = k, \beta = \gamma u_\infty$. Thus, from (30) we obtain

$$c_1 = u_\infty + \frac{\alpha(t)}{\gamma} + \frac{L\alpha(t)}{k}, \quad c_2 = -\frac{\alpha(t)}{k}$$

and therefore (27) takes the form

$$u(x,t) = v(x,t) + \frac{\alpha(t)}{k}(L - x) + \frac{\alpha(t)}{\gamma} + u_\infty. \tag{44}$$

Newton's Law of cooling – Example 2 (continued)

Plugging (44) into (43), and having the *Observation* on slide 53 in mind, we arrive at

$$\begin{cases} v_t = \kappa v_{xx} + f(x,t) - \left[\dfrac{L-x}{k} + \dfrac{1}{\gamma}\right]\alpha'(t) & 0 < x < L, \quad t > 0 \\ v_x(0,t) = 0 & t > 0 \\ \gamma v(L,t) + k v_x(L,t) = 0 & t > 0 \\ v(x,0) = \phi(x) - \left[\dfrac{L-x}{k} + \dfrac{1}{\gamma}\right]\alpha(0) - u_\infty & 0 < x < L \end{cases} \tag{45}$$

Now that we have homogeneous boundary conditions, we may solve for v through eigenfunction expansion as usual, and then obtain u from (44).

The eigenfunctions of problem (42)

Here we give the details of solving problem (42). The solution of problem (45) is along similar lines and is left as a homework problem.

We begin by examining the homogeneous PDE corresponding to (42), and the associated boundary conditions:

$$\begin{cases} v_t = \kappa v_{xx} & 0 < x < L, \quad t > 0 \\ v(0,t) = 0 & t > 0 \\ \gamma v(L,t) + k v_x(L,t) = 0 & t > 0 \end{cases} \tag{46}$$

We look for a separable solution of the form $v(x,t) = X(x)T(t)$. We get:

$$T'(t)X(x) = \kappa T(t)X''(x), \quad X(0)T(t) = 0, \quad \gamma X(L)T(t) + kX'(L)T(t) = 0$$

which simplifies to

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)}, \quad X(0) = 0, \quad hX(L) + X'(L) = 0 \tag{47}$$

where $h = \gamma/k$.

The eigenfunctions of problem (42) – slide 2

The first of equations (47) implies that

$$\frac{T'(t)}{\kappa T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2$$

for some constant λ. Therefore $T'(t) + \kappa\lambda^2 T(t = 0$ and

$$X''(x) + \lambda^2 X(x) = 0, \quad X(0) = 0, \quad hX(L) + X'(L) = 0, \tag{48}$$

whence

$$T(t) = Ce^{-\kappa\lambda^2 t}, \quad X(x) = A\sin\lambda x + B\cos\lambda x.$$

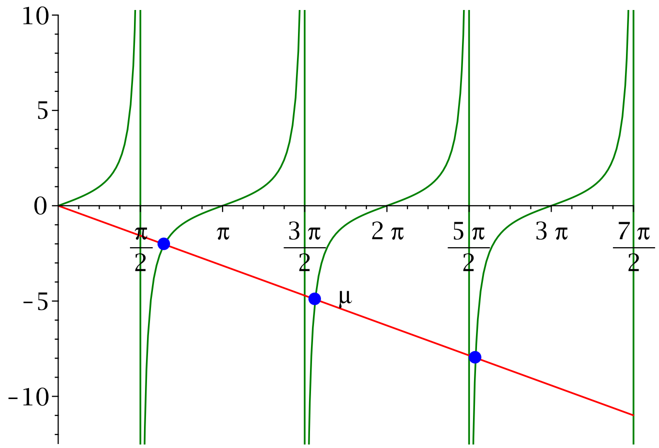
The boundary condition $X(0) = 0$ implies that $B = 0$. Therefore $X(x) = A\sin\lambda x$. The boundary condition at $x = L$ says that $hA\sin\lambda L + \lambda A\cos\lambda L = 0$, that is, $\tan\lambda L = -\frac{1}{h}\lambda$. We rewrite this as $\tan\lambda L = -\frac{1}{hL}\lambda L$ and then let $\mu = \lambda L$ and arrive at $\tan\mu = -\frac{1}{hL}\mu$.

Conclusion: Need to solve the transcendental equation

$$\tan\mu = -\frac{1}{hL}\mu \tag{49}$$

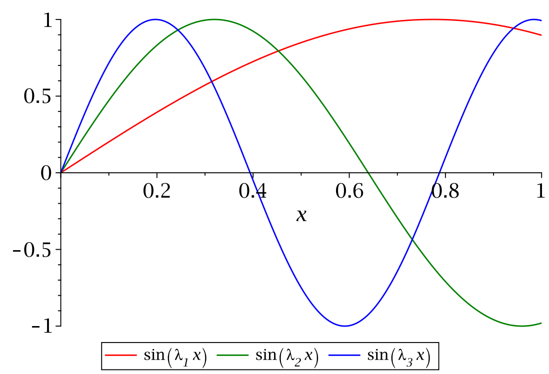
numerically to determine μ. Then $\lambda = \mu/L$.

The eigenfunctions of problem (42) – slide 3



The graphs of $\tan \mu$ and $-\frac{1}{hL}\mu$ plotted together. We have taken $L = 1$, $h = 1$ for the purposes of this illustration. The intersection of the graphs mark the solutions of (49). The first five positive roots are $\mu = 2.0288, 4.9132, 7.9787, 11.0855, 14.2074$.

