Notes on traffic flow

for

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1. The derivation of the traffic flow equation

As an idealized mathematical model of traffic on a one-way highway, we let $\rho(x,t)$ be the *traffic density* (number of cars per unit length of the highway) at the location x at time t. Thus, the number of cars within the stretch $a < x < b$ is $\int_a^b \rho(x,t)\,dx$. We also write $q(x,t)$ for *traffic flow*, defined as the number of cars passing the location x at time t, per unit time.

The principle of conservation of cars: The rate of change of the number of cars within a stretch $a < x < b$ of the highway equals the rate at which cars enter the location $x = a$ minus the rate at which cars leave the location $x = b$, that is

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = q(a,t) - q(b,t). \tag{1}$$

In view of the elementary calculus property

$$q(a,t) - q(b,t) = \int_b^a \frac{\partial}{\partial x} q(x,t)\,dx = -\int_a^b \frac{\partial}{\partial x} q(x,t)\,dx,$$

we may rewrite (1) as

$$\frac{d}{dt}\int_a^b \rho(x,t)\,dx = -\int_a^b \frac{\partial}{\partial x} q(x,t)\,dx,$$

and further rearrange it into

$$\int_a^b \frac{\partial}{\partial t}\rho(x,t)\,dx + \int_a^b \frac{\partial}{\partial x} q(x,t)\,dx = 0.$$

Combining the two integrals we arrive at

$$. \int_a^b \Big(\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} q(x,t)\Big)\,dx = 0. \tag{2}$$

Since (2) holds for any choice of a and b, it follows that the integrand is zero. Indeed, if the integrand were nonzero,[1] then we could pick a and b to bracket a region where the integrand is strictly positive. Then the integral, which measures the area under the curve, would be positive, which would contradict (2). We conclude that

$$\frac{\partial}{\partial t}\rho(x,t) + \frac{\partial}{\partial x} q(x,t) = 0. \tag{3}$$

This partial differential equation expresses the principle of conservation of cars. That, however, is not sufficient on its own since it consists of one equation in two unknowns ρ and q. In an effort to move forward, we introduce a new unknown, $u(x,t)$, that expresses

[1] Note the "proof by contradiction" argument here.

the velocity of cars at the location x at time t. In fact, the quantities ρ, q, and u are related through

$$q = \rho u. \tag{4}$$

To see that, consider traffic flowing past an observation station A located at a point x along the highway. The car that passes A at time t, will travel a distance of $u\Delta t$ during a brief time interval Δt, and will find itself at a point B on the highway. The segment of length $u\Delta t$ between A and B will be filled with $\rho u\Delta t$ cars since the density ρ is the number of cars per unit length. But those cars must have gone past the station A during the time interval Δt. Since the traffic flow q measures the number of cars going by per unit time, the number of those cars is $q\Delta t$. We conclude that $\rho u\Delta t = q\Delta t$, whence $\rho u = q$, which is what equation (4) asserts.

Through equation (4), we may eliminate q from equation (3) and obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0. \tag{5}$$

This alternative form of the statement conservation of cars is still insufficient since it consists of one equation in two unknowns ρ and u. We need to look more deeply.

A crucial observation is that *the traffic density determines the traffic velocity.* To see that, consider the extreme case of essentially zero density. Then the cars travel without interference from each other. In an idealized world, they will travel at the maximum allowable speed, say $u_{\max}$. At the other extreme, when the highway is crowded with bumper-to-bumper traffic, let's call the corresponding density $\rho_{\max}$, the velocity is in effect zero.

In summary, the velocity varies from $u_{\max}$ to zero as the density goes from zero to $\rho_{\max}$, that is, $u = \phi(\rho)$ for some function ϕ such that $\phi(0) = u_{\max}$ and $\phi(\rho_{\max}) = 0$. For the purpose of this study, we take ϕ to be a linear function, as shown in Figure 1. The equation of that line is

$$u = u_{\max}\left(1 - \frac{\rho}{\rho_{\max}}\right). \tag{6}$$

Plugging this into (5) we get

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(u_{\max}\rho\left(1 - \frac{\rho}{\rho_{\max}}\right)\right) = 0$$

which simplifies to

$$\frac{\partial \rho}{\partial t} + u_{\max}\frac{\partial}{\partial x}\left(\rho - \frac{\rho^2}{\rho_{\max}}\right) = 0,$$

and thus

$$\frac{\partial \rho}{\partial t} + \left(1 - \frac{2\rho}{\rho_{\max}}\right)\frac{\partial \rho}{\partial x} = 0. \tag{7}$$

This PDE, in the single unknown ρ, is our working model of traffic flow.

