A Guided Tour of

Analytical Mechanics

with animations in MAPLE

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Contents

Preface

Unless otherwise specified, by "solving a problem" I mean performing all the steps laid out below:

1. Select configuration parameters.
2. Define the position vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots$ of the point masses in terms of the *generalized coordinates* $q_1, q_2, \ldots$.
3. Compute the velocities of the point masses:

$$\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j} \dot{q}_j, \quad i = 1, 2, \ldots.$$

4. Compute the kinetic energy $T = \frac{1}{2} \sum_i m_i \|\boldsymbol{v}_i\|^2$, the potential energy V, and the Lagrangian $L = T - V$.
5. Form the equations of motion (a system of second order differential equations (DEs)) in the unknowns $q_1(t), q_2(t), \ldots$:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}, \quad j = 1, 2, \ldots.$$

   If done by hand, this step would be the most labor-intensive part of the calculations. The calculations can get unbearably complex and can easily lead to formulas that fill more than one page. Fortunately we can relegate the tedious computations to MAPLE.[1]

6. Solve the system of DEs. Except for a few special cases, such system are generally not solvable in terms of elementary function. One solves them numerically with the help of specialized software such as MAPLE (or MATHEMATICA).

   The software replaces the continues time variable t by a closely spaced "time ticks" $t_0, t_1, t_2, \ldots$ which span the time interval of interest, say $[0, T]$, and then it applies some rather sophisticated numerical algorithms to evaluate the unknowns $q_1(t), q_2(t), \ldots$ at those time ticks. The result may be presented as:

   (a) a table of numbers; but that's not very illuminating, so it's rarely done that way;

[1] Nowadays MAPLE and MATHEMATICA are the two dominant *Computer Algebra Systems*. If you are familiar with MATHEMATICA, you should be able to translate the MAPLE commands in this book into the equivalent MATHEMATICA commands.

(b) as a set of plots of q_j versus t. This is the most common way. Both MAPLE (and MATHEMATICA) can do this easily; or

(c) as a computer animation, which is the most "user friendly" choice but which takes some work—and a certain amount of know-how—to produce. I will show you how to do this in MAPLE.

Chapter 1

An introduction through examples

This chapter introduces some of the basic ideas involved in the Lagrangian formulation of dynamics through examples. You will need to take some of the statements and formulas for granted since they won't be formally introduced until several chapters later. The objective here is to acquire some "gut feeling" for the subject which can help to motivate some of the abstract concepts that come later.

1.1 ▪ The simple pendulum à la Newton

A *pendulum*, specifically a *simple pendulum*, is a massless rigid rod of fixed length ℓ, one end of which is attached to, and can swing about, an immobile pivot, and to the other end of which is attached a point of mass m, called *the bob*.[2] The force of gravity tends to pull the pendulum down so that to bring the free end to the lowest possible position, called the pendulum's *stable equilibrium configuration*. A pendulum can stay motionless in the stable equilibrium configuration forever. If disturbed slightly away from the equilibrium, however, it will oscillate back and forth about it, indefinitely in principle if there are no frictional/dissipative effects. Figure 1.1 shows a simple pendulum at a generic position where the rod makes an angle φ relative to the vertical.

The pendulum may also be balanced in an inverted position, obtained by turning it upward about the pivot by 180 degrees (remember that the connecting rod is rigid.) That position, which admittedly is difficult to achieve in practice, is called the pendulum's *unstable equilibrium configuration*. A pendulum can stay motionless in the unstable equilibrium configuration forever, in principle. If disturbed slightly away from that equilibrium, however, it will move away from it in general.

The stable and unstable equilibria are the only possible equilibrium position of a simple pendulum. The pendulum cannot stay motionless at an angle, say at 45 degrees, relative to the vertical.

A pendulum's *initial condition*, that is, its state at time zero, completely determine its future motion. I am assuming here that the only external action on the pendulum is the force of gravity. The initial condition consists of a pair of data items, one being the initial angle that the rod makes relative to stable equilibrium position, and the other is the initial velocity with which the bob is set into motion.

As a specific instance, consider the case where the rod's initial angle is zero, and the

[2] The pendulum of a grandfather clock is a reasonably good example of such a pendulum, albeit the rod is not massless, and the mass attached to the end of it is not literally a point mass.

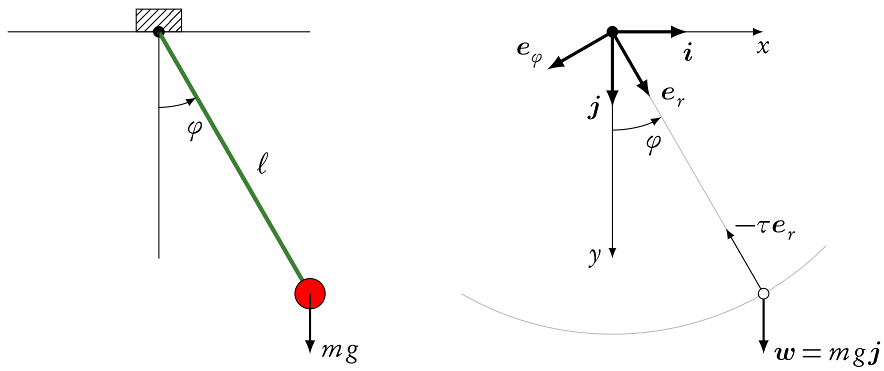


Figure 1.1: On the left is a depiction of the physical shape of the pendulum. On the right we see the mathematical machinery devised to analyze the pendulum's motion. The unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ are attached to the fixed Cartesian coordinates system and are stationary; the unit vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\varphi$ move with the pendulum. The weight of the bob is $\boldsymbol{w} = mg\boldsymbol{j}$.

bob's initial velocity is small. Then the pendulum will oscillate back and forth about the stable configuration, similar to what we see in a grandfather clock. If the initial velocity is slightly larger, the pendulum will undergo wider oscillations. If, however, the initial velocity is sufficiently large, the pendulum will not oscillate at all. It will swing about pivot, reach the unstable equilibrium position at the top and go past it, fall down from the other side, and return to its initial position, having made a complete 360 degree rotation about the pivot. At this point the pendulum finds itself in the same condition that it had at the initial time, therefore it will repeat what it did the first time around. In the absence of energy dissipating factors, the rotations about the pivot will continue indefinitely.

To make a mathematical model of the pendulum, we introduce the Cartesian coordinates xy with the origin at the pendulum's pivot, and the y axis pointing down. We also introduce the stationary unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ along the x and y axes, and the moving unit vectors $\boldsymbol{e}_r$ along the pendulum's rod and $\boldsymbol{e}_\varphi$ which is perpendicular to it, as shown in Figure 1.1. It it evident that the vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\varphi$ may be expressed as linear combinations of the vectors $\boldsymbol{i}$ and $\boldsymbol{j}$:

$$\boldsymbol{e}_r = \boldsymbol{i}\sin\varphi + \boldsymbol{j}\cos\varphi, \qquad \boldsymbol{e}_\varphi = -\boldsymbol{i}\cos\varphi + \boldsymbol{j}\sin\varphi.$$

Furthermore, let us observe that their time derivatives are related through

$$\dot{\boldsymbol{e}}_r = \boldsymbol{i}\dot\varphi\cos\varphi - \boldsymbol{j}\dot\varphi\sin\varphi = -\dot\varphi\boldsymbol{e}_\varphi, \qquad \dot{\boldsymbol{e}}_\varphi = \boldsymbol{i}\dot\varphi\sin\varphi + \boldsymbol{j}\dot\varphi\cos\varphi = \dot\varphi\boldsymbol{e}_r. \tag{1.1}$$

The bob's position vector $\boldsymbol{r}(t)$ relative to the origin is $\boldsymbol{r} = \ell\boldsymbol{e}_r$, where ℓ is the length of the rod, and therefore the bob's velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$ and acceleration $\boldsymbol{a} = \dot{\boldsymbol{v}}$ may be computed easily with the help of (1.1):

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = (\ell\boldsymbol{e}_r)^{\cdot} = -\ell\dot\varphi\boldsymbol{e}_\varphi, \qquad \boldsymbol{a} = \dot{\boldsymbol{v}} = (-\ell\dot\varphi\boldsymbol{e}_\varphi)^{\cdot} = -\ell\ddot\varphi\boldsymbol{e}_\varphi - \ell\dot\varphi\dot{\boldsymbol{e}}_\varphi = -\ell\ddot\varphi\boldsymbol{e}_\varphi - \ell\dot\varphi^2\boldsymbol{e}_r.$$

We see that the bob's acceleration has a component along $\boldsymbol{e}_\varphi$ and another along $\boldsymbol{e}_r$.

Newton's law of motion asserts that $m\boldsymbol{a} = \boldsymbol{F}$, where $\boldsymbol{F}$ is the resultant of all forces acting on the bob. Referring to Figure 1.1 we see that the forces acting on the bob consist of weight $\boldsymbol{w}$ and the tension $-\tau\boldsymbol{e}_r$ along the rod,[3] where τ generally varies with time and is unknown. It follows that

$$m(-\ell\ddot{\varphi}\boldsymbol{e}_\varphi - \ell\dot{\varphi}^2\boldsymbol{e}_r) = \boldsymbol{w} - \tau\boldsymbol{e}_r.$$

The weight, however, is $\boldsymbol{w} = mg\boldsymbol{j}$, where m is the mass of the bob and g is the acceleration due to gravity. We replace $\boldsymbol{w}$ with its decomposition $\boldsymbol{w} = mg\boldsymbol{j} = (mg\cos\varphi)\boldsymbol{e}_r + (mg\sin\varphi)\boldsymbol{e}_\varphi$ in the equation of motion, and collect the coefficients of $\boldsymbol{e}_r$ and and $\boldsymbol{e}_\varphi$, and arrive at

$$\big[m\ell\ddot{\varphi} + mg\sin\varphi\big]\boldsymbol{e}_\varphi + \big[m\ell\dot{\varphi}^2 + mg\cos\varphi - \tau\big]\boldsymbol{e}_r = \boldsymbol{0}.$$

Since $\boldsymbol{e}_r$ and $\boldsymbol{e}_\varphi$ are orthogonal, hence linearly independent, each of the expressions in the square brackets is zero. We conclude that

$$m\ell\ddot{\varphi} + mg\sin\varphi = 0, \qquad m\ell\dot{\varphi}^2 + mg\cos\varphi - \tau = 0. \tag{1.2}$$

The first equation is a second order differential equation in the unknown φ. It has a unique solution for any initial condition $\{\varphi(0), \dot{\varphi}(0)\}$, although the solution is not expressible in terms of elementary functions. In practice, one solves the equation through a numerical approximation algorithm on a computer. Once the solution $\varphi(t)$ is obtained, it may be substituted in the second equation to evaluate the tension $\tau(t)$ in the rod, should it be of interest.