We write μ_n, $n = 1, 2, \ldots$ for the roots of the equation (49). The corresponding values of λ are $\lambda_n = \mu_n/L$, and the solution of (48) are $X_n(x) = \sin \lambda_n x$.



The eigenfunctions of problem (42) – slide 4

The Sturm–Liouville Theory. The problem (48) that we just solved, is a special case of what is know as the *Sturm–Liouville problem*:

$$\begin{cases} \Big(p(x)X'(x)\Big)' + q(x)X(x) + \lambda w(x)X(x) = 0, \\ \alpha_1 X(a) + \alpha_2 X'(a) = 0, \\ \beta_1 X(b) + \beta_2 X'(b) = 0. \end{cases} \tag{50}$$

The Sturm–Liouville Theory, dating back to 1837, states that under certain conditions (see Wikipedia for the precise requirements) the boundary value problem (50) has infinitely many *eigenvalues* λ_n which may be ordered as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty,$$

and corresponding to each λ_n there is a unique (up to a multiplicative constant) nonzero *eigenfunctions* $X_n(x)$. The eigenfunctions, after appropriate scaling, satisfy the *orthogonality condition*

$$\int_a^b w(x)X_m(x)X_n(x)\,dx = \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n \end{cases}$$

Any function $\phi(x)$ on the interval (a, b) may be expressed as the infinite sum $\phi(x) = \sum_{n=1}^{\infty} c_n X_n(x)$, where $c_n = \int_a^b w(x)\phi(x)X_n(x)\,dx$.

The Fourier sine series in 2D

The Fourier sine series in 2D

In Slide 21 we learned how to expand a function $\phi(x)$ into the Fourier sine series. Here we generalize the idea to functions of two variables. Specifically, let us consider a function $\phi(x, y)$ on the square $(O, L) \times (0, L)$. For any fixed value of y, this is a function of the single variable x, and therefore we may apply the formulas (12), (13), and (14) on Slide 21 to obtain:

$$\phi(x, y) = \sum_{n=1}^{\infty} b_n(y) \sin \lambda_n x, \tag{51}$$

where

$$b_n(y) = \frac{2}{L} \int_0^L \phi(x, y) \sin \lambda_n x \, dx, \quad \text{and} \quad \lambda_n = \frac{n\pi}{L}. \tag{52}$$

The Fourier sine series in 2D – continued

The function $b_n(y)$ itself may be expanded into a Fourier sine series, as in

$$b_n(y) = \sum_{m=1}^{\infty} a_{mn} \sin \lambda_m y \tag{53}$$

where

$$a_{mn} = \frac{2}{L} \int_0^L b_n(y) \sin \lambda_m y \, dy$$

Substituting for $b_n(y)$ from (52), this becomes

$$a_{mn} = \frac{4}{L^2} \int_0^L \int_0^L \phi(x, y) \sin \lambda_n x \sin \lambda_m y \, dx \, dy.$$

Furthermore, substituting $b_n(y)$ from (53) into (51) we see that

$$\phi(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin \lambda_n x \sin \lambda_m y.$$

The Fourier sine series in 2D – summary

To summarize the calculations of the previous two slides: A function $\phi(x, y)$ on the square $(0, L) \times (0, L)$ may be expanded into two-dimensional Fourier sine series as

$$\phi(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin \lambda_n x \sin \lambda_m y. \tag{54}$$

where

$$a_{mn} = \frac{4}{L^2} \int_0^L \int_0^L \phi(x, y) \sin \lambda_n x \sin \lambda_m y \, dx \, dy. \tag{55}$$

These are the two-dimensional versions of the formulas on Slide 21.

Heat conduction in two dimensions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y) = 0$$

Heat conduction in two dimensions

The equation of heat conduction $\partial u/\partial t = \kappa \partial^2 u/\partial x^2 + f(x,t)$ generalizes to two spatial dimensions as

$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x,y,t),$$

where the temperature u is a function of three variables, $u = u(x,y,t)$. When the heat generation term $f(x,y,t)$ and the boundary conditions are independent of time t, the temperature stabilizes to the *steady state* distribution, $u(x,y)$, and therefore $\partial u/\partial t$ drops out and we are left with

$$\kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + f(x,y) = 0.$$

Dividing through κ and renaming $\frac{1}{\kappa} f(x,y)$ as $f(x,y)$, we arrive at:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y) = 0. \qquad \textit{(Poisson's equation)}$$

Solving the heat equation in 2D

Let us look at the heat conduction problem in the square $S = (0, L) \times (0, L)$ with zero boundary conditions along the edges:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y) = 0 \qquad \text{in } S, \tag{56a}$$

$$u(x, 0) = u(x, L) = u(0, y) = u(L, y) = 0 \qquad \text{for all } 0 < x < L,\ 0 < y < L. \tag{56b}$$

To solve that boundary value problem, we expand the known function $f(x, y)$ and the unknown function $u(x, y)$ into Fourier sine series according to (54)

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{mn} \sin \lambda_n x \sin \lambda_m y, \quad f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{mn} \sin \lambda_n x \sin \lambda_m y,$$

The coefficients c_{mn} are calculated according to (55):

$$c_{mn} = \frac{4}{L^2} \int_0^L \int_0^L f(x, y) \sin \lambda_n x \sin \lambda_m y \, dx \, dy, \tag{57}$$

but the coefficients a_{mn} are unknown and are to be determined.