In the calculations that follow, the parenthesized coefficient in (7) is too bulky to carry around, and therefore we introduce a shorthand for it:

$$c(\rho) = 1 - \frac{2\rho}{\rho_{\max}}, \tag{8}$$

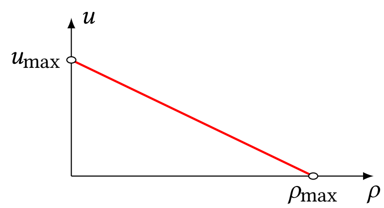


FIGURE 1. The function $u = \phi(\rho)$ models traffic velocity versus density.

whereby (7) take the compact form

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0. \tag{9}$$

2. The initial value problem and the characteristics

We wish to understand and analyze the initial value problem associated with (9), that is

$$\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0 \qquad -\infty < x < \infty, \quad t > 0, \tag{10a}$$

$$\rho(x, 0) = f(x) \qquad -\infty < x < \infty, \tag{10b}$$

where $f(x)$ is a given traffic density at time $t = 0$.

To see how the traffic evolves, consider an observer moving along the highway so that his/her position at time t is an as yet unspecified function $x(t)$. The traffic density observed by the observer at time t is $\rho\big(x(t), t\big)$. The rate of change of the observed value may be calculated through the chain rule:

$$\frac{d}{dt}\rho\big(x(t), t\big) = \frac{\partial \rho}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \rho}{\partial t} \cdot \frac{dt}{dt}.$$

But $dt/dt = 1$. Therefore

$$\frac{d}{dt}\rho\big(x(t), t\big) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \cdot \frac{dx}{dt}.$$

We see that if the observer moves so that $dx/dt = c(\rho)$, then we have

$$\frac{d}{dt}\rho\big(x(t), t\big) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}c(\rho) = 0,$$

due to (10a). This tells us that if the observer moves so that $dx/dt = c(\rho)$, then *he/she will be observing an unchanging density.* But if the density is unchanging, then $c(\rho)$ is unchanging, and consequently dx/dt is unchanging (since $dx/dt = c(\rho)$), which means that the observer's path in space-time is a straight line!

That straight line in space-time is called a *characteristic line*. The value of dx/dt (that is, $c(\rho)$) along that line is called the *characteristic's speed*. Figure 2 illustrates what the characteristic lines may look like in a space-time diagram. Note that the speed of a characteristic is determined by the value that $c(\rho)$ take at $t = 0$ since whatever that value, it remains constant along the characteristic. For instance, the characteristic line drawn in blue intersects the x axis at $x = \xi$ where the density $f(\xi)$ is given in (10b), and therefore the characteristic's speed is $c\big(f(\xi)\big)$. We conclude that the equation of the characteristic line is $x = \xi + c\big(f(\xi)\big)\, t$. Since f generally may vary along the x axis, the characteristics generally propagate at varying speed, hence they have different slopes.

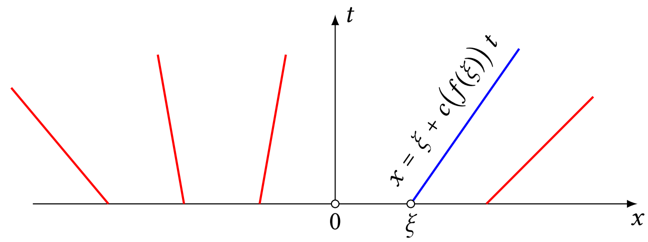


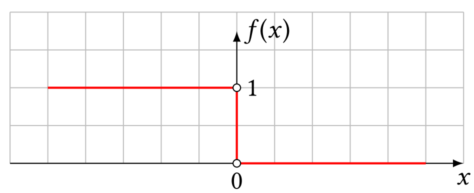
FIGURE 2. Sample characteristic lines.