1.2 ▪ The simple pendulum à la Euler

In the previous section we assumed, without explanation, that the force within the pendulum's rod points along the rod; see Figure 1.1 where that force is shown as the vector $-\tau\boldsymbol{e}_r$.

That assumption seems to be so "obvious" that many textbooks on mechanics and its applications present it without as much as a comment. A close scrutiny, however, shows that the assumption is far from obvious, and in fact, it is not a logical consequence of any of Newton's laws of motion. Antman [2] presents a critical analysis of this issue and concludes that the proper approach is through an application of *Euler's law of motion*, which states that the rate of change of the pendulum's angular momentum equals the resultant torque applied to it.

1.3 ▪ The simple pendulum à la Lagrange

In this section we rederive the differential equation of motion of the simple pendulum through Lagrange's analytical approach. We no longer need the vectors $\boldsymbol{e}_r$ and $\boldsymbol{e}_\varphi$. Instead, we write the bob's position vector $\boldsymbol{r}$ directly in terms of its $\boldsymbol{i}$ and $\boldsymbol{j}$ components:

$$\boldsymbol{r} = (\ell\sin\varphi)\boldsymbol{i} + (\ell\cos\varphi)\boldsymbol{j},$$

and then differentiate to find the velocity

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = (\ell\dot{\varphi}\cos\varphi)\boldsymbol{i} - (\ell\dot{\varphi}\sin\varphi)\boldsymbol{j}.$$

It follows that that $||\boldsymbol{v}||^2 = \ell^2\dot{\varphi}^2$.

To proceed further, we introduce a few definitions and assertions whose motivations and explanations will emerge only in subsequent chapters.

[3]The assertion that the force exerted on the bob by the rod lies along the rod requires justification. See the next section for elaboration.

- A the *kinetic energy* T of a point mass m moving with velocity $\boldsymbol{v}$ is $T = \frac{1}{2}m\|\boldsymbol{v}\|^2$. In the case of the pendulum this is $T = \frac{1}{2}m\ell^2\dot{\varphi}^2$.
- The *potential energy* V of a point mass m in a constant gravitational field equals mgh where g is the acceleration due to gravity, and h is its height *above* an arbitrarily selected reference point. In the case of the pendulum, the elevation of the bob relative to the lowest point in its path is $h = \ell(1 - \cos\varphi)$, therefore $V = mg\ell(1 - \cos\varphi)$.
- The *Lagrangian* L of a mechanical system is the difference between its kinetic and potential energies, that is, $L = T - V$. In the case of the pendulum we have:

$$L(\varphi, \dot{\varphi}) = \frac{1}{2}m\ell^2\dot{\varphi}^2 - mg\ell(1 - \cos\varphi). \tag{1.3}$$

As the notation above indicates, we are viewing the Lagrangian L as a function two variables φ and $\dot{\varphi}$. It should be emphasized that φ and $\dot{\varphi}$ are considered independent variables here.[4]

The Lagrangian completely characterizes a mechanical system. It incorporates the system's parameters, geometry, and physics, all in one neat bundle. Beyond this point the analysis of the system's motion is pure calculus—or analysis, as Lagrange called it in his *Mécanique Analytique*—with no need to refer to the system's components and geometry.

According to Lagrange's theory which we will later study in detail, the equation of motion of a mechanical system whose Lagrangian depends on two variables φ and $\dot{\varphi}$, is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) = \frac{\partial L}{\partial \varphi}. \tag{1.4}$$

In the case of pendulum we have:

$$\frac{\partial L}{\partial \dot{\varphi}} = m\ell^2\dot{\varphi}, \qquad \frac{\partial L}{\partial \varphi} = -mg\ell\sin\varphi,$$

and therefore the equation of motion is

$$(m\ell^2\dot{\varphi})^{\cdot} = -mg\ell\sin\varphi,$$

or equivalently,

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0, \tag{1.5}$$

which agrees with the first equation in (1.2). The second of those equations may be obtained through the Lagrangian approach as well, but we will not get into that right now.

1.4 ▪ The double pendulum

A *double pendulum* is obtained by suspending a second pendulum from the bob of a first pendulum, as shown in the left diagram in Figure 1.2. The double pendulum's geometric configuration is specified through the two angles φ_1 and φ_2 that the rods make relative to the vertical.

[4] If you find the notation $\dot{\varphi}$ confusing in that regard, consider renaming it to ω, as in

$$L(\varphi, \omega) = \frac{1}{2}m\ell^2\omega^2 - mg\ell(1 - \cos\varphi).$$

Now L is a function of two independent variables φ and ω.

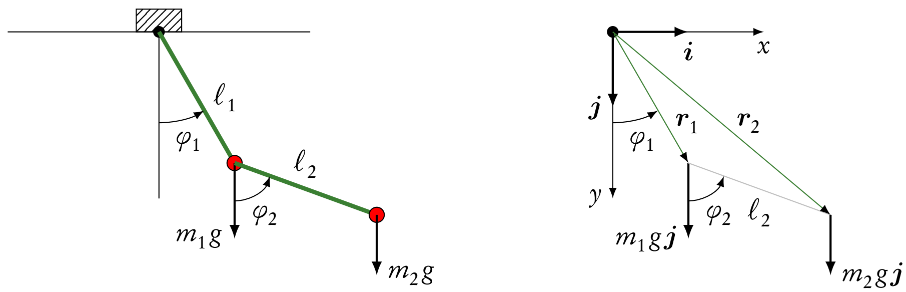


Figure 1.2: On the left is a depiction of the physical shape of the double pendulum. On the right we see the pendulum's mathematical model given by the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of the two bobs.

To make a mathematical model of a double pendulum, we follow the ideas sketched in the previous section. Specifically, we introduce the xy Cartesian coordinates and the stationary unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ as shown in Figure 1.2, and then express the position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of the two bobs in terms of their components relative to $\boldsymbol{i}$ and $\boldsymbol{j}$:

$$\boldsymbol{r}_1 = (\ell_1 \sin\varphi_1)\boldsymbol{i} + (\ell_1 \cos\varphi_1)\boldsymbol{j}, \qquad \boldsymbol{r}_2 = \boldsymbol{r}_1 + (\ell_2 \sin\varphi_2)\boldsymbol{i} + (\ell_2 \cos\varphi_2)\boldsymbol{j}. \tag{1.6}$$

Then we find the velocities of the bobs through differentiation:

$$\boldsymbol{v}_1 = (\ell_1 \dot\varphi_1 \cos\varphi_1)\boldsymbol{i} - (\ell_1 \dot\varphi_1 \sin\varphi_1)\boldsymbol{j}, \quad \boldsymbol{v}_2 = \boldsymbol{v}_1 + (\ell_2 \dot\varphi_2 \cos\varphi_2)\boldsymbol{i} - (\ell_2 \dot\varphi_2 \sin\varphi_2)\boldsymbol{j}.$$

We see that $\|\boldsymbol{v}_1\|^2 = \ell_1^2 \dot\varphi_1^2$. Computing $\|\boldsymbol{v}_2\|^2$ takes only a little bit more work. We observe that $\boldsymbol{v}_2 = \boldsymbol{v}_1 + \tilde{\boldsymbol{v}}$, where $\tilde{\boldsymbol{v}} = (\ell_2 \dot\varphi_2 \cos\varphi_2)\boldsymbol{i} - (\ell_2 \dot\varphi_2 \sin\varphi_2)\boldsymbol{j}$. Therefore

$$\begin{aligned}
\|\boldsymbol{v}_2\|^2 &= \|\boldsymbol{v}_1\|^2 + \|\tilde{\boldsymbol{v}}\|^2 + 2\boldsymbol{v}_1 \cdot \tilde{\boldsymbol{v}} \\
&= \ell_1^2\dot\varphi_1^2 + \ell_2^2\dot\varphi_2^2 + 2\big[(\ell_1\dot\varphi_1\cos\varphi_1)\boldsymbol{i} - (\ell_1\dot\varphi_1\sin\varphi_1)\boldsymbol{j}\big]\cdot\big[(\ell_2\dot\varphi_2\cos\varphi_2)\boldsymbol{i} - (\ell_2\dot\varphi_2\sin\varphi_2)\boldsymbol{j}\big] \\
&= \ell_1^2\dot\varphi_1^2 + \ell_2^2\dot\varphi_2^2 + 2\ell_1\ell_2\dot\varphi_1\dot\varphi_2(\cos\varphi_1\cos\varphi_2 + \sin\varphi_1\sin\varphi_2). \\
&= \ell_1^2\dot\varphi_1^2 + \ell_2^2\dot\varphi_2^2 + 2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\cos(\varphi_2 - \varphi_1).
\end{aligned}$$

We conclude that the double pendulum's kinetic energy is

$$\begin{aligned}
T &= \frac{1}{2}m_1\ell_1^2\dot\varphi_1^2 + \frac{1}{2}m_2\big[\ell_1^2\dot\varphi_1^2 + \ell_2^2\dot\varphi_2^2 + 2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\cos(\varphi_2-\varphi_1)\big] \\
&= \frac{1}{2}(m_1+m_2)\ell_1^2\dot\varphi_1^2 + \frac{1}{2}m_2\ell_2^2\dot\varphi_2^2 + m_2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\cos(\varphi_2-\varphi_1).
\end{aligned}$$

As to the potential energy, let us recall that a mass's potential energy in a constant gravitational field is the product of its weight and its elevation above a certain reference point. In the case of a double pendulum, it is easiest to set the reference point at the origin of the coordinates; see Figure 1.2. Then the $\boldsymbol{j}$ components of the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ provide the elevations of the bobs *below* the reference point, therefore their elevations *above* the reference point will require a sign reversal. Referring to (1.6) we see that

$$V = -m_1 g\cos\varphi_1 - m_2 g\big[\ell_1\cos\varphi_1 + \ell_2\cos\varphi_2\big] = -(m_1+m_2)g\cos\varphi_1 - m_2 g\ell_2\cos\varphi_2.$$