Solving the heat equation in 2D – continued

To determine the coefficients a_{mn} in the expansion of $u(x, y)$, we calculate the partial derivatives of that expansion, as in

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\lambda_n^2 a_{mn} \sin\lambda_n x \sin\lambda_m y,$$

$$\frac{\partial^2 u}{\partial y^2} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\lambda_m^2 a_{mn} \sin\lambda_n x \sin\lambda_m y,$$

and substitute these, along with the series expansion of $f(x, y)$, into the PDE (56a). We get

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\lambda_n^2 a_{mn} \sin\lambda_n x \sin\lambda_m y + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} -\lambda_m^2 a_{mn} \sin\lambda_n x \sin\lambda_m y + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{mn} \sin\lambda_n x \sin\lambda_m y = 0.$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \Big[-(\lambda_n^2 + \lambda_m^2) a_{mn} + c_{mn}\Big] \sin\lambda_n x \sin\lambda_m y = 0.$$

Solving the heat equation in 2D – continued

It follows that $-(\lambda_n^2 + \lambda_m^2) a_{mn} + c_{mn} = 0$, and therefore

$$a_{mn} = \frac{c_{mn}}{\lambda_n^2 + \lambda_m^2}.$$

Consequently

$$u(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{c_{mn}}{\lambda_n^2 + \lambda_m^2} \sin \lambda_n x \sin \lambda_m y, \tag{58}$$

where the coefficients c_{mn} are given in (57).

A worked out example

Let us calculate the the temperature $u(x, y)$ in problem (56) on Slide 73 where the heat generation is uniform $f(x, y) = 1$ throughout the square.

We begin with calculating the coefficients c_{mn} of the Fourier expansion of $f(x, y)$ through the formula (55):

$$
\begin{aligned}
c_{mn} &= \frac{4}{L^2}\int_0^L\int_0^L 1 \times \sin\lambda_n x \sin\lambda_m y\, dx\, dy \\
&= \frac{4}{L^2}\left(\int_0^L \sin\lambda_n x\, dx\right)\left(\int_0^L \sin\lambda_m y\, dy\right) \\
&= \frac{4}{L^2}\left(-\frac{1}{\lambda_n}\cos\lambda_n x\Big|_0^L\right)\left(-\frac{1}{\lambda_m}\cos\lambda_n y\Big|_0^L\right) \\
&= \frac{4}{\lambda_m\lambda_n L^2}\Big(-\cos\lambda_n L + 1\Big)\Big(-\cos\lambda_m L + 1\Big).
\end{aligned}
$$

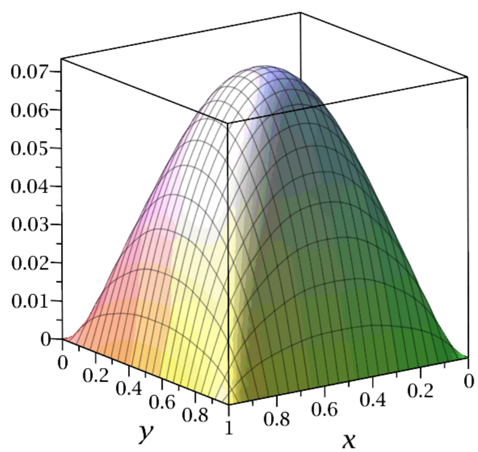
A worked out example – continued

Then according to (58) we get

$$u(x,y) = \frac{4}{L^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(1-\cos\lambda_m L)(1-\cos\lambda_n L)}{\lambda_m\lambda_n(\lambda_n^2+\lambda_m^2)}\sin\lambda_n x\sin\lambda_m y.$$

Considering that $\lambda_n = n\pi/L$ and that $\cos\lambda_n L = \cos n\pi = (-1)^n$, this takes the form

$$u(x,y) = \frac{4L^2}{\pi^4}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left(1-(-1)^m\right)\left(1-(-1)^n\right)}{mn(m^2+n^2)}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L}.$$



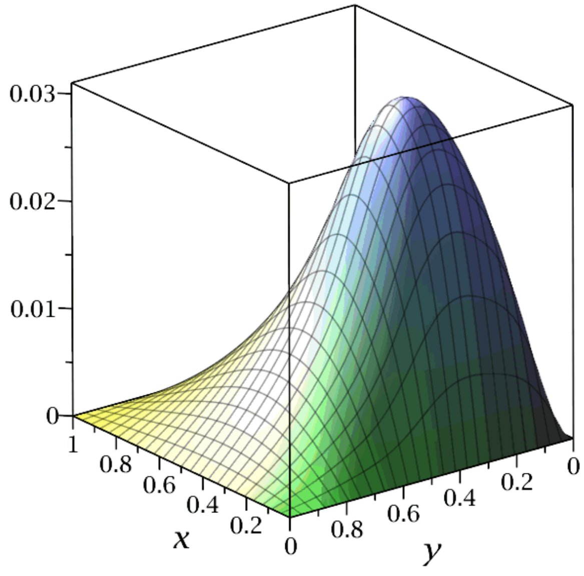
Movie made with $L = 1$ and ∞ set to 8

Exercise

Calculate the solution $u(x, y)$ of the steady-state heat conduction problem (56) on Slide 73, assuming that heat is generated only in the lower-left quarter of the domain, that is,

$$f(x,y) = \begin{cases} 1 & \text{if } 0 < x < L/2 \text{ and } 0 < y < L/2, \\ 0 & \text{otherwise,} \end{cases}$$