3. EXAMPLES

On this section we provide detailed solutions of three illustrative examples of the use of characteristics to analyze the initial value problem (10).

3.1. **Example 1.** Consider the special case of the initial value problem (10) where $u_{\max} = 1$, $\rho_{\max} = 1$, and $f(x)$ is the step function

$$f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0, \end{cases}$$



This corresponds to a traffic that begins to move after the red light turns green. Based on the given data, we have $c(\rho) = 1 - 2\rho$. We see that $c(0) = 1$. Therefore, the characteristics that originate on the positive x axis propagate with velocity 1. We also see that $c(1) = -1$. Therefore, the characteristics that originate on the negative x axis propagate with velocity -1. The corresponding space-time diagram is shown in Figure 3. We see that the diagram is divided into three distinct regions. Within region 1 we have $\rho(x,t) = 1$ since density is constant along the characteristics and each characteristic originates on the negative x axis where the density is 1. For the same reason, within region 3 we have $\rho(x,t) = 0$. The characteristics drawn in red separate the three regions. Their equations are $x = -t$ and $x = t$. Thus, region 1 corresponds to $x < -t$, region 2 corresponds to $-t < x < t$, and region 3 corresponds to $x > t$.

Within the "fan" region 2 we calculate $\rho(x,t)$ as follows. Pick any point (x,t) within region 2. The equation of characteristic through it is $x = c(\rho)t$, where ρ is the (constant) density along that characteristic. Since $c(\rho) = 1 - 2\rho$, the equation of the characteristic takes the form $x = (1 - 2\rho)t$. We solve this for ρ and obtain $\rho = \frac{1}{2}\big(1 - \frac{x}{t}\big)$. We conclude that

$$\rho(x,t) = \begin{cases} 1 & \text{if } x < -t, \\ \frac{1}{2}\big(1 - \frac{x}{t}\big) & \text{if } -t < x < t, \\ 0 & \text{if } x > t. \end{cases} \tag{11}$$

Remark: The solution expressed in (11) does not stand under scrutiny. If it is the solution of the initial value problem (10), shouldn't $\rho(x,0) = f(x)$? Yes, it should, but (11) is

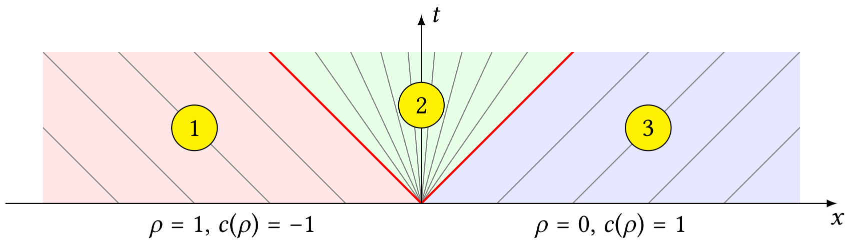


FIGURE 3. The space-time diagram for Example 1.

undefined at $t = 0$ since it involves a division by t. To remedy the situation, we need to be explicit about the need for special treatment of the $t = 0$ case. Thus, a pedantically correct expression of the solution would be

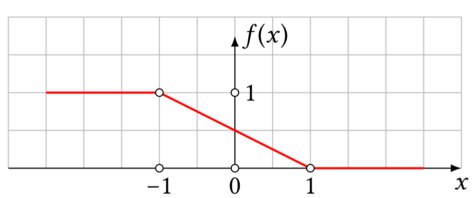
$$\rho(x,t) = \begin{cases} f(x) & \text{if } t = 0, \\ \begin{cases} 1 & \text{if } x < -t, \\ \frac{1}{2}\left(1 - \frac{x}{t}\right) & \text{if } -t < x < t, \\ 0 & \text{if } x > t. \end{cases} & \text{if } t > 0. \end{cases} \tag{12}$$

Normally people prefer the simpler but questionable representation (11) over the precise but elaborate version (12), taking it for granted that (11) is meant for $t > 0$. We will make that implicit assumption when presenting the solutions to this section's remaining examples. Beware, however, that a computer program is generally not as forgiving may balk at accepting (11).