Thus, the double pendulum's Lagrangian, $L = T - V$, takes the form

$$\begin{aligned}
L(\varphi_1,\varphi_2,\dot\varphi_1,\dot\varphi_2) = \frac{1}{2}(m_1+m_2)\ell_1^2\dot\varphi_1^2 + \frac{1}{2}m_2\ell_2^2\dot\varphi_2^2 + m_2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\cos(\varphi_2-\varphi_1) \\
+ (m_1+m_2)g\cos\varphi_1 + m_2 g\ell_2\cos\varphi_2.
\end{aligned}$$

In the previous section's simple pendulum, the Lagrangian $L(\varphi,\dot\varphi)$ was a function two variables. In the present case, the Lagrangian $L(\varphi_1,\varphi_2,\dot\varphi_1,\dot\varphi_2)$ is a function of four variables. In general, if a mechanical system's geometric configuration is specified through n variables $q_1,\dots,q_n$, then its Lagrangian is a function of $2n$ variables $q_1,\dots,q_n,\dot q_1,\dots,\dot q_n$. The equivalent of the single equation of motion (1.4) now is a system of n equations, called the mechanical system's *Euler–Lagrange equations*:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)=\frac{\partial L}{\partial q_i},\qquad i=1,\dots,n.$$

The variable $q_1,\dots,q_n$ are called the system's *generalized coordinates*.

Applied to the case of double pendulum, the Euler–Lagrange equations lead to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_1}\right)=\frac{\partial L}{\partial \varphi_1},\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi_2}\right)=\frac{\partial L}{\partial \varphi_2}.$$

To evaluate these explicitly, we begin by computing

$$\begin{aligned}
\frac{\partial L}{\partial \dot\varphi_1}&=(m_1+m_2)\ell_1^2\dot\varphi_1+m_2\ell_1\ell_2\dot\varphi_2\cos(\varphi_2-\varphi_1),\\
\frac{\partial L}{\partial \dot\varphi_2}&=m_2\ell_2^2\dot\varphi_2+m_2\ell_1\ell_2\dot\varphi_1\cos(\varphi_2-\varphi_1),\\
\frac{\partial L}{\partial \varphi_1}&=m_2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\sin(\varphi_2-\varphi_1)-(m_1+m_2)g\sin\varphi_1,\\
\frac{\partial L}{\partial \varphi_2}&=-m_2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\sin(\varphi_2-\varphi_1)-m_2g\ell_2\sin\varphi_2.
\end{aligned}$$

We conclude that the differential equations of motion are

$$\begin{aligned}
&\left[(m_1+m_2)\ell_1^2\dot\varphi_1+m_2\ell_1\ell_2\dot\varphi_2\cos(\varphi_2-\varphi_1)\right]^{\cdot}\\
&\qquad\qquad=m_2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\sin(\varphi_2-\varphi_1)-(m_1+m_2)g\sin\varphi_1,\\
&\left[m_2\ell_2^2\dot\varphi_2+m_2\ell_1\ell_2\dot\varphi_1\cos(\varphi_2-\varphi_1)\right]^{\cdot}\\
&\qquad\qquad=-m_2\ell_1\ell_2\dot\varphi_1\dot\varphi_2\sin(\varphi_2-\varphi_1)-m_2g\ell_2\sin\varphi_2.
\end{aligned}$$

Exercises

1.1. **Pendulum with a mobile pivot.** Figure 1.3 shows a pendulum whose pivot is allowed to move horizontally without friction. The pivot has mass m_1 while the bob has mass m_2. Find the equations of motion of the pendulum.

1.2. **A spherical pendulum.** The motion of the simple pendulum of length ℓ introduced in this chapter was confined to a single vertical plane, and therefore the pendulum's bob moved along a circular arc of radius ℓ. If off-plane motions are permitted, then the bob will move on a sphere of radius ℓ centered at the pivot. In that setting the pendulum is called a *spherical pendulum*; see Figure 1.4.
Derive the equations of motion of the spherical pendulum.

1.3. **Bead on a spinning hoop.** A circular wire hoop of radius R spins about a vertical diameter at a constant angular velocity Ω. A bead of mass m can slide without friction along the hoop. The hoop's radius that connects to the bead makes an angle

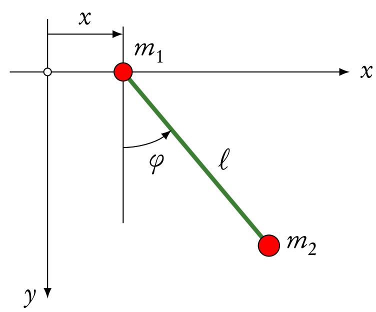


Figure 1.3: Pendulum with a horizontally mobile pivot (Exercise 1.1).

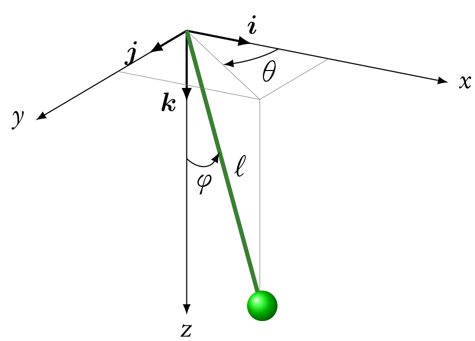


Figure 1.4: A spherical pendulum (Exercise 1.2).

of $\varphi(t)$ with respect to the vertical; see Figure 1.5. Find the differential equation satisfied by φ.

1.4. **A governor mechanism.** Figure 1.6 is a schematic drawing of a (simplified) Watt governor which was invented for the automatic control of the speed of steam engines. Our version consists of four massless rigid links of length ℓ each, hinged at their ends to form a rhombus. The vertex O remains motionless, while the sleeve at vertex S can slide on the device's vertical shaft, thereby change the rhombus's shape. Two balls of mass m_1 each are attached to the vertices A and B. The sleeve's mass is m_2. The entire assembly rotates at a constant angular speed Ω about the vertical shaft. Find the differential equation satisfied by the angle φ marked on the diagram.

1.5. **Two masses on a string.** A particle P of mass m_1 lies on a smooth horizontal table and is attached to a long, inextensible string which passes through a smooth hole O in the table and hangs down. The other end of the string carries a particle Q of mass m_2; see the illustration in Figure 1.7.
The particle P is positioned at the point $(a, 0, 0)$ in the xyz coordinates shown, and given a horizontal initial velocity perpendicular to the x axis. Find the differential equations of motion.
Hint: Let $\rho(t)$ and $\varphi(t)$ be P's position at time t in polar coordinates as seen in Figure 1.7. The equations of motions constitute a system of differential in $\rho(t)$ and $\varphi(t)$.

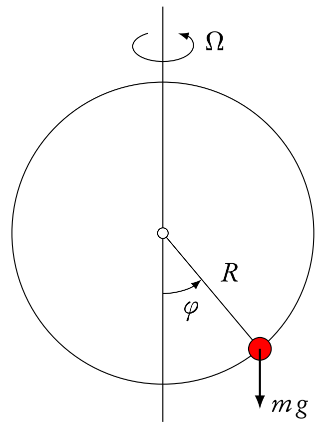


Figure 1.5: Bead on a rotating hoop (Exercise 1.3).

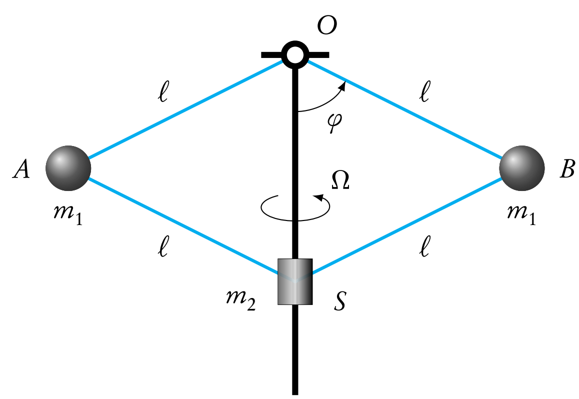


Figure 1.6: A simplified Watt governor (Exercise 1.4).

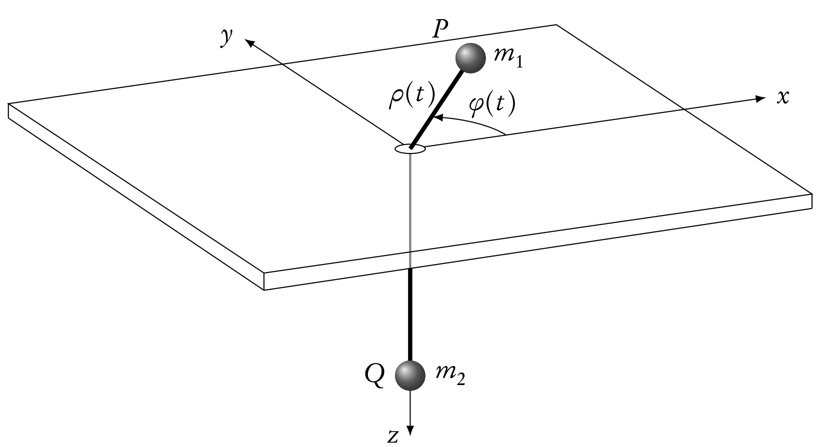


Figure 1.7: The point P slides on the table. The point Q moves vertically (Exercise 1.5).

Chapter 2

Work and potential energy

Work is the product of force and its displacement. To be precise, the infinitesimal work dW performed in displacing a force $\boldsymbol{F}$ by an infinitesimal distance $d\boldsymbol{r}$ is $dW = \boldsymbol{F} \cdot d\boldsymbol{r}$. If the point of the application of the force moves along a path C in space, then the work performed along the path is the line integral

$$W = \int_C \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{2.1}$$

If you are repositioning a massive desk in an office, for example, then work measures the amount of effort exerted by you in performing the task.

Expanding upon the moving of the desk scenario, suppose that you intend to move the desk from a point A to a point B. It should be obvious that the amount of work performed will vary, depending on the path along which you move the desk between A and B. Chances are that the shortest (straight line) path will require lesser effort than a long path that winds around the office.