Here is what the solution looks like:

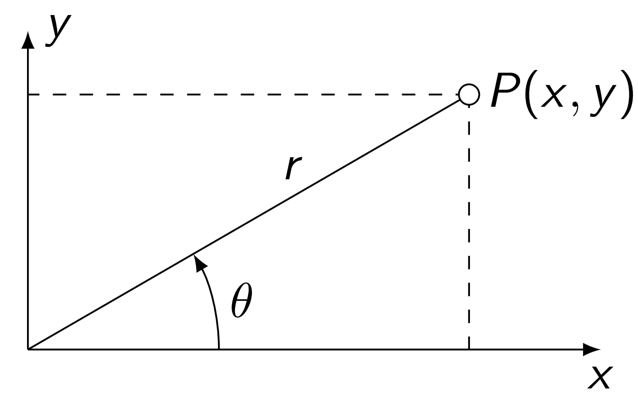


Movie made with $L = 1$ and ∞ set to 8

From Cartesian to polar

Change of coordinates from Cartesian to polar

The point P at (x, y) in Cartesian coordinates is represented as (r, θ) in polar coordinates where r, called the *radial coordinate* or the *radius*, is the distance of the point from the origin O, and θ, called the *angular coordinate* or the *polar coordinate*, is the rotation angle, measured counterclockwise, of the ray OP away from the positive x axis



$$x = r\cos\theta, \quad y = r\sin\theta$$

$$x = r\cos\theta \quad \overset{\partial/\partial x}{\Longrightarrow} \quad 1 = \frac{\partial r}{\partial x}\cos\theta - r\sin\theta\frac{\partial\theta}{\partial x}$$

$$y = r\sin\theta \quad \overset{\partial/\partial x}{\Longrightarrow} \quad 0 = \frac{\partial r}{\partial x}\sin\theta + r\cos\theta\frac{\partial\theta}{\partial x}$$

Solve for $\partial r \partial x$ and $\partial\theta\partial x$:

$$\frac{\partial r}{\partial x} = \cos\theta, \quad \frac{\partial\theta}{\partial x} = -\frac{1}{r}\sin\theta, \quad \text{and similarly} \quad \frac{\partial r}{\partial y} = \sin\theta, \quad \frac{\partial\theta}{\partial y} = \frac{1}{r}\cos\theta.$$

First derivatives in polar coordinates

A function $u(x, y)$ expressed in the Cartesian coordinates may be evaluated at the corresponding polar coordinates as $u(r\cos\theta, r\sin\theta)$. The result of the evaluation is a function $U(r, \theta)$, where

$$u(x, y) = u(r\cos\theta, r\sin\theta) = U(r, \theta).$$

Then by the *chain rule*

$$\frac{\partial u}{\partial x} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial U}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial U}{\partial \theta}\sin\theta$$

$$\frac{\partial u}{\partial y} = \frac{\partial U}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial U}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial U}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial U}{\partial \theta}\cos\theta$$

Coming up next...the calculation of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ and (homework) $\frac{\partial^2 u}{\partial x \partial y}$.

The derivative u_{xx} in polar coordinates

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial U}{\partial \theta}\sin\theta\right) \\
&= \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial U}{\partial \theta}\sin\theta\right)\cos\theta - \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial U}{\partial \theta}\sin\theta\right)\sin\theta \\
&= \left(\frac{\partial^2 U}{\partial r^2}\cos\theta + \frac{1}{r^2}\frac{\partial U}{\partial \theta}\sin\theta - \frac{1}{r}\frac{\partial^2 U}{\partial r\partial \theta}\sin\theta\right)\cos\theta \\
&\qquad - \frac{1}{r}\left(\frac{\partial^2 U}{\partial r\partial \theta}\cos\theta - \frac{\partial U}{\partial r}\sin\theta - \frac{1}{r}\frac{\partial^2 U}{\partial \theta^2}\sin\theta - \frac{1}{r}\frac{\partial U}{\partial \theta}\cos\theta\right)\sin\theta \\
&= \frac{\partial^2 U}{\partial r^2}\cos^2\theta - \frac{2}{r}\frac{\partial^2 U}{\partial r\partial \theta}\sin\theta\cos\theta + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\sin^2\theta \\
&\qquad + \frac{1}{r}\frac{\partial U}{\partial r}\sin^2\theta + \frac{2}{r^2}\frac{\partial U}{\partial \theta}\sin\theta\cos\theta
\end{aligned}
$$

The derivative u_{yy} in polar coordinates

$$
\begin{aligned}
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \\
&= \frac{\partial}{\partial y}\left(\frac{\partial U}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial U}{\partial \theta}\cos\theta\right) \\
&= \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial U}{\partial \theta}\cos\theta\right)\sin\theta + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial U}{\partial \theta}\cos\theta\right)\cos\theta \\
&= \left(\frac{\partial^2 U}{\partial r^2}\sin\theta - \frac{1}{r^2}\frac{\partial U}{\partial \theta}\cos\theta + \frac{1}{r}\frac{\partial^2 U}{\partial r\partial \theta}\cos\theta\right)\sin\theta \\
&\qquad + \frac{1}{r}\left(\frac{\partial^2 U}{\partial r\partial \theta}\sin\theta + \frac{\partial U}{\partial r}\cos\theta + \frac{1}{r}\frac{\partial^2 U}{\partial \theta^2}\cos\theta - \frac{1}{r}\frac{\partial U}{\partial \theta}\sin\theta\right)\cos\theta \\
&= \frac{\partial^2 U}{\partial r^2}\sin^2\theta + \frac{2}{r}\frac{\partial^2 U}{\partial r\partial \theta}\sin\theta\cos\theta + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\cos^2\theta \\
&\qquad + \frac{1}{r}\frac{\partial U}{\partial r}\cos^2\theta - \frac{2}{r^2}\frac{\partial U}{\partial \theta}\sin\theta\cos\theta
\end{aligned}
$$