Having obtained the solution $\rho(x,t)$ of the problem, we may sketch the graph of the density at any time t. For instance, plugging $t = 1$ in the solution, we get

$$\rho(x,1) = \begin{cases} 1 & \text{if } x < -1, \\ \frac{1}{2}(1 - x) & \text{if } -1 < x < 1, \\ 0 & \text{if } x > 1. \end{cases}$$

Here is the corresponding graph.



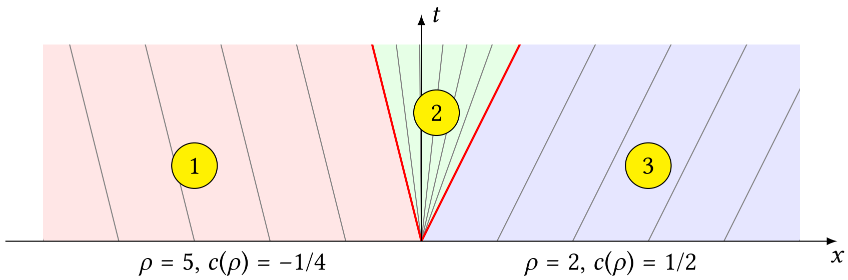
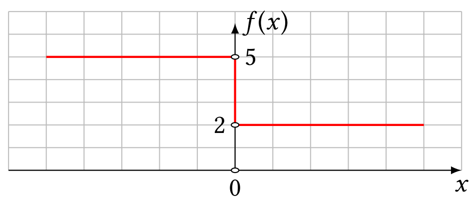


FIGURE 4. The space-time diagram for Example 2.

3.2. **Example 2.** Let us solve the initial value problem (10) with the data $u_{\max} = 1$, $\rho_{\max} = 8$, and

$$f(x) = \begin{cases} 5 & \text{if } x < 0, \\ 2 & \text{if } x > 0, \end{cases}$$

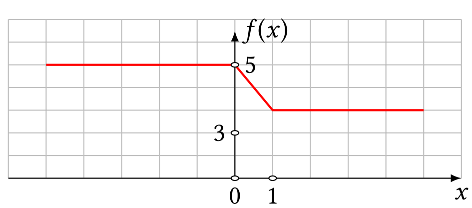


Based on the given data, we have $c(\rho) = 1 - \frac{1}{4}\rho$. We see that $c(2) = 1/2$. Therefore, the characteristics that originate on the positive x axis propagate with velocity 1/2. We also see that $c(5) = -1/4$. Therefore, the characteristics that originate on the negative x axis propagate with velocity $-1/4$. The corresponding space-time diagram is shown in Figure 4. As in the previous example, the diagram is divided into three regions. Within region 1 we have $\rho(x,t) = 5$. Within region 3 we have $\rho(x,t) = 2$. The characteristics drawn in red separate the three regions. Their equations are $x = -\frac{1}{4}t$ and $x = \frac{1}{2}t$. Thus, region 1 corresponds to $x < -\frac{1}{4}t$, region 2 corresponds to $-\frac{1}{4}t < x < \frac{1}{2}t$, and region 3 corresponds to $x > \frac{1}{2}t$.

We calculate the density $\rho(x,t)$ in the fan region as we did in the previous example. The equations of characteristics in that region are $x = c(\rho)t = \left(1 - \frac{1}{4}\rho\right) t$. We solve this for ρ and obtain $\rho = 4\left(1 - \frac{x}{t}\right)$. We conclude that

$$\rho(x,t) = \begin{cases} 5 & \text{if } x < -\frac{1}{4}t, \\ 4\left(1 - \frac{x}{t}\right) & \text{if } -\frac{1}{4}t < x < \frac{1}{2}t \\ 2 & \text{if } x > \frac{1}{2}t. \end{cases}$$

3.3. **Example 3.** Here we solve the initial value problem (10) with the data $u_{\max} = 4$, $\rho_{\max} = 8$, and $f(x)$ as in



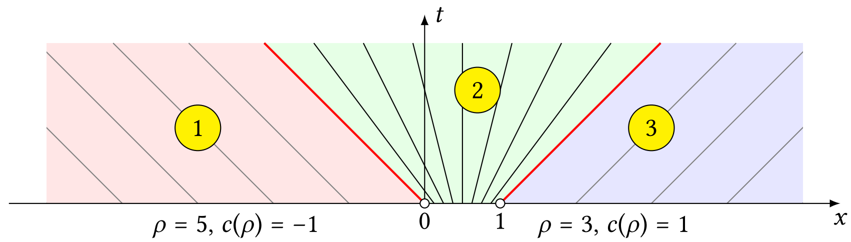


FIGURE 5. The space-time diagram for Example 3.