There are many interesting and important situations where, unlike the moving of the desk example, the work performed in goings from a point A to a point B is *independent of the path* taken between A and B. The most elementary example is the raising or lowering of a weight. To see how it works, set up a Cartesian coordinate system in space so that the x and y axes are horizontal, and the z axis points up. Let $\boldsymbol{r}_a = (x_a, y_a, z_a)$ and $\boldsymbol{r}_b = (x_b, y_b, z_b)$ be the position vectors[5] of the starting and ending points A and B, and let $\boldsymbol{r} = \langle x, y, z\rangle$ be the position vector of a generic point along a path $C(A,B)$ with endpoints A and B. Suppose that we move an object of mass m along that path. The force of the object's weight is $\boldsymbol{F} = \langle 0, 0, -mg\rangle$, where g is the acceleration of gravity. The work performed along the path is

$$\begin{aligned} W &= \int_{C(A,B)} \boldsymbol{F} \cdot d\boldsymbol{r} \\ &= \int_{C(A,B)} \langle 0, 0, -mg\rangle \cdot \langle dx, dy, dz\rangle = \int_{C(A,B)} -mg\,dz = -mg(z_b - z_a). \end{aligned}$$

We see that the work in moving the weight from A to B is expressed in terms the z coordinates of the endpoints, thus it is the same on all conceivable paths that go from A to B.

[5] A *position vector* of a point $P(x, y,)$ is the vector $\boldsymbol{r} = \langle x, y, z\rangle$ that extends from the origin to the point P.

To generalize, consider a (possibly position dependent) force field $\boldsymbol{F}(\boldsymbol{r})$ with the property that the work performed in going from a given point A to an arbitrary point $\boldsymbol{r}$ in space is independent of the path from A to $\boldsymbol{r}$. Let us write $V(\boldsymbol{r})$ for the negative of the value of that integral, that is,

$$V(\boldsymbol{r}) = -\int_{C(A,\boldsymbol{r})} \boldsymbol{F}(\boldsymbol{r}') \cdot d\boldsymbol{r}'. \tag{2.2}$$

The function V defined this way is called *the potential function*, or simply *the potential*, of the the vector field $\boldsymbol{F}$. Equivalently, the vector field $\boldsymbol{F}$ is said to be *derived from a potential*. In (2.2) I have written $\boldsymbol{r}'$ for the dummy variable of integration in order to distinguish it from the position vector $\boldsymbol{r}$ which designates the path's endpoint. The minus sign does not have a deep significance; it's convenient to build it into the definition since it leads to more pleasing forms of general statements, such as the one on conservation of energy.

Theorem 2.1. *Consider a continuous vector field $\boldsymbol{F}$ defined in an open and connected domain $\mathcal{D}$ in the n-dimensional space, and suppose that $\boldsymbol{F}$ possesses a potential function V as in* (2.2)*. Then V is differentiable and $\boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r})$.*

Proof. By definition, the gradient ∇V of a function V at a point $\boldsymbol{r}$ is the vector with the property that for any unit vector $\boldsymbol{e}$, the directional derivative of V in the direction of $\boldsymbol{e}$ is given by $\nabla V(\boldsymbol{r}) \cdot \boldsymbol{e}$. That is,

$$\nabla V(\boldsymbol{r}) \cdot \boldsymbol{e} = \lim_{h \to 0} \frac{V(\boldsymbol{r} + h\boldsymbol{e}) - V(\boldsymbol{r})}{h}.$$

To simplify the discussion, let us write P and Q for the points in space corresponding to the position vectors $\boldsymbol{r}$ and $\boldsymbol{r} + h\boldsymbol{e}$, as illustrated in Figure 2.1. Pick a path $C(A, \boldsymbol{r})$ to evaluate V at P, then extend that path as a straight line segment to Q to evaluate V at Q. Then the difference $V(Q) - V(P)$ amounts to an integration along the straight segment PQ:

$$V(\boldsymbol{r} + h\boldsymbol{e}) - V(\boldsymbol{r}) = V(Q) - V(P) = -\int_0^h \boldsymbol{F}(\boldsymbol{r} + \xi\boldsymbol{e}) \cdot \boldsymbol{e}\, d\xi,$$

Then it follows that

$$\frac{V(\boldsymbol{r} + h\boldsymbol{e}) - V(\boldsymbol{r})}{h} = -\frac{1}{h}\int_0^h \boldsymbol{F}(\boldsymbol{r} + \xi\boldsymbol{e}) \cdot \boldsymbol{e}\, d\xi = -\boldsymbol{F}(\boldsymbol{r} + \hat{\xi}\boldsymbol{e}) \cdot \boldsymbol{e}$$

for some $\hat{\xi} \in (0, h)$, the latter assertion being a consequence of the Mean Value Theorem for integrals; see e.g., Stewart [13].

As h goes to zero, so does $\hat{\xi}$ because $\hat{\xi} \in (0, h)$. It follows that

$$\nabla V(\boldsymbol{r}) \cdot \boldsymbol{e} = \lim_{h \to 0} \frac{V(\boldsymbol{r} + h\boldsymbol{e}) - V(\boldsymbol{r})}{h} = -\boldsymbol{F}(\boldsymbol{r}) \cdot \boldsymbol{e}.$$

Since this holds for every $\boldsymbol{e}$, it follows that $\nabla V(\boldsymbol{r}) = -\boldsymbol{F}(\boldsymbol{r})$. □

Remark 2.1. Let is point out the roles that the theorem's technical assumptions play in the proof:

- The *continuity* of the vector field $\boldsymbol{F}$ enters at the point where the Mean Value Theorem is applied. That theorem is not true without continuity.

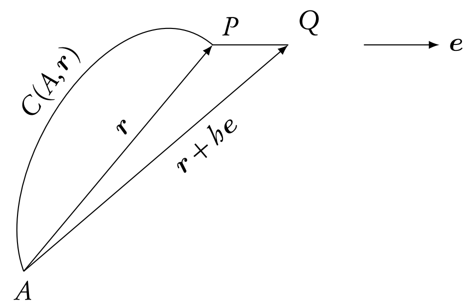


Figure 2.1: If $V(P)$ is evaluated by integration along the path $C(A,\boldsymbol{r})$, then $V(Q)$ may be evaluated by integrating along that same path, and then continuing along the straight line segment PQ of length h in the direction $\boldsymbol{e}$.

- The assumption that the domain $\mathscr{D}$ is *connected* is needed to ensure that a path may be established between any pair of points in $\mathscr{D}$. That's what enabled us to sketch the curve $C(A,\boldsymbol{r})$ that connects the points A and P in Figure 2.1.
- The assumption that the domain $\mathscr{D}$ is *open* means that a ball of positive radius may be placed around any point $P \in \mathscr{D}$ so that the ball lies entirely within $\mathscr{D}$. It's that property which ensures that moving away from P by a small distance h, as we did in Figure 2.1, we land safely on a point Q which lies within $\mathscr{D}$.

Remark 2.2. Adding a constant to the potential function V does not affect the equality $\boldsymbol{F}(\boldsymbol{r}) = \nabla V(\boldsymbol{r})$. Thus, a vector field's potential, if it has one, is defined modulo an additive constant.

Example 2.2. Earlier in the this section we observed that the force field corresponding to an object's weight is $\boldsymbol{F} = \langle 0, 0, -mg \rangle$. We see that $\boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r})$ where $V(x,y,z) = mgz$. We will use mgz as the potential of a weight throughout these notes. Note the effect of the minus sign in (2.2); in its absence the potential of a weight would have been $-mgz$.

Example 2.3. In the previous example we assumed that the acceleration of gravity g is a constant. That's a good assumption if the changes in height during the motion are small relative to the radius of the Earth. In general, the gravitational force that a point mass M exerts on a point mass m drops as the inverse square of the distance. Specifically, Newton's law of gravitation says

$$\boldsymbol{F}(\boldsymbol{r}) = -\left(\frac{GMm}{\|\boldsymbol{r}\|^2}\right)\frac{\boldsymbol{r}}{\|\boldsymbol{r}\|}. \tag{2.3}$$

where $\boldsymbol{r}$ is m's position vector relative to M, and G is the *universal gravitational constant*. The inverse square law is manifested through the $\|\boldsymbol{r}\|^2$ term that appears in the denominator inside the parentheses. The factor $\boldsymbol{r}/\|\boldsymbol{r}\|$ is a unit vector that points from M to m. It is possible to show (see Exercise **??**) that $\boldsymbol{F}$ is derived from a potential.

Theorem 2.4. *Suppose the force field $\boldsymbol{F}$ is derived from a potential V. Then the work per-*

formed in moving the force along any path from a point $\boldsymbol{r}_a$ to $\boldsymbol{r}_b$ is given by

$$W = V(\boldsymbol{r}_a) - V(\boldsymbol{r}_b). \tag{2.4}$$

Proof. We have $\boldsymbol{F}(\boldsymbol{r}) = -\nabla V(\boldsymbol{r})$ therefore

$$\begin{aligned}\boldsymbol{F}\cdot d\boldsymbol{r} = -\nabla V(\boldsymbol{r})\cdot d\boldsymbol{r} = -\Big\langle \frac{\partial V}{\partial x_1}, \dots \frac{\partial V}{\partial x_n}\Big\rangle \cdot \langle dx_1, \dots, dx_n\rangle \\ = -\Big(\frac{\partial V}{\partial x_1}dx_1 + \cdots + \frac{\partial V}{\partial x_n}dx_n\Big) = -dV,\end{aligned}$$

therefore

$$W = \int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_{\boldsymbol{r}_a}^{\boldsymbol{r}_b} -dV = V(\boldsymbol{r}_a) - V(\boldsymbol{r}_b).$$

□

Exercises

2.1. Verify that the gravitational potential field $\boldsymbol{F}(\boldsymbol{r})$ in (2.3) is derived from the potential

$$V(\boldsymbol{r}) = \frac{GMm}{||\boldsymbol{r}||}.$$

Chapter 3

A single particle in a conservative force field

3.1 ▪ The principle of conservation of energy

Newton's law of motion, $\boldsymbol{F} = m\ddot{\boldsymbol{r}}$ relates the acceleration $\ddot{\boldsymbol{r}}$ of a point of constant mass m subjected to a force $\boldsymbol{F}$. If the force is derived from a potential $V(\boldsymbol{r})$, that is, $\boldsymbol{F} = -\nabla V$, then the law of motion takes the form

$$m\ddot{\boldsymbol{r}} = -\nabla V(\boldsymbol{r}). \tag{3.1}$$

Multiplying this through by the velocity $\dot{\boldsymbol{r}}$

$$m\dot{\boldsymbol{r}} \cdot \ddot{\boldsymbol{r}} = -\nabla V(\boldsymbol{r}) \cdot \dot{\boldsymbol{r}},$$

and then rearranging

$$\frac{1}{2} m(\dot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}})^{\cdot} + \big(V(\boldsymbol{r})\big)^{\cdot} = 0,$$

we arrive at

$$\frac{d}{dt}\Big(\frac{1}{2} m\|\dot{\boldsymbol{r}}\|^2 + V(\boldsymbol{r})\Big) = 0,$$

which tells us that the quantity

$$E = \frac{1}{2} m\|\dot{\boldsymbol{r}}\|^2 + V(\boldsymbol{r}) \tag{3.2}$$

remains constant during the motion. The constant E is called the particle's *mechanical energy* (or just the *energy* for short). The first term on the right-hand side of (3.2) is called the *kinetic energy*; the second term is called the *potential energy*. The constancy of E in a motion is called the *principle of conservation of energy*.