The Laplacian in polar coordinates

The expression $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ and its three-dimensional version $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ are ever-present in mathematical models stemming from physics, and encompass heat conduction, fluid and solid mechanics, electromagnetism, relativity, and cosmology. That expression is called the *Laplacian* of a function u and is written Δu (notation popular among mathematicians) and $\nabla^2 u$ (notation popular among engineers and physicists). We have seen how the Laplacian plays a fundamental role in describing heat conduction. So far we have dealt with the Laplacian expressed in Cartesian coordinates. Equipped with the calculations of the preceding two slides, we may express the Laplacian in polar coordinates by summing the expressions for $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$ calculated there. There is great deal of cancellation/simplification and we arrive at

$$\Delta u = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} \tag{59}$$

The Fourier series

$$f(x) = A_0 + \sum_{n=1}^{\infty} \Big(A_n \cos nx + B_n \sin nx \Big)$$

The Fourier series

Up to this point we have focused on the *Fourier sine series* which was formally defined on Slide 21. The Fourier sine series works best with functions $f(x)$ defined on an interval $(0, L)$ that satisfy zero boundary conditions, that is $f(0) = f(L) = 0$.

In this section we introduce the general Fourier series which works for all functions, regardless of any boundary conditions. To simplify the algebra, we limit the presentation to functions on the interval $(-\pi, \pi)$. Extending the conclusions to arbitrary intervals (a, b) is pretty straightforward.

Here is the general Fourier series for functions defined on the interval $(-\pi, \pi)$:

$$f(x) = A_0 + \sum_{n=1}^{\infty}\Big(A_n \cos nx + B_n \sin nx\Big). \qquad (60)$$

We skip the technical details here, but suffice to say that such a representation is possible for just about any function $f(x)$ that you would normally run across. In the next few slides we focus on how to determine the As and Bs for a given f.

Calculating the Fourier series' coefficients

The value of A_0 is easy to determine: just integrate (60) over $(-\pi, \pi)$ and note that for any positive integer n we have

$$\int_{-\pi}^{\pi} \cos nx \, dx = \frac{1}{n} \sin nx \Big|_{-\pi}^{\pi} = \frac{1}{n}\Big[\sin n\pi - \sin(-n\pi)\Big] = 0, \tag{61a}$$

$$\int_{-\pi}^{\pi} \sin nx \, dx = -\frac{1}{n} \cos nx \Big|_{-\pi}^{\pi} = -\frac{1}{n}\Big[\cos n\pi - \cos(-n\pi)\Big] = 0. \tag{61b}$$

Consequently, $\int_{-\pi}^{\pi} f(x)\, dx = 2\pi A_0$, and therefore

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx.$$

Remark: It is worth noting that A_0 calculated above is precisely the average value of $f(x)$ over the interval $(-\pi, \pi)$.

Calculating the Fourier series' coefficients (continued)

To calculate the remaining As and Bs, we observe that for all positive integers m and n we have

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{62a}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases} \tag{62b}$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0 \tag{62c}$$

Going back to (60), multiply both sides by $\cos mx$, where m is a positive integer, and integrate. We get

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = A_0 \int_{-\pi}^{\pi} \cos mx \, dx + \sum_{n=1}^{\infty} \left(A_n \int_{-\pi}^{\pi} \cos mx \cos nx \, dx + B_n \int_{-\pi}^{\pi} \cos mx \sin nx \, dx \right). \tag{63}$$

Calculating the Fourier series' coefficients (continued)

The coefficients of A_0 and B_n in (63) are zero due to (61a) and (62c). The coefficients of A_n are all zero by (62a) except when $n = m$ in which case the coefficient is π. Thus, (63) collapses to $\int_{-\pi}^{\pi} f(x) \cos mx \, dx = A_m \pi$, whence

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 1, 2, \dots.$$

To determine the coefficients B_n, we multiply (60) by $\sin mx$ and integrate. Repeating the reasoning above, we arrive at

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \dots.$$

The Fourier series – Summary

Here we summarize the findings of this section.