We see that the equation of the slanted part of f is $5 - 2x$, therefore $f(x)$ may be expressed as

$$f(x) = \begin{cases} 5 & \text{if } x < 0, \\ 5 - 2x & \text{if } 0 < x < 1, \\ 3 & \text{if } x > 1. \end{cases}$$

Based on the given data, we have $c(\rho) = 4 - \rho$. We see that $c(3) = 1$, Therefore, the characteristics that originate in $x > 1$ propagate with velocity 1. We also see that $c(5) = -1$. Therefore, the characteristics that originate on the negative x axis propagate with velocity -1. The corresponding space-time diagram is shown in Figure 5. As in the previous examples, the diagram is divided into three regions. Within region 1 we have $\rho(x,t) = 5$. Within region 3 we have $\rho(x,t) = 3$. The characteristics drawn in red separate the three regions. Their equations are $x = -t$ and $x = t + 1$. Thus, region 1 corresponds to $x < -t$, region 2 corresponds to $-t < x < t + 1$, and region 3 corresponds to $x > t + 1$.

To calculate the density $\rho(x,t)$ within region 2, we pick a point (x,t) in that region and let $x = c(\rho)t + \xi = (4 - \rho)t + \xi$ be the characteristic line that passes through it, where ρ is the (constant) density along the characteristic. The characteristic line intersects the x axis at $x = \xi$. The density at ξ is $\rho f(\xi) = 5 - 2\xi$. Therefore $\xi = \frac{1}{2}(5 - \rho)$. It follows that $x = (4 - \rho)t + \frac{1}{2}(5 - \rho)$. We solve this for ρ and obtain

$$\rho = \frac{8t + 5 - 2x}{1 + 2t},$$

and therefore

$$\rho(x,t) = \begin{cases} 5 & \text{if } x < -t, \\ \frac{8t+5-2x}{1+2t} & \text{if } -t < x < t + 1, \\ 3 & \text{if } x > t + 1. \end{cases}$$

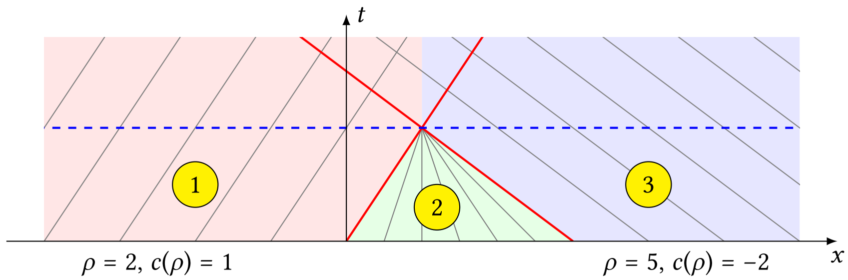
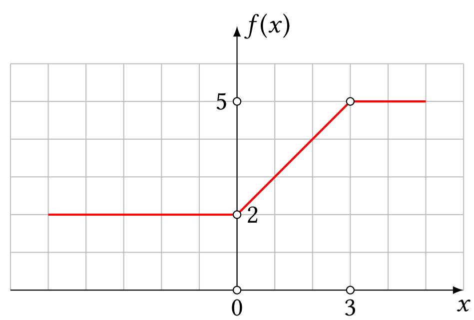


Figure 6. A preliminary sketch of the space-time diagram. A shock forms when the red characteristic lines meet. The diagram is *not valid* above the dotted blue line.

