Math 404, Fall 2020
Homework #4

1. (3 points) On Slide 80 it is shown that in going from the Cartesian (x, y) coordinates to the polar (r, θ) coordinates, we have

$$\frac{\partial r}{\partial x} = \cos\theta, \qquad \frac{\partial \theta}{\partial x} = -\frac{1}{r}\sin\theta.$$

It is also stated there without proof that

$$\frac{\partial r}{\partial y} = \sin\theta, \qquad \frac{\partial \theta}{\partial y} = \frac{1}{r}\cos\theta.$$

Show that's indeed the case.

2. (5 points)

The polar representation of the derivatives $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ have been calculated on slides 82 and 83. Complete the picture by calculating the polar representation of the mixed derivative $\frac{\partial^2 u}{\partial x \partial y}$.

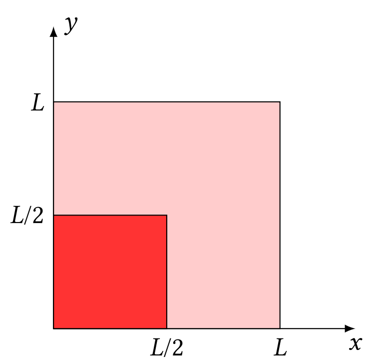
3. (10 points) Calculate the solution $u(x, y)$ of the steady-state heat conduction problem

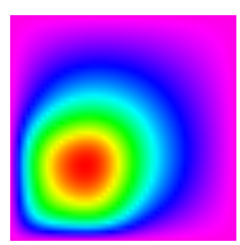
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x, y) = 0$$

in the square $(0, L) \times (0, L)$, where the heat source $f(x, y)$ is

$$f(x, y) = \begin{cases} 1 & \text{if } 0 < x < L/2 \text{ and } 0 < y < L/2, \\ 0 & \text{otherwise,} \end{cases}$$

and the temperature is maintained at zero all around the square's boundary.





The red square in the diagram on the left shows the part of the plate that is being heated. The picture on the right shows the resulting temperature distribution in the plate according to my calculations. Can you produce a picture like that based on your calculations? That will be get you 5 bonus points.