Remark 3.1. The kinetic and potential energies don't remain constant during the motion; it's their *sum* that does. Therefore the reduction of one is accompanied by the increase of the other. It helps to think of this as a conversion of one form of energy to the other. The myriad of motion phenomena encountered in our daily experiences are manifestations of such interplay between the kinetic and potential energies.

Remark 3.2. The conservation of the total energy E is a consequence of the assumption that the force field $\boldsymbol{F}$ is derived from a potential. This should explain the alternative name, a *conservative force field*, which is commonly used to refer to a force field derived from a potential.

Remark 3.3. Had we chosen against putting the minus sign in the definition (2.2), the principle of conservation of energy would have stated that the *difference* between the kinetic and potential energies remains constant, which is not as appealing as saying that their sum remains a constant.

3.2 ▪ The scalar case

The rest of this chapter is devoted to a study of the scalar version of equation (3.1), that is,

$$m\ddot{x} = -V'(x), \tag{3.3}$$

where x is scalar, and V' is the derivative of a potential V. In addition to the obvious applications in one-dimensional dynamics, this equation occurs in quite a number of other context which are far from one-dimensional motions. The equation of motion of a simple pendulum (1.5), for instance, falls in this category, but the motion is certainly not linear. We will more on this in Section 3.4.

The previous section's statement on conservation of energy, which in the scalar case takes the form

$$E = \frac{1}{2}m\dot{x}^2 + V(x), \tag{3.4}$$

is a first order differential equation in the unknown $x(t)$, and which may be solved, in principle, through a separation of variables. We have $\dot{x}^2 = \frac{2}{m}\big(E - V(x)\big)$, therefore $\dot{x} = \pm\sqrt{\frac{2}{m}\big(E - V(x)\big)}$, and hence

$$\int \frac{dx}{\sqrt{2\big(E - V(x)\big)}} = \pm\int dt = \pm t + C.$$

Expect for the most trivial cases, the integral on the left is impossible to evaluate in terms of elementary functions. It is possible, however, to obtain quite an adequate "feel" for the solution, without performing any integration at all, through a *phase plane analysis* of the equation.

To explain the idea, consider the potential function V whose graph is shown in Figure 3.1(a). Regard the solution $x(t)$ of the differential equation (3.3) as the abscissa of a point P that moves along the horizontal axis in that figure according to the equation's dynamics. Then the point Q with coordinates $\big(x(t), V(x(t)\big)$ slides on the graph of V accordingly. The length of the line segment PQ equals the potential energy $V(x)$. We extend that segment upward to a point R so the the length of QR equals the kinetic energy $\frac{1}{2}m\dot{x}^2$. Since the sum of the kinetic and potential energies remains a constant E during the motion, the locus of the point R is the horizontal line $V = E$, as marked on the figure.

The point Q cannot rise above the horizontal line $V = E$ during the motion because the nonnegative length of the line segment QR (which equals $\frac{1}{2}m\dot{x}^2$) prevents it. Consequently, the motion of Q is confined to the graph's red-colored arc. We refer to that arc as a *potential well corresponding to the energy* E and we think of Q as a point that has fallen into the well and is unable to get out.

At the edges of the potential well the potential energy equals E and the kinetic energy, and therefore the velocity $\dot{x}$, are zero. In the interior, where the potential energy

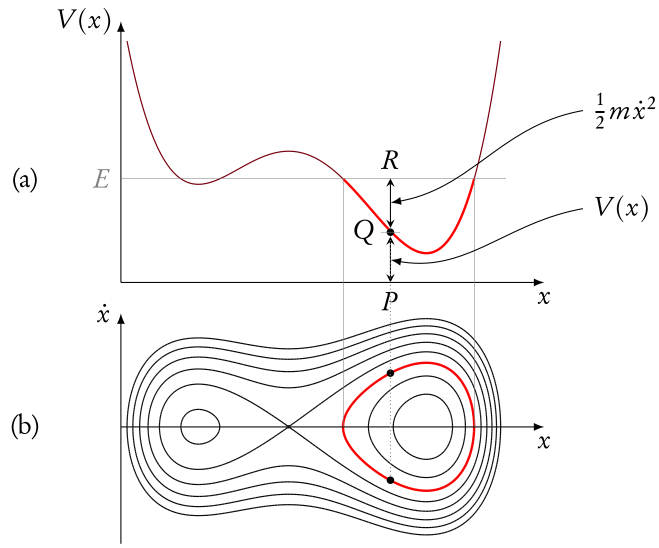


Figure 3.1: The dynamics of the equation $\ddot{x} + V'(x) = 0$ is completely determined by the potential function V. The figure on top shows the graph of $V(x)$, and an energy well corresponding to an energy level E. The coordinate x is confined to the energy well shown in red. Since the total energy is conserved, as the potential energy drops below E within the well, the kinetic energy increases, resulting in the phase portrait shown in the bottom figure.

drops below E, the kinetic energy, and therefore the velocity squared, $\dot{x}^2$, are positive. We conclude that as we traverse the potential well from left to right, the velocity begins at zero, increases gradually (in absolute value) to a maximum, then drops back to zero at the rightmost end. The sign of the velocity may be positive or negative since the only information we are getting from Figure 3.1(a) is on the square of the velocity.

This observation leads to the red oval in the diagram shown in Figure 3.1(b). The horizontal axis in that figure is the same as the x in Figure 3.1(a). The vertical axis is the velocity $\dot{x}$. Observe that at the leftmost and rightmost points of the oval, which correspond to the extremes of the potential well, the velocity is zero, and in between in rises to a maximum (or falls to a minimum), as we expect. The oval is symmetric with respect to the x axis because a given $\dot{x}^2$ yields two velocities $\pm|\dot{x}|$.

The red oval constructed in the previous discourse depends on the choice of the energy level E. It should be clear that lowering E slightly will shrink the oval, and raising E slightly, will expand it. The black curves in Figure 3.1(b) are the result of selecting various values of E.

Figure 3.1(b) is called the *phase diagram* or *phase portrait* of the differential equation (3.3). The curves in it are called *orbits*. An alternative to the geometric construction of the orbits carried out above, we may equally well view them as implicitly defined curves in the x–$\dot{x}$ plane through the equation (3.4). Varying the parameter E produces the family of all orbits, some of which are shown in Figure 3.1(b).

Yet another way of viewing the orbits is as level curves of the the surface defined by the function

$$E(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + V(x).$$

Two views of the surface $E(x, \dot{x})$ corresponding to the potential $V(x)$ of Figure 3.1(a) are

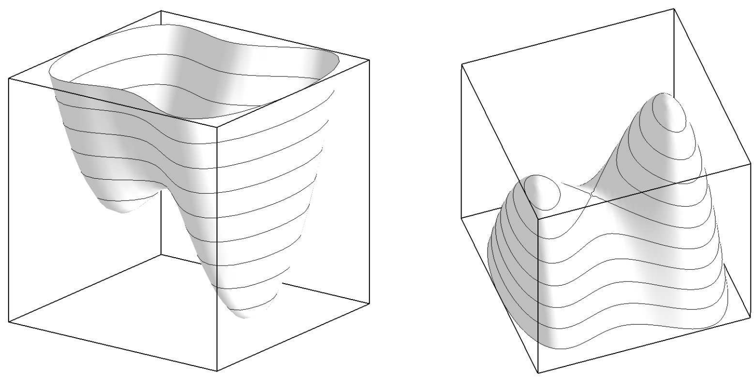


Figure 3.2: Two views of the energy surface $E(x, \dot{x})$ corresponding to the potential $V(x)$ of Figure 3.1(a). The graph on the right has been turned upside down to bring out some of the details which are hidden in the conventional view on the left.

shown in Figure 3.2.

3.3 ▪ Stability

We have seen that the dynamics of the equation (3.3) dictate that a hypothetical particle trapped in an energy well cannot escape. The closer the energy level E is to the bottom of the well, the lesser room there is for the particle to maneuver.[6] In the extreme case when the particle's energy matches that of the well's bottom, the particle cannot move at all. We express this by saying that the bottom of the well is a *stable equilibrium point* of the differential equation (3.3). If the energy level is increases just slightly, the particle will move along an oval around the equilibrium point. Referring to the construction of the phase portrait from the potential V, it should be evident that *a local minimum of V corresponds to a stable equilibrium*. Similarly, a local maximum of V corresponds to a *saddle* on the energy surface (see Figure 3.2) therefore *a local maximum of V corresponds to an unstable equilibrium*.