For all practical purposes, any function $f(x)$ defined in the interval $(-\pi, \pi)$ may be expressed as

$$f(x) = A_0 + \sum_{n=1}^{\infty} \Big(A_n \cos nx + B_n \sin nx \Big), \tag{64a}$$

where

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \tag{64b}$$

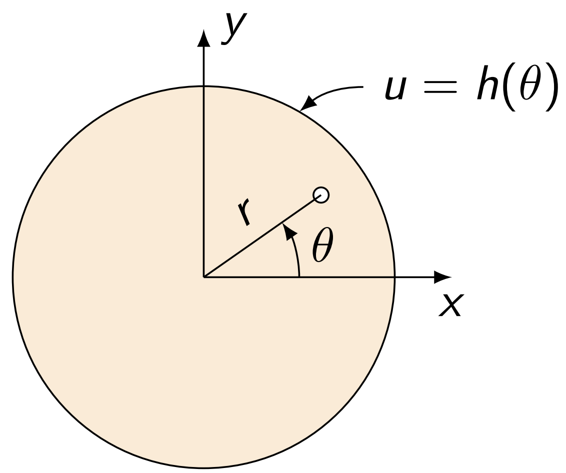
$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad n = 1, 2, \ldots, \tag{64c}$$

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx, \quad n = 1, 2, \ldots. \tag{64d}$$

Steady-state heat conduction in a disk

Steady-state heat conduction in a disk

Consider a thin circular disk of radius a, insulated on its flat faces, and exposed all around its peripheral edge.



We install a polar coordinate system (r, θ) in the plane of the disk, with the origin at the disk's center, and we impose a prescribed temperature $f(\theta)$, $-\pi < \theta < \pi$ around the edge and wait until the temperature stabilizes to a steady-state $u(r, \theta)$. Mathematically, this is described as a boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, & 0 < r < a, \quad -\pi < \theta < \pi \\ u(a, \theta) = h(\theta). & -\pi < \theta < \pi \\ u(r, \theta) \text{ is } 2\pi\text{-periodic in } \theta \\ u(0, \theta) \text{ is finite} \end{cases} \tag{65}$$

Separation of variables

We look for a solution $u(r, \theta) = R(t)\Psi(\theta)$. Plugging this into the PDE we obtain

$$R''(t)\Psi(\theta) + \frac{1}{r}R'(t)\Psi(\theta) + \frac{1}{r^2}R(t)\Psi''(\theta) = 0,$$

and then we separate the variables:

$$\frac{r^2 R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\Psi''(\theta)}{\Psi(\theta)} \tag{66}$$

The left-hand side involves r only, and the right-hand side involves θ only. Therefore each side is a constant. The constant may be negative, zero, or positive.

A negative constant, say $-\lambda^2$, is not interesting since the Ψ equation becomes $-\frac{\Psi''(\theta)}{\Psi(\theta)} = -\lambda^2$, that is, $\Psi''(\theta) - \lambda^2\Psi(\theta) = 0$ whose general solution is $\Psi(\theta) = A\cosh\lambda\theta + B\sinh\lambda\theta$. But such a function is not periodic in θ, and therefore the periodicity condition in (65) cannot be met.

On the other hand, the zero or positive choices for the separation constant are both viable and lead to interesting results.

The case of a zero separation constant

Let's consider the case where the separation constant, that is, the common value of the two sides of (66), is zero. Then we would have

$$\Psi''(\theta) = 0, \qquad r^2 R''(r) + rR'(r) = 0.$$

The solution of the Ψ equation is $\Psi(\theta) = A\theta + B$. The periodicity requirement on Ψ forces A to be zero, therefore we are left with $\Psi(\theta) = B$. In other words, $\Psi(\theta)$ is any constant function. That certainly satisfies the periodicity condition.

To solve the R equation, we rewrite it as $r^2 R''(r) + rR'(r) = 0$, and thus $\frac{R''(r)}{R'(r)} = -\frac{1}{r}$, and integrate and get $\ln R'(r) = \ln c_1 - \ln r$ which simplifies to $\ln\big(rR'(r)\big) = \ln c_1$, that is $rR'(r) = c_1$. Therefore $R'(r) = c_1/r$ and consequently

$$R(r) = c_1 \ln r + c_2. \tag{67}$$

We are forced to take $c_1 = 0$, otherwise the function would blow up as r approaches zero, violating the finiteness requirement stated in (65).
Conclusion: When the separation constant is zero, the only acceptable solution is $\Psi(\theta) = \text{constant}$, $R(r) = \text{constant}$, and therefore $u(r, \theta) = \text{constant}$.

The case of a positive separation constant

Let's consider the case where the separation constant, that is, the common value of the two sides of (66) is positive, say λ^2. Then we would have

$$\Psi''(\theta) + \lambda^2\Psi(\theta) = 0, \qquad r^2R''(r) + rR'(r) - \lambda^2 R(r) = 0. \tag{68}$$

The general solution of the Ψ equation is $\Psi(\theta) = A\cos\lambda\theta + B\sin\lambda\theta$, whence $\Psi'(\theta) = -A\lambda\sin\lambda\theta + B\lambda\cos\lambda\theta$.

We are interested in the range $-\pi < \theta < \pi$. The solution $u(r,\theta)$ will be continuous and smooth across the negative x axis if $\Psi(-\pi) = \Psi(\pi)$ and $\Psi'(-\pi) = \Psi'(\pi)$, that is

$$\begin{aligned} A\cos(-\lambda\pi) + B\sin(-\lambda\pi) &= A\cos(\lambda\pi) + B\sin(\lambda\pi), \\ -A\lambda\sin(-\lambda\pi) + B\lambda\cos(-\lambda\pi) &= -A\lambda\sin(\lambda\pi) + B\lambda\cos(\lambda\pi). \end{aligned}$$

These two equations simplify to $B\sin\lambda\pi = 0$ and $A\sin\lambda\pi = 0$, respectively. If $\sin\lambda\pi$ is nonzero, then both A and B are zero, and that results in the trivial solution $\Psi(\theta) = 0$. We conclude that $\sin\lambda\pi = 0$, and therefore $\lambda\pi = n\pi$ for all positive integers n.