4. Shock waves

Consider the initial value problem (10) where $u_{\max} = 3$, $\rho_{\max} = 6$, and $f(x)$ is as in

$$f(x) = \begin{cases} 2 & \text{if } x < 0, \\ 2 + x & \text{if } 0 < x < 3, \\ 5 & \text{if } x > 3. \end{cases}$$



Based on the given data, we have $c(\rho) = 3 - \rho$. We see that $c(5) = -2$, Therefore, the characteristics that originate in $x > 3$ propagate with velocity -2. We also see that $c(2) = 1$. Therefore, the characteristics that originate on the negative x axis propagate with velocity 1. A preliminary version of the corresponding space-time diagram is shown in Figure 6. We see that the characteristic lines that originate at $x = 0$ and $x = 3$ (drawn in red) intersect at some time t_0 marked with the dotted blue line. The one coming from $x = 0$ carries with it the signal that $\rho = 2$. The one coming from $x = 3$ carries with it the signal that $\rho = 5$. This creates a conflict at the intersection. What is the density there?

The conflict is not merely at the point where the red characteristics intersect. *All left-moving and right-moving characteristics intersect above the dotted blue line*, bringing with them conflicting signals. We will address this issue further down, but for now let us complete the analysis of the solution within the time interval $0 < t < t_0$, before the first conflict develops.

We begin by noting that the equation of the characteristic line that originates at $x = 0$ is $x = t$, while the equation of the characteristic line that originates at $x = 3$ is $x = 3 - 2t$.

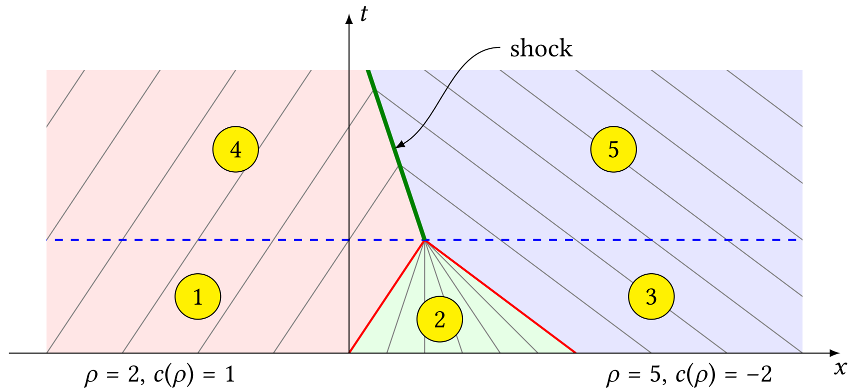


FIGURE 7. The completed space-time diagram is valid both below and above the dotted blue line. The shock that forms above that line marks the interface between two regions of differing densities.

The lines intersect at $x = 1$, $t = 1$. Therefore the regions 1 and 3 are characterized by

$$\begin{aligned}
\text{region } 1 &= \Big\{(x,t) \,:\, t < 1, \quad x < t\Big\}, \\
\text{region } 2 &= \Big\{(x,t) \,:\, t < 1, \quad t < x < 3 - 2t\Big\}, \\
\text{region } 3 &= \Big\{(x,t) \,:\, t < 1, \quad x > 3 - 2t\Big\}.
\end{aligned}$$

As in the previous examples, density $\rho(x, t)$ is 2 in the region 1, and it is 5 in the region 3. We calculate the density within region 2 as before, by taking a point (x, t) within that region an considering a characteristic line $x = c(\rho)t + \xi$ through it, where ρ is the (constant) density along that characteristic. We know that $c(\rho) = 3 - \rho$, and that the density at the intercept ξ is $\rho = f(\xi) = 2 + \xi$, whence $\xi = \rho - 2$. Therefore the equation of the characteristic takes the form $x = (3 - \rho)t + \rho - 2$. Solving this for ρ, we obtain:

$$\rho = \frac{x - 3t + 2}{1 - t}.$$

We conclude that

$$\rho(x,t) = \begin{cases} 2 & \text{in region } 1, \\ \frac{x-3t+2}{1-t} & \text{in region } 2, \\ 5 & \text{in region } 3. \end{cases} \tag{13}$$

Now we turn to the situation above $t = t_0$. The conflict there is resolved by introducing a line of discontinuity, called a *shock*, shown in Figure 7. Characteristics from the left and from the right impinge on the shock, bringing with them conflicting density information originating at the initial data. As a result, the density is discontinuous across the shock. Specifically, the density is 2 in region 4, and 5 in region 5. The density jumps from 2 to 5 across the shock.

