Math 404, Fall 2020
Homework #10

For your convenience, I begin this homework assignment with a quick summary of the explicit and implicit finite difference schemes for solving the heat equation. The homework question comes at the very end. In fact, there are two questions there. The second one is optional but it will earn you bonus points if you do it.

1. The finite difference discretization

We wish to solve the initial boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad a < x < b, \quad 0 < t < T, \tag{1a}$$

$$u(x,0) = f(x), \qquad a < x < b, \tag{1b}$$

$$u(a,t) = \alpha(t), \qquad 0 < t < T, \tag{1c}$$

$$u(b,t) = \beta(t), \qquad 0 < t < T, \tag{1d}$$

for the unknown function u. The initial condition $f(x)$, the boundary conditions $\alpha(t)$ and $\beta(t)$, and the upper limit in time, T, are given.

In a finite-difference approximation, we subdivide the space interval $[a, b]$ into n equal-length segments, and subdivide the time interval $[0, T]$ into m equal-length segments. This imposes an $(m+1)\times(n+1)$ grid the domain of u as seen in Figure 1. The grid spacing in the x direction is $\Delta x = (b-a)/n$, and the grid spacing in the t direction is $\Delta t = T/m$. We write x_j, $j = 1, 2, \dots, n+1$ for the x coordinates of the grid points, and t_i, $i = 1, 2, \dots, m+1$ for the t coordinates of the grid points. In particular

$$x_1 = a, \quad x_{n+1} = b, \quad t_1 = 0, \quad t_{m+1} = T.$$

We refer to the grid points through their indices (i, j), where i increases in the t direction and j increases in the x directions. We write $u_{i,j}$ for the value of $u(x,t)$ at the node (i, j), that is,

$$u_{i,j} = u(x_j, t_i).$$

At the grid point (i, j) the partial derivative $\partial u/\partial t$ may be approximated as

$$\left.\frac{\partial u}{\partial t}\right|_{(x_j,t_i)} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta t}, \qquad \textit{(forward difference)} \tag{2a}$$

or

$$\left.\frac{\partial u}{\partial t}\right|_{(x_j,t_i)} \approx \frac{u_{i,j} - u_{i-1,j}}{\Delta t}. \qquad \textit{(backward difference)} \tag{2b}$$

Replacing the $\partial u/\partial t$ term in the PDE with (2a) leads to the so-called *explicit finite difference scheme*, while replacing it with (2b) leads to the so-called *implicit finite difference scheme*, as we shall see.

As to the PDE's second order derivative $\partial^2 u/\partial x^2$, we replace it with

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_j,t_i)} \approx \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta x)^2}, \tag{3}$$

as demonstrated in class.

Be sure to examine each term in equations (2a), (2b), and (3) and see how they are related to the corresponding grid points in Figure 1.

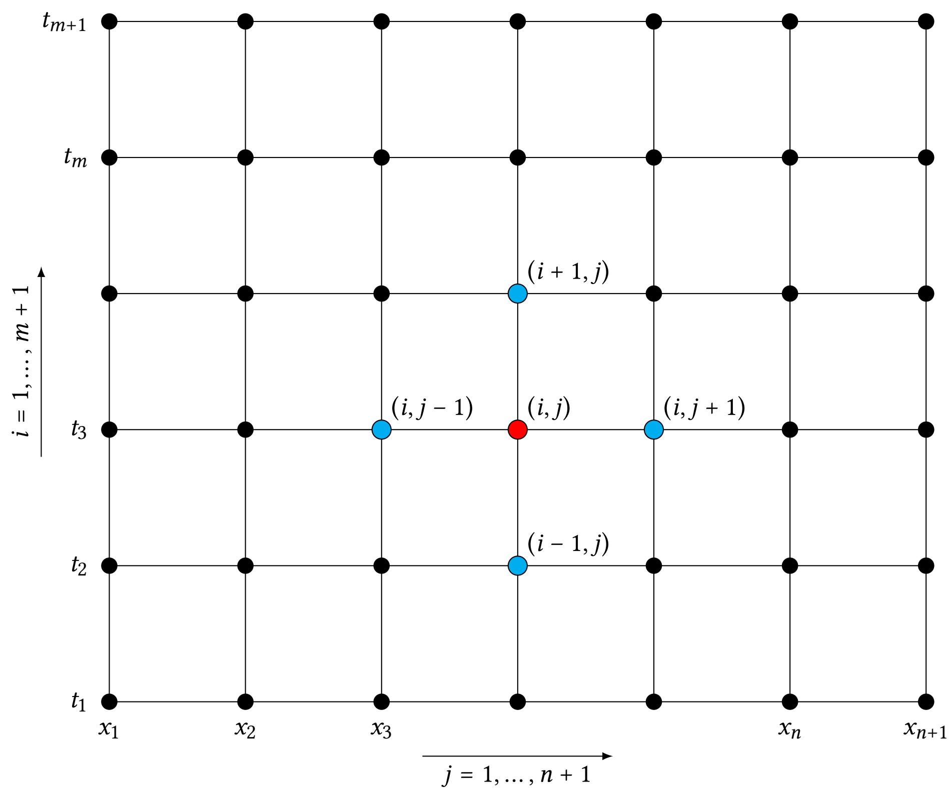

FIGURE 1. The finite difference grid.