3.4 ▪ The phase portrait of a simple pendulum

The previous section's treatment of the scalar equation of motion has greater applicability than it may appear on the surface. The variable $x(t)$ there need not be the coordinate of a moving point along a straight line. For instance, according to (1.5) on page 4, the motion of a simple pendulum is described by the differential equation

$$\ddot{\varphi} + \frac{g}{\ell} \sin \varphi = 0, \tag{3.5}$$

where $\varphi = \varphi(t)$ is the angle the the pendulum's rod makes relative to the downward pointing vertical, as depicted in Figure 3.3 on the left. Although the motion of the pendulum takes places in two dimensions, the equation of motion (3.5) is exactly of the form (3.3) with $V'(\varphi) = g/\ell \sin \varphi$, hence $V(\varphi) = g/\ell(1 - \cos \varphi)$, therefore the analysis of the pre-

[6] I am assuming a round-bottomed energy well here. In a flat-bottomed energy well the particle can move around no matter how shallow the well.

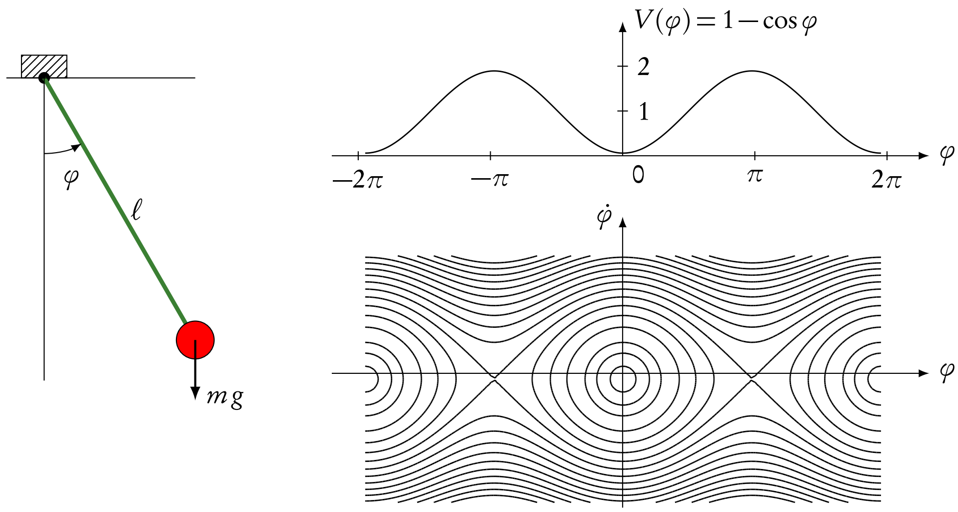


Figure 3.3: The pendulum is shown on the left. The graph of the potential function $V(\varphi) = g/\ell(1 - \cos\varphi)$ (with $g/\ell = 1$) is shown at top right. The corresponding phase portrait is shown at bottom right. The function V has a period of 2π, therefore it would have sufficed to limit the plots to the range $-\pi \le \varphi \le \pi$. Outside of that range, things repeat by periodicity.

vious section applies.[7] The graph of $V(\varphi)$ and the resulting phase portrait, constructed according to the previous section's guidelines, are shown in Figure 3.3 on the right.

The configuration of a pendulum is completely specified by the angle φ at any instant. The configuration corresponding to $\varphi + 2k\pi$ is exactly the same as that of φ for any integer k. In other words, the pendulum's configuration is determined by $\varphi \mod 2\pi$. In particular, the left and right edges of the phase portrait in Figure 3.3 correspond to the same configuration. This is best visualized by wrapping the phase portrait into a cylinder and gluing the left and right edges together. This is illustrated in Figure 3.4.

Exercises

3.1. Analyze the stability of the spinning hoop of Exercise 1.3. Show that the lower equilibrium is stable if Ω is small, and unstable if it is large. Find the value of Ω where the transition takes place.

3.2. Plot representative phase portraits for the two cases of the problem above.

[7] As noted earlier, the potential function is defined within an additive arbitrary constant, therefore the "1" in $1 - \cos\varphi$ is immaterial; its inclusion makes $V(0) = 0$ which is nice but of no special consequence.

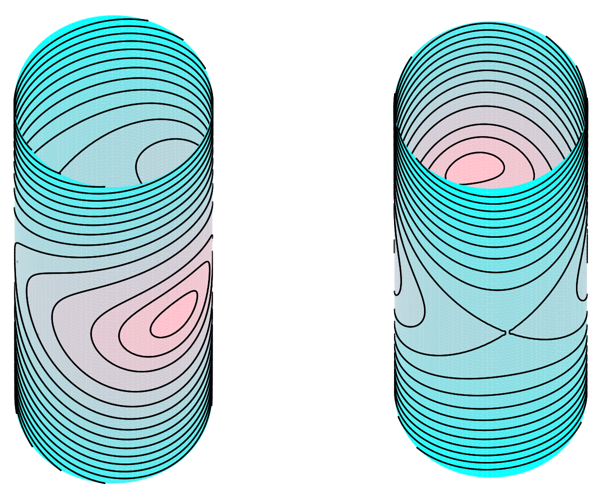


Figure 3.4: Two views of the pendulum's phase portrait as wrapped into a cylinder to emphasize that the pendulum's configuration depends on $\varphi \mod 2\pi$.

Chapter 4

The Kapitsa pendulum

4.1 ▪ The inverted pendulum

The pendulum in Figure 4.1 consists of a massless rod of length ℓ, a point mass m as the bob, and a pivot which oscillates vertically according to $y = a\cos\omega t$, where a and ω are prescribed constants. In Exercise 4.1 you will show that the equation of motion is

$$\ddot{\varphi} + \frac{g}{\ell}\sin\varphi + \frac{a\omega^2}{\ell}\sin\varphi\cos\omega t = 0, \tag{4.1}$$

where φ is the angle of the rod relative to the vertical, as shown.

If the amplitude a of the pivot's oscillation is small, and if ω is not too large, then we expect the system to behave similar to an ordinary pendulum, albeit with somewhat jittery oscillations. In particular, the lower equilibrium $\varphi = 0$ would be stable and the upper equilibrium $\varphi = \pi$ would be unstable. A graph of the the solution $\varphi(t)$ of the pendulum's equation of motion with smallish a and ω is shown on Figure 4.1.

It is the purpose of this chapter to show that as ω is increases beyond a critical threshold, the pendulum's behavior changes drastically. Specifically, the lower equilibrium becomes unstable and the upper equilibrium becomes stable. Thus, the pendulum turns around by 180 degrees, points upward, and oscillates about the $\varphi = \pi$ position! Figure 4.2 shows the solution of the pendulum's equation of motion for a relatively fast ω. Note that the oscillation now take place about the upper equilibrium $\varphi = \pi$. The pendulum is standing upright, pointing up!

4.2 ▪ Averaging out the fast oscillations

To explain this interesting phenomenon, Introduce the a hypothetical "nominal rod" which connects the origin to the bob, and let ψ be its angle relative to the vertical, as shown in the schematic diagram in Figure 4.1. The let $\delta = \varphi - \psi$ be the angle between the real rod and the nominal rod. According to the Law of Sines applied to the triangle shown in the figure, we have:

$$\frac{\sin\psi}{\ell} = \frac{\sin\delta}{a\cos\omega t}$$

that is,

$$\sin\delta = \frac{a}{\ell}\sin\psi\cos\omega t.$$

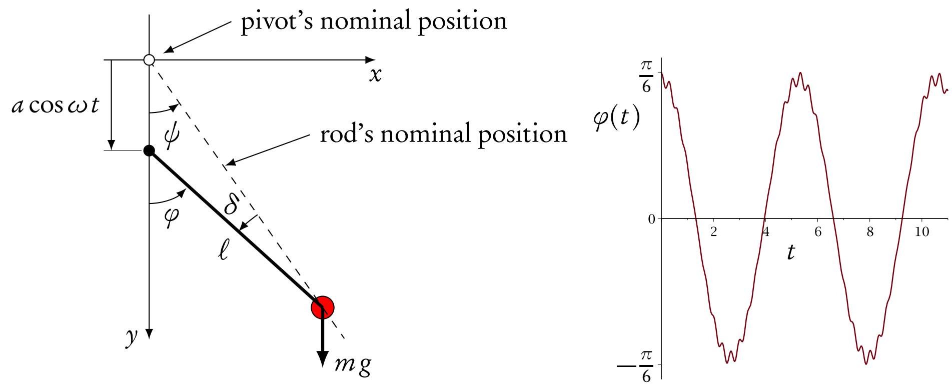


Figure 4.1: The pivot oscillates vertically according to $a\cos\omega t$ about the pendulum's nominal pivot. In the schematic diagram on the left the pivot's displacement is exaggerated; we assume that a/ℓ is very small in our computations. When the pendulum's arm makes a "nominal" angle ψ with the vertical, the angle actually oscillates rapidly in the range $\psi \pm \delta$. The graph on the right is that of the angle $\varphi(t)$ obtained by solving the differential equation (4.1) with parameters $\ell = 1$, $g = 1$, $a = 0.05$, and $\omega = 20$, and initial conditions $\varphi(0) = 5\pi/6$, $\dot\varphi(0) = 0$. The oscillation about the lower equilibrium position ($\varphi = 0$) is stable since $\omega^2 < 2g\ell/a$.

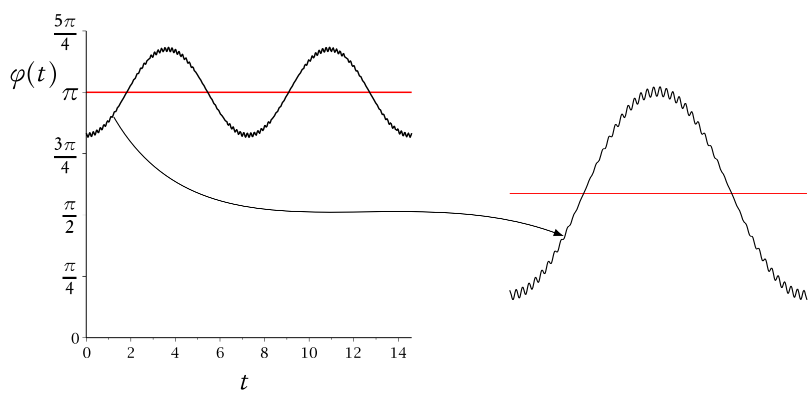


Figure 4.2: The solution of the differential equation (4.1) with parameters $\ell = 1$, $g = 1$, $a = 0.05$, and $\omega = 40$, and initial conditions $\varphi(0) = \pi/6$, $\dot\varphi(0) = 0$. The oscillation about the upper equilibrium position ($\varphi = \pi$) is stable since $\omega^2 > 2g\ell/a$. The figure on the right is an enlarged copy of a portion of the graph on the left. We see that $\varphi(t)$ consists of high-frequency, small-amplitude oscillations riding on a slowly oscillating function.