The questions that remain to be resolved is: what is the the equation of the shock? The shock has been drawn as a straight line in the diagram. How do we know that it is a straight line?

The answers to these questions emerge from applying the *principle of conservation of cars*—the shock does not create or destroy cars. To work out the details, let us pick a point P along the shock and let v be the shock's speed at that point. Let us write ρ_{left} and u_{left} for the traffic density and velocity just to the left of P. Similarly, let ρ_{right} and u_{right} for the traffic density and velocity just to the right of P.

The traffic approaches P from the left with velocity $u_{\text{left}} - v$, and it moves away from P with velocity $u_{\text{right}} - v$. Recalling the discussion that led to (4), the number of cars arriving at P per unit time is $\rho_{\text{left}}(u_{\text{left}} - v)$, and the number of cars pulling away from P per unit time is $\rho_{\text{right}}(u_{\text{right}} - v)$. We conclude that

$$\rho_{\text{left}}(u_{\text{left}} - v) = \rho_{\text{right}}(u_{\text{right}} - v),$$

which we solve for v:

$$v = \frac{\rho_{\text{right}} u_{\text{right}} - \rho_{\text{left}} u_{\text{left}}}{\rho_{\text{right}} - \rho_{\text{left}}}. \tag{14}$$

This is called the *Rankine–Hugoniot jump condition* and was derived in the late 1880's in the context of gas dynamics. (That was before there were automobiles, highways, and traffic as we know it.) The equation may be expressed succinctly as

$$v = \frac{\text{jump in } (\rho u)}{\text{jump in } \rho}. \tag{15}$$

Let us apply (14) to determine the shock speed in our problem. Since $u_{\max} = 3$ and $\rho_{\max} = 6$, the velocity equation (6) takes the form

$$u = 3\left(1 - \frac{\rho}{6}\right) = 3 - \frac{1}{2}\rho.$$

We have $\rho_{\text{left}} = 2$, and therefore $u_{\text{left}} = 2$. We also have $\rho_{\text{right}} = 5$, and therefore $u_{\text{right}} = 1/2$. Then from (14) we get

$$v = \frac{(5)(1/2) - (2)(2)}{5 - 2} = -\frac{1}{2}.$$

We conclude that the shock propagates with velocity $-1/2$, which incidentally shows that the shock is a straight line since its speed is a constant. The shock originates at $(1, 1)$, therefore its equation is $x = -\frac{1}{2}(t - 1) + 1$, which simplifies to $x = -\frac{1}{2}t + \frac{3}{2}$. It follows that

$$\text{region } 4 = \left\{(x, t) \,:\, t \geq 1, \quad x < -\frac{1}{2}t + \frac{3}{2}\right\},$$
$$\text{region } 5 = \left\{(x, t) \,:\, t \geq 1, \quad x > -\frac{1}{2}t + \frac{3}{2}\right\},$$

and consequently,

$$\rho(x, t) = \begin{cases} 2 & \text{in region } 4\,, \\ 5 & \text{in region } 5\,. \end{cases} \tag{16}$$

This, together with (13), presents a complete solution to our initial value problem. Figure 8 shows the graphs of $\rho(x, t)$ at times $t = 1/2$ and $t = 5$.

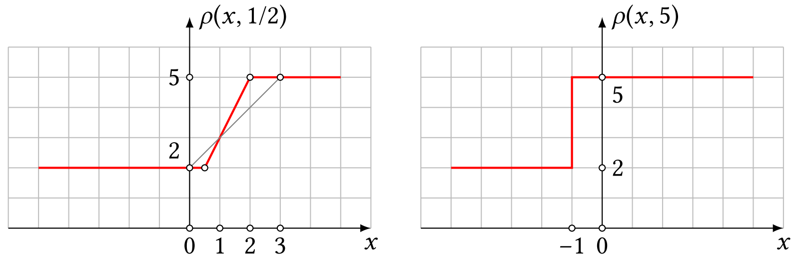


FIGURE 8. Graphs of the density $\rho(x, t)$ of the solution of Example 3's initial value problem, plotted at $t = 1/2$ and $t = 5$. The slanted gray line in the diagram on the left depicts the initial the initial value $\rho(x, 0)$.