2. THE EXPLICIT SCHEME

Let us replace the partial derivatives in the PDE (1) with the finite difference approximations (2a) and (3). We get:

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta t} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta x)^2}, \qquad i = 1, \ldots, m, \quad j = 2, \ldots, n. \tag{4}$$

This equation is known as the *explicit finite difference scheme* for the heat equation. It enables us to calculate $u_{i+1,j}$ at time t_{i+1} in terms of the values of u at the previous time t_i. Thus, we may march forward in time beginning with $t = 0$ where the value of u is known from the initial condition in (1b).

Isolating $u_{i+1,j}$ in the equation above, we get

$$u_{i+1,j} = u_{i,j} + \frac{\Delta t}{(\Delta x)^2}\Big[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\Big].$$

To simplify the notation, we introduce

$$r = \frac{\Delta t}{(\Delta x)^2}. \tag{5}$$

And then, combining the two $u_{i,j}$ we arrive at

$$u_{i+1,j} = ru_{i,j-1} + (1 - 2r)u_{i,j} + ru_{i,j+1}. \tag{6}$$

To truly appreciate what this equation says, let us evaluate it for $j = 2, \dots, n$:

$$\begin{aligned}
u_{i+1,2} &= ru_{i,1} + (1-2r)u_{i,2} + ru_{i,3},\\
u_{i+1,3} &= ru_{i,2} + (1-2r)u_{i,3} + ru_{i,4},\\
u_{i+1,4} &= ru_{i,3} + (1-2r)u_{i,4} + ru_{i,5},\\
&\cdots\\
u_{i+1,n} &= ru_{i,n-1} + (1-2r)u_{i,n} + ru_{i,n+1}.
\end{aligned}$$

We pad this system of equations from the top and from the bottom by the two equations

$$\begin{aligned}
u_{i+1,1} &= \alpha(t_{i+1}),\\
u_{i+1,n+1} &= \beta(t_{i+1})
\end{aligned}$$

which express the values of $u_{i+1,1}$ and $u_{i+1,n}$ which are known from the boundary conditions. We get

$$\begin{aligned}
u_{i+1,1} &= \alpha(t_{i+1}),\\
u_{i+1,2} &= ru_{i,1} + (1-2r)u_{i,2} + ru_{i,3},\\
u_{i+1,3} &= ru_{i,2} + (1-2r)u_{i,3} + ru_{i,4},\\
u_{i+1,4} &= ru_{i,3} + (1-2r)u_{i,4} + ru_{i,5},\\
&\cdots\\
u_{i+1,n} &= ru_{i,n-1} + (1-2r)u_{i,n} + ru_{i,n+1},\\
u_{i+1,n+1} &= \beta(t_{i+1}).
\end{aligned}$$

Finally, we cast the equations into a matrix form:

$$\begin{bmatrix} u_{i+1,1}\\ u_{i+1,2}\\ u_{i+1,3}\\ u_{i+1,4}\\ \vdots\\ u_{i+1,n}\\ u_{i+1,n+1} \end{bmatrix} = \begin{bmatrix} 0 & & & & & & & \\ r & (1-2r) & r & & & & & \\ 0 & r & (1-2r) & r & & & & \\ 0 & 0 & r & (1-2r) & r & & & \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & r & (1-2r) & r & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{i,1}\\ u_{i,2}\\ u_{i,3}\\ u_{i,4}\\ \vdots\\ u_{i,n}\\ u_{i,n+1} \end{bmatrix} + \begin{bmatrix} \alpha(t_{i+1})\\ 0\\ 0\\ 0\\ \vdots\\ 0\\ \beta(t_{i+1}) \end{bmatrix}. \tag{7}$$

In this equation we clearly see how the values of u at time t_i are related to the values of u at time t_{i+1}. In class we learned how to enter this equation in Matlab and get numbers out of it.

Remark 1. As noted in class, equation (7) produces a faithful representation of the solution of the initial value problem (1) provided that $r \le 1/2$. It is likely to produce junk otherwise.