The case of a positive separation constant (continued)

We conclude that the Ψ functions of interest are $\Psi_n(\theta) = A_n \cos n\theta + B_n \sin n\theta$, $n = 1, 2, \ldots$.

We return to (68) now and evaluate the R equation with $\lambda = n$. We get $r^2 R''(r) + rR'(r) - n^2 R(r) = 0$. This ODE is called *Euler's equation* and there is a well-know trick for solving it. Specifically, We try a solution of the form $R(r) = r^\alpha$ for a yet unspecified exponent α. Plugging this into the ODE we see that $\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2 r^\alpha = 0$, whence $\alpha(\alpha-1) + \alpha - n^2 = 0$, which simplifies to $\alpha^2 = n^2$. We conclude that $\alpha = \pm n$, and therefore the general solution of Euler's equation is

$$R(r) = c_1 r^{-n} + c_2 r^n. \tag{69}$$

We are forced to take $c_1 = 0$, otherwise the function would blow up as r approaches zero, violating the finiteness requirement stated in (65).

Conclusion: When the separation constant is positive, it has to be an integer, and the function $u(r, \theta) = r^n (A_n \cos n\theta + B_n \sin n\theta)$ satisfies the PDE in (65).

A Fourier series representation of the solution

Putting together the results of the preceding slides, we arrive at the following candidate for the solution of the boundary value problem (65):

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \Big(A_n \cos n\theta + B_n \sin n\theta \Big). \tag{70}$$

This solution candidate satisfies the PDE, the periodicity, and finiteness requirements. It remains to pick the As and Bs in order for meet the prescribed boundary condition $u(a,\theta) = h(\theta)$, that is,

$$h(\theta) = A_0 + \sum_{n=1}^{\infty} a^n \Big(A_n \cos n\theta + B_n \sin n\theta \Big).$$

The form of this expression happens to match precisely that of the general Fourier series formalism summarized on Slide 90. Applying equations (64) to the case at hand, we see that;

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta)\, d\theta, \;\; A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} h(\theta) \cos n\theta \, d\theta, \;\; B_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} h(\theta) \sin n\theta \, d\theta. \tag{71}$$

A worked out problem

Let's solve the boundary value problem (70) when

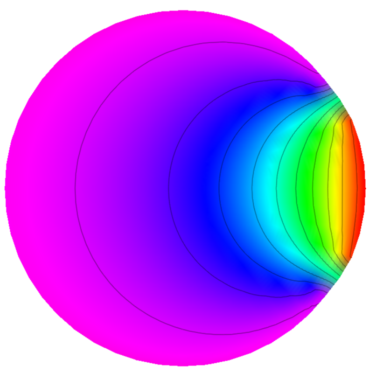
$$h(\theta) = \begin{cases} 1 & \text{if } |\theta| < \pi/6, \\ 0 & \text{otherwise.} \end{cases}$$

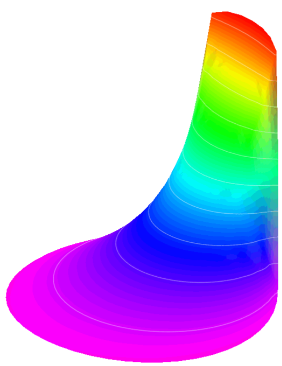
We evaluate the As and Bs according to (71) with the given h. We obtain:

$$A_0 = \frac{1}{6}, \quad A_n = \frac{2}{n\pi a^n} \sin \frac{n\pi}{6}, \quad B_n = 0.$$

Then the solution (70) takes the form

$$u(r,\theta) = \frac{1}{6} + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sin \frac{n\pi}{6} \right) \left(\frac{r}{a} \right)^n \cos n\theta.$$

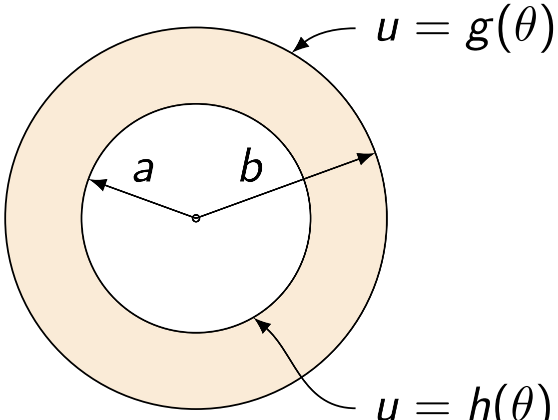




Illustrations made with $a = 1$, and ∞ set to 50

Heat conduction on an annulus

Consider a thin annulus of inner and outer radii a and b, respectively, insulated on its flat faces, and exposed on its inner and outer peripheral edges where the temperature is fixed at $h(\theta)$ around the inner edge, and $g(\theta)$ around the outer edge. Here θ is the angular coordinate in a polar coordinate system affixed to the annulus at its center.



The resulting steady-state temperature field, $u(r,\theta)$, is the solution of the boundary value problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0, & a < r < b, \quad -\pi < \theta < \pi \\ u(a,\theta) = h(\theta). & -\pi < \theta < \pi \\ u(b,\theta) = g(\theta). & -\pi < \theta < \pi \\ u(r,\theta) \text{ is } 2\pi\text{-periodic in } \theta \end{cases} \tag{72}$$

Heat conduction on an annulus – continued

We separate the variables in (72) just as we did in the case of heat conduction on a disk. In fact, most of the calculation there carries over here with only small changes.
Specifically, recall that in Slide 94 we dismissed the logarithmic term in (67) to avoid blowup at $r = 0$. But that is of no concern in an annulus since $r = 0$ varies from a to b, and does not hit zero. Therefore we retain the full solution given in (67) in the current calculation.