Then the assumption $a \ll \ell$ implies that $\sin\delta \ll 1$, therefore $\sin\delta \approx \delta$. We conclude that

$$\delta \approx \frac{a}{\ell}\sin\psi\cos\omega t. \tag{4.2}$$

We are going to need δ's second derivative soon, so let's calculate it right now. We have:

$$\dot{\delta} \approx \frac{a}{\ell}\big[\dot{\psi}\cos\psi\cos\omega t - \omega\sin\psi\sin\omega t\big],$$
$$\ddot{\delta} \approx \frac{a}{\ell}\big[\ddot{\psi}\cos\psi\cos\omega t - \dot{\psi}^2\sin\psi\cos\omega t - 2\omega\dot{\psi}\cos\psi\sin\omega t - \omega^2\sin\psi\cos\omega t\big].$$

We are interested in high frequency oscillations of the pivot, that is, $\omega \gg 1$. Therefore, the term with ω^2 in the expression above dominates the rest. We conclude that

$$\ddot{\delta} \approx -\frac{a\omega^2}{\ell}\sin\psi\cos\omega t.$$

Now we go to the differential equation (4.1) and replace φ by $\psi + \delta$, and replace $\sin\varphi$ with its Taylor series approximation

$$\sin\varphi = \sin(\psi + \delta) \approx \sin\psi + \delta\cos\psi.$$

We get:

$$\ddot{\psi} + \ddot{\delta} + \frac{g}{\ell}\big[\sin\psi + \delta\cos\psi\big] + \frac{a\omega^2}{\ell}\big[\sin\psi + \delta\cos\psi\big]\cos\omega t = 0.$$

We multiply out everything and replace $\ddot{\delta}$ with the expression obtained above, and arrive at

$$\ddot{\psi} - \frac{a\omega^2}{\ell}\sin\psi\cos\omega t + \frac{g}{\ell}\sin\psi + \frac{g}{\ell}\delta\cos\psi + \frac{a\omega^2}{\ell}\sin\psi\cos\omega t + \frac{a\omega^2}{\ell}\delta\cos\psi\cos\omega t = 0.$$

The second and fifth terms cancel, leaving us with

$$\ddot{\psi} + \frac{g}{\ell}\sin\psi + \frac{g}{\ell}\delta\cos\psi + \frac{a\omega^2}{\ell}\delta\cos\psi\sin\omega t = 0.$$

Then we substitute for δ from (4.2):

$$\ddot{\psi} + \frac{g}{\ell}\sin\psi + \frac{ag}{\ell^2}\sin\psi\cos\psi\cos\omega t + \frac{a^2\omega^2}{\ell^2}\sin\psi\cos\psi\cos^2\omega t = 0. \tag{4.3}$$

In the graphs of $\varphi(t)$ in figures 4.1 and 4.2 we see that the period $2\pi/\omega$ of the pivot's oscillations is much smaller than the oscillations of the pendulum itself. Consequently, within one such time period, the value of ψ and its derivatives are essentially constants. On the basis of this observations, we average (4.3) over one $2\pi/\omega$ period, where we regard ψ as constant. Since

$$\frac{1}{2\pi/\omega}\int_0^{2\pi/\omega}\cos\omega t\,dt = 0, \qquad \frac{1}{2\pi/\omega}\int_0^{2\pi/\omega}\cos^2\omega t\,dt = \frac{1}{2},$$

we get

$$\ddot{\psi} + \frac{g}{\ell}\sin\psi + \frac{a^2\omega^2}{2\ell^2}\sin\psi\cos\psi = 0,$$

or the equivalent

$$\ddot{\psi} + \frac{g}{\ell}\Big[1 + \frac{a^2\omega^2}{2\ell g}\cos\psi\Big]\sin\psi = 0. \tag{4.4}$$

In comparison with the equation (3.5) of the motion of an ordinary unforced pendulum, the Kapitsa pendulum sees an “effective acceleration of gravity” given by

$$g\Big[1+\frac{a^2\omega^2}{2\ell g}\cos\psi\Big].$$

If the pendulum's motion is in the $0<\psi<\pi/2$ regime, the quantity in the square brackets is greater than 1, therefore vibrating the support is tantamount to increasing the acceleration of gravity.[8] If, however, the pendulum's motion is in the $\pi/2<\psi<\pi$ regime, then the effective acceleration of gravity may become negative if the coefficient of $\cos\psi$ is sufficiently large. The latter will happen if ω is sufficiently large. That's tantamount to reversing the direction of gravity, which sort of explains why the pendulum turns upright.

4.3 ▪ Stability analysis

The effective equation of motion (4.4) of the Kapitsa pendulum is of the type (3.3) which was studied in Chapter 3. Comparing the two, we see that

$$V'(\psi)=\frac{g}{\ell}\sin\psi+\frac{a^2\omega^2}{2\ell^2}\sin\psi\cos\psi, \tag{4.5}$$

whence

$$V(\psi)=\frac{g}{\ell}(1-\cos\psi)+\frac{a^2\omega^2}{4\ell^2}\sin^2\psi \tag{4.6}$$

The analysis presented in Chapter 3 is based entirely on the shape of V's graph. Therefore we proceed to analyze the shape.

The equation's equilibria are the roots of the equation $V'(\psi)=0$. Upon factorizing the equation as

$$\Big[\frac{g}{\ell}+\frac{a^2\omega^2}{2\ell^2}\cos\psi\Big]\sin\psi$$

we see that the roots are the solutions of

$$\sin\psi=0 \quad\text{and}\quad \cos\psi=-\frac{2gl}{a^2\omega^2}.$$

The first equation yields $\psi=0$ and $\psi=\pi$ as roots. (It suffices to look for roots in the $0\le\psi\le\pi$ range.) The second equation yields a root $\hat{\psi}$ given by

$$\hat{\psi}=\cos^{-1}\Big(-\frac{2gl}{a^2\omega^2}\Big) \tag{4.7}$$

if and only if $\omega^2>2g\ell/a^2$. (If that ratio equals to 1 then the root is π, which duplicates what we have already found.)

Table 4.1 lists the critical points of the function $V(\psi)$, along with the values of V, V', and V'' at those points. We see that:

- $V''(0)>0$ regardless of the parameter values, therefore the hanging-down equilibrium, $\psi=0$, is always stable;

[8] Don't take this literally; the acceleration of gravity in (3.5) is a constant while the effective acceleration of gravity in (4.4) depends on $\cos\psi$, therefore is not a constant.

ψ	0	$\hat{\psi}$	π
$V(\psi)$	0	$V(\hat{\psi})$	$\frac{2g}{\ell}$
$V'(\psi)$	0	0	0
$V''(\psi)$	$\frac{g}{l}\left[\frac{a^2\omega^2}{2g\ell}+1\right]$	$-\frac{a^2\omega^2}{2\ell^2}\sin^2\psi$	$\frac{g}{l}\left[\frac{a^2\omega^2}{2g\ell}-1\right]$

Table 4.1: The analysis of the critical points of the function $V(\psi)$ defined in (4.6). The critical point $\hat{\psi}$, defined in (4.7) exists if and only if $\omega^2 > 2g\ell/a^2$.

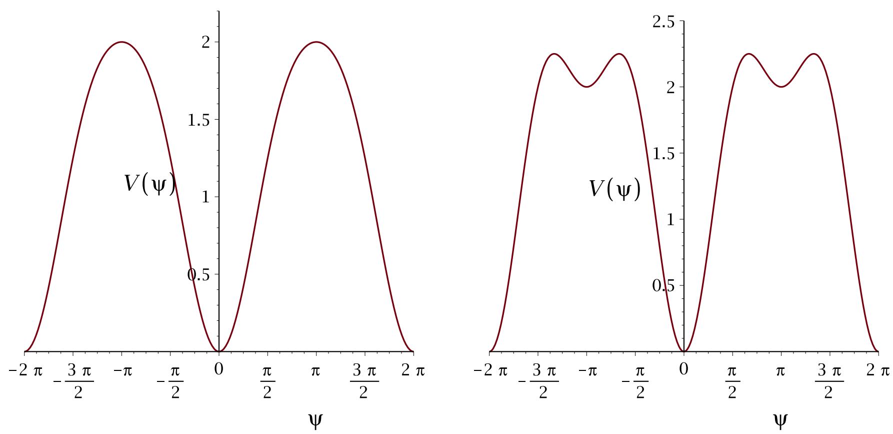


Figure 4.3: Two representative graphs of the function $V(\psi)$ in (4.6) with the parameters $\ell = 1$, $g = 1$, $a = 0.05$. On the left we have taken $\omega = 20$, which leads to $2gl/(a^2\omega^2) = 2 > 1$. This corresponds to a stable equilibrium at $\psi = 0$ and an unstable equilibrium at $\psi = \pi$. On the left we have taken $\omega = 40$, which leads to $2gl/(a^2\omega^2) = 1/2 < 1$. This corresponds to stable equilibria at $\psi = 0$ and $\psi = \pi$, and an unstable equilibrium at $\psi = 2\pi/3$.

- $V''(\hat{\psi}) < 0$ regardless of the parameter values, therefore the equilibrium $\hat{\psi} = 0$, if it exists, is unstable; and
- if $\omega^2 < 2g\ell/a^2$ then $V''(\pi) < 0$, therefore the inverted equilibrium, $\psi = \pi$, is unstable; but if $\omega^2 > 2g\ell/a^2$ then $V''(\pi) > 0$, therefore the inverted equilibrium, $\psi = \pi$, is stable.

Figure 4.3 shows the graphs of $V(\psi)$ for two representative cases. The graphs are plotted over the range $[-2\pi, 2\pi]$ to give a clear sense of their nature; only the range $[0, \pi]$ is of true relevance to us.

Exercises

4.1. Derive the equation of motion (4.1) of Kapitsa's pendulum.

4.2. **Horizontally oscillating pivot.** Consider a pendulum similar to Kapitsa's, but whose pivot oscillates horizontally rather than vertically. Derive the equation of motion and do a stability analysis.