3. The implicit scheme

Let us replace the partial derivatives in (1a) with the finite difference approximations (2b) and (3). We get:

$$\frac{u_{i,j} - u_{i-1,j}}{\Delta t} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta x)^2}, \qquad i = 1, \dots, m, \quad j = 2, \dots, n. \tag{8}$$

This is known as the *implicit finite difference scheme* for the heat equation. The reason for calling it *implicit* is that, unlike the previous sections *explicit* scheme, it **does not** express

u at a given time in terms of the values of u at the previous times. To see this clearly, take Δt to the right-hand side:

$$u_{i,j} - u_{i-1,j} = \frac{\Delta t}{(\Delta x)^2}\Big[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\Big],$$

and then let $r = \Delta t/(\Delta x)^2$ as we did in (5),

$$u_{i,j} - u_{i-1,j} = r\Big[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\Big].$$

Then rearrange the terms as:

$$-ru_{i,j-1} + (1+2r)u_{i,j} - ru_{i,j+1} = u_{i-1,j}.$$

Observe that this fails to express u at time t_i in terms of u at time t_{i-1}. Rather, it express *linear combination of the u values* at time t_i in terms of the value of u at time t_{i-1}.

But all is not lost. Let us write out explicitly what the equation says for $j = 2, 3, \dots, n$:

$$\begin{aligned}
-ru_{i,1} + (1+2r)u_{i,2} - ru_{i,3} &= u_{i-1,2},\\
-ru_{i,2} + (1+2r)u_{i,3} - ru_{i,4} &= u_{i-1,3},\\
-ru_{i,3} + (1+2r)u_{i,4} - ru_{i,5} &= u_{i-1,4},\\
&\cdots\\
-ru_{i,n-1} + (1+2r)u_{i,n} - ru_{i,n+1} &= u_{i-1,n}.
\end{aligned}$$

We pad the equations, as before by the values supplied by the boundary conditions, and we arrive at

$$\begin{aligned}
u_{i,1} &= \alpha(t_i),\\
-ru_{i,1} + (1+2r)u_{i,2} - ru_{i,3} &= u_{i-1,2},\\
-ru_{i,2} + (1+2r)u_{i,3} - ru_{i,4} &= u_{i-1,3},\\
-ru_{i,3} + (1+2r)u_{i,4} - ru_{i,5} &= u_{i-1,4},\\
&\cdots\\
-ru_{i,n-1} + (1+2r)u_{i,n} - ru_{i,n+1} &= u_{i-1,n},\\
u_{i,n+1} &= \beta(t_i).
\end{aligned}$$

Finally, we cast this into a matrix form:

$$\begin{bmatrix}
1 & & & & & & & \\
-r & (1+2r) & -r & & & & & \\
0 & -r & (1+2r) & -r & & & & \\
0 & 0 & -r & (1+2r) & -r & & & \\
\vdots & & & & & & & \\
0 & 0 & 0 & 0 & -r & (1+2r) & -r & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix} u_{i,1} \\ u_{i,2} \\ u_{i,3} \\ u_{i,4} \\ \vdots \\ u_{i,n} \\ u_{i,n+1} \end{bmatrix}
=
\begin{bmatrix} \alpha(t_i) \\ u_{i-1,2} \\ u_{i-1,3} \\ u_{i-1,4} \\ \vdots \\ u_{i-1,n} \\ \beta(t_i) \end{bmatrix} \tag{9}$$

This equation has the form $Ax = b$, were A is a tridiagonal matrix, x is the (unknown) vector of the solution at time t_i, and the right-hand side b is a known vector which is constructed from the values of the solution at time t_{i-1}, and the boundary conditions. We see that calculating the unknown vector calls for solving a linear system of equations. We have seen in class how encode this system into Matlab, and that we solve it by the command $x = A\backslash b$.

4. Adding a heat source

Early in the semester, we saw that in the presence of a heat source, the PDE (1a) changes to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + F(x,t).$$

Here we wish to modify the results of the previous sections to include $F(x,t)$ in the finite difference formulation.

Let $F_{i,j} = F(x_j, t_i)$. Then the explicit scheme in (4) takes the form

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta t} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{(\Delta x)^2} + F_{i,j} \qquad i = 1,\dots,m, \quad j = 2,\dots,n.$$

Multiplying through by Δt and recalling the definition of r in (5), we get

$$u_{i+1,j} = u_{i,j} + r\Big[u_{i,j-1} - 2u_{i,j} + u_{i,j+1}\Big] + F_{i,j}\Delta t$$

which we rearrange into

$$u_{i+1,j} = ru_{i,j-1} + (1-2r)u_{i,j} + ru_{i,j+1} + F_{i,j}\Delta t \tag{10}$$

This is how equation (6) changes when we add a heat source.

Homework problem #1. (8pts) *The explicit scheme with a heat source.*

Examine the calculations that lead from equation (6) to the matrix form (7). Do the equivalent calculation beginning with equation (10) and obtain the corresponding matrix form.

Homework problem #2. (optional, 8 bonus pts) *The implicit scheme with a heat source.*

Examine the calculations that lead from equation (8) to the matrix form (9). Derive the matrix formulation when a heat source is present.