Similarly, on Slide 96 we dismissed the r^{-n} in (69), but we retain it in the current calculation since r does not approach zero. Then, the equivalent of the representation (70) in the case of annulus becomes

$$u(r, \theta) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} \Big(\big(A_n r^n + B_n r^{-n} \big) \cos n\theta + \big(C_n r^n + D_n r^{-n} \big) \sin n\theta \Big). \quad (73)$$

Heat conduction on an annulus – continued

Applying the boundary conditions leads to

$$h(\theta) = A_0 + B_0 \ln a + \sum_{n=1}^{\infty} \left(\left(A_n a^n + B_n a^{-n} \right) \cos n\theta + \left(C_n a^n + D_n a^{-n} \right) \sin n\theta, \right) \tag{74a}$$

$$g(\theta) = A_0 + B_0 \ln b + \sum_{n=1}^{\infty} \left(\left(A_n b^n + B_n b^{-n} \right) \cos n\theta + \left(C_n b^n + D_n b^{-n} \right) \sin n\theta. \right). \tag{74b}$$

The calculation of the coefficients proceeds as before. We observe that both of the equations above match the general Fourier series formalism from Slide 90. Applying equations (64) to (74a) yields

$$A_0 + B_0 \ln a = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(\theta)\, d\theta, \tag{75a}$$

$$A_n a^n + B_n a^{-n} = \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \cos n\theta \, d\theta, \tag{75b}$$

$$C_n a^n + D_n a^{-n} = \frac{1}{\pi} \int_{-\pi}^{\pi} h(\theta) \sin n\theta \, d\theta. \tag{75c}$$

Heat conduction on an annulus – continued

Similarly, applying equations (64) to (74b) yields

$$A_0 + B_0 \ln b = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta, \tag{76a}$$

$$A_n b^n + B_n b^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos n\theta\,d\theta, \tag{76b}$$

$$C_n b^n + D_n b^{-n} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin n\theta\,d\theta. \tag{76c}$$

We solve the set of six equations in (75) and (76) for the six unknowns A_0, B_0, A_n, B_n, C_n, D_n, and obtain:

$$A_0 = \frac{\bar{h}\ln b - \bar{g}\ln a}{\ln(b/a)}, \quad B_0 = \frac{\bar{g} - \bar{h}}{\ln(b/a)}, \tag{77a}$$

$$A_n = \frac{1}{\Delta_n}\left[-b^{-n}H_c^{(n)} + a^{-n}Gc\right], \quad B_n = \frac{1}{\Delta_n}\left[b^n H_c^{(n)} - a^n G_c^{(n)}\right], \tag{77b}$$

$$C_n = \frac{1}{\Delta_n}\left[-b^{-n}H_s^{(n)} + a^{-n}Gs\right], \quad D_n = \frac{1}{\Delta_n}\left[b^n H_s^{(n)} - a^n G_s^{(n)}\right]. \tag{77c}$$

. . . continued on next slide

Heat conduction on an annulus – continued

...continued from the previous slide where

$$\Delta_n = \pi\left[\left(\frac{b}{a}\right)^n - \left(\frac{a}{b}\right)^n\right], \quad \bar{h} = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\,d\theta, \quad \bar{g} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\,d\theta,$$

$$H_c^{(n)} = \int_{-\pi}^{\pi} h(\theta)\cos n\theta\,d\theta, \quad G_c^{(n)} = \int_{-\pi}^{\pi} g(\theta)\cos n\theta\,d\theta,$$

$$H_s^{(n)} = \int_{-\pi}^{\pi} h(\theta)\sin n\theta\,d\theta, \quad G_s^{(n)} = \int_{-\pi}^{\pi} g(\theta)\sin n\theta\,d\theta.$$

We plug the coefficients calculated in equations (77) into (73) and regroup the terms to arrive at the solution

$$u(r,\theta) = \frac{\bar{h}\ln b - \bar{g}\ln a}{\ln(b/a)} + \frac{\bar{g} - \bar{h}}{\ln(b/a)} r$$
$$+ \sum_{n=1}^{\infty} \frac{1}{\Delta_n}\left(\left[\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right] H_c^{(n)} + \left[\left(\frac{a}{r}\right)^n - \left(\frac{r}{a}\right)^n\right] G_c^{(n)}\right)\cos n\theta$$
$$+ \sum_{n=1}^{\infty} \frac{1}{\Delta_n}\left(\left[\left(\frac{b}{r}\right)^n - \left(\frac{r}{b}\right)^n\right] H_s^{(n)} + \left[\left(\frac{a}{r}\right)^n - \left(\frac{r}{a}\right)^n\right] G_s^{(n)}\right)\sin n\theta.$$

Graphics

Here is a sample of the general solution obtained in the previous slide. The annulus's inner and outer radii are $a = 0.5$, $b = 3$, and the boundary conditions are

$$h(\theta) = 0, \qquad g(\theta) = \begin{cases} 1 & \text{if } |\theta| < \pi/3, \\ 0 & \text{otherwise.} \end{cases}$$