Chapter 5

Lagrangian mechanics

5.1 ▪ Newtonian mechanics

Let $\boldsymbol{r}(t)$ be the position vector at time t of a particle (point mass) of constant mass m moving in the three-dimensional space under the influence of a force $\boldsymbol{f}(t)$. According to Newton, the equation of motion is $m\ddot{\boldsymbol{r}} = \boldsymbol{f}$, where, to simplify the notation, I have written $\boldsymbol{r}$ and $\boldsymbol{f}$ for $\boldsymbol{r}(t)$ and $\boldsymbol{f}(t)$. A superimposed dot on a variable indicates the time derivative of that variable. Thus, $\dot{\boldsymbol{r}}$ is the particle's velocity and $\ddot{\boldsymbol{r}}$ is its acceleration.

The motion of a collection of N particles is given as a set N vectorial equations

$$m_k \ddot{\boldsymbol{r}}_k = \boldsymbol{f}_k, \quad k = 1, 2, \ldots, N, \tag{5.1}$$

where m_k is the mass of the kth particle, $\boldsymbol{r}_k$ is its position vector, and $\boldsymbol{f}_k$ is the resultant of all forces acting on m_k.

Example 5.1. Consider and idealized "dumbbell" consisting of two particles of masses m_1 and m_2, connected with a rigid massless rod, as shown in Figure 5.1(a). In the free flight of the dumbbell, as when it is tossed up in the air, the force exerted on m_1 is the resultant of the (known) weight vector $m_1\boldsymbol{g}$ and the (unknown) push/pull $\boldsymbol{f}_{12}$ the rod. That is, $\boldsymbol{f}_1 = m_1\boldsymbol{g} + \boldsymbol{f}_{12}$.

Let us write $\boldsymbol{r}_k = \langle r_{k,1}, r_{k,2}, r_{k,3}\rangle$ and $\boldsymbol{f}_k = \langle f_{k,1}, f_{k,2}, f_{k,3}\rangle$ for the Cartesian representations of the vectors $\boldsymbol{r}_k$ and $\boldsymbol{f}_k$. Then the N vector equations above may equivalently be viewed as $3N$ scalar equations

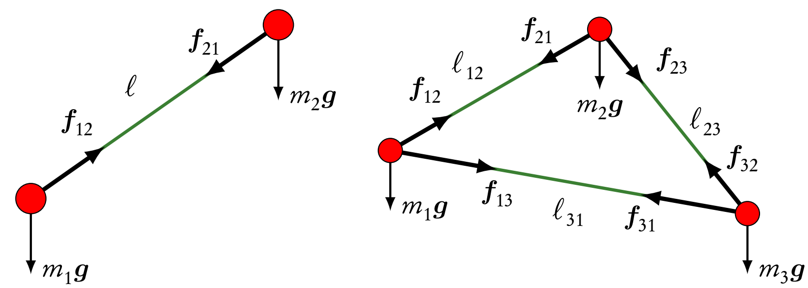
$$m_k \ddot{r}_{k,j} = f_{k,j}, \quad j = 1, 2, 3, \quad k = 1, 2, \ldots, N. \tag{5.2}$$

The following obvious trick flattens the doubly-indexed variables into singly-index quantities and results in a significant algebraic simplification. We introduce the vectors

$$\boldsymbol{x} = \langle \underbrace{r_{1,1}, r_{1,2}, r_{1,3}}_{\boldsymbol{r}_1}, \underbrace{r_{2,1}, r_{2,2}, r_{2,3}}_{\boldsymbol{r}_2}, \ldots \underbrace{r_{N,1}, r_{N,2}, r_{N,3}}_{\boldsymbol{r}_N} \rangle, \tag{5.3a}$$

$$\boldsymbol{f} = \langle \underbrace{f_{1,1}, f_{1,2}, f_{1,3}}_{\boldsymbol{f}_1}, \underbrace{f_{2,1}, f_{2,2}, f_{2,3}}_{\boldsymbol{f}_2}, \ldots \underbrace{f_{N,1}, f_{N,2}, f_{N,3}}_{\boldsymbol{f}_N} \rangle, \tag{5.3b}$$

$$\boldsymbol{m} = \langle \underbrace{m_1, m_1, m_1}_{m_1}, \underbrace{m_2, m_2, m_2}_{m_2}, \ldots, \underbrace{m_N, m_N, m_N}_{m_N} \rangle, \tag{5.3c}$$



(a) A idealized dumbbell (b) a rigid triangle

Figure 5.1: This idealized dumbbell on the left consists of two points masses connected through a massless rigid rod of length ℓ. In free flight, the force exerted on m_1 is the resultant of the weight vector $m_1 \boldsymbol{g}$ and the push/pull of the rod $\boldsymbol{f}_{12}$. The rigid triangle on the right consists of three point masses connected through a massless rigid rods.

and write (5.2) as

$$m_i \ddot{x}_i = f_i, \quad i = 1, 2, \ldots, 3N. \tag{5.4}$$

The change from (5.2) to (5.4) may seem merely cosmetic, but it entails a major change of philosophy and opens the doors to Lagrangian mechanics, as we shall see. Specifically, we view (5.4) as the differential equation of a motion of a point $\boldsymbol{x}$ in $\mathbf{R}^{3N}$. According to (5.3a), knowing the position of the single point $\boldsymbol{x} \in \mathbf{R}^{3N}$ is equivalent to knowing the positions of the N points $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N$ in the (physical) three-dimensional space. Thus, the study of the motion of a system of N points in the three-dimensional space is equivalent to the study of the motion of a *single point* in the abstract $\mathbf{R}^{3N}$. Specifying an $\boldsymbol{x}$ in the $\mathbf{R}^{3N}$ amounts to specifying the geometrical configuration of the particle system.

Definition 5.2. *The* $3N$*-dimensional space introduced above is called the mechanical system's* Cartesian configuration space. *In analogy with Newton's equations of motion, the vectors* $\boldsymbol{x}, \boldsymbol{f}, \boldsymbol{m}$ *defined in* (5.3) *are called* the position, the force, *and* the mass *of the single abstract "particle" moving in the configuration space.*

Remark 5.1. Although it is tempting to think of the equation of motion (5.4) as a generalization of Newton's equation $m\ddot{\boldsymbol{x}} = \boldsymbol{f}$ to $\mathbf{R}^{3N}$, the analogy is imperfect. The true generalization would have been

$$m\ddot{x}_i = f_i, \quad i = 1, 2, \ldots, 3N,$$

involving only a *single* m. In contrast, (5.4)'s fictitious "particle" exhibits different masses along different coordinate directions.

5.2 ▪ Holonomic constraints

The motion of a particle in the three-dimensional physical space traces a curve, as in the arc of a thrown ball, or the orbit of a planet. The motion of N particle then traces N curves in the three-dimensional space. The position x in configuration space, defined in (5.3a), merges the coordinates of the N particles into one, therefore the motion of the entire N-particle system appears as a single curve in the configuration space. We call that curve the system's *orbit in the configuration space*. When there is no risk of confusion, we will simply call it the *orbit*.

If there are no impediments in placing the particles independently in arbitrary positions in space, then the orbit of the system of N particle may reach any point in the configuration space—all is needed is the application of an appropriate force to get there. If, however, the relative movements of the points are constrained, as in the dumbbell of Figure 5.1(a), only a subset of the configuration space may be reached.

Example 5.3. Let $\boldsymbol{r}_1 = \langle r_{1,1}, r_{1,2}, r_{1,3} \rangle$ and $\boldsymbol{r}_2 = \langle r_{2,1}, r_{2,2}, r_{2,3} \rangle$. be the position vectors of the dumbbell of Figure 5.1(a). Then, according to (5.3a) the position vector $\boldsymbol{x} \in R^6$ is given by

$$\boldsymbol{x} = \langle r_{1,1}, r_{1,2}, r_{1,3}, r_{2,1}, r_{2,2}, r_{2,3} \rangle.$$

The constraint of the fixed length ℓ of the connecting rod is expressed as $||\boldsymbol{r}_1 - \boldsymbol{r}_2|| = \ell$, or more explicitly, as

$$(r_{1,1} - r_{2,1})^2 + (r_{1,2} - r_{2,2})^2 + (r_{1,3} - r_{2,3})^2 = \ell^2,$$

that is,

$$(x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2 - \ell^2 = 0. \tag{5.5}$$

This defines a 5-dimensional "surface"—a *manifold* is the technical term—embedded in $\mathbf{R}^6$. The point orbit cannot roam arbitrarily in $\mathbf{R}^6$; it is constrained to stay on that manifold.

Example 5.4. Figure 5.1(b) shows three point masses connected with three massless rigid rods, and thus forming a rigid triangle. The position vectors $\boldsymbol{r}_i$, $i = 1, 2, 3$, of the masses are constrained through the three constraint equations

$$||\boldsymbol{r}_1 - \boldsymbol{r}_2|| = \ell_{12}, \qquad ||\boldsymbol{r}_2 - \boldsymbol{r}_3|| = \ell_{23}, \qquad ||\boldsymbol{r}_3 - \boldsymbol{r}_1|| = \ell_{31},$$

which, in terms of the extended variable

$$\begin{aligned} \boldsymbol{x} &= \langle r_{1,1}, r_{1,2}, r_{1,3}, r_{2,1}, r_{2,2}, r_{2,3}, r_{3,1}, r_{3,2}, r_{3,3}, \rangle \\ &= \langle x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, \rangle \end{aligned}$$

take on the form

$$\begin{aligned} (x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2 &= \ell_{12}^2, \\ (x_4 - x_7)^2 + (x_6 - x_8)^2 + (x_7 - x_9)^2 &= \ell_{23}^2, \\ (x_7 - x_1)^2 + (x_8 - x_2)^2 + (x_9 - x_3)^2 &= \ell_{31}^2. \end{aligned}$$

These confine the triangle's orbit in the configuration space to a 6-dimensional manifold embedded in $\mathbf{R}^9$.

Example 5.5. Reconsider the previous example with a added twist. Suppose that the triangle's rods are equipped with remote-controlled motors with may be activated to vary the rods' lengths as desired during the flight. The previous example's constraint equations take the form

$$\begin{aligned} (x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2 &= \ell_{12}(t)^2, \\ (x_4 - x_7)^2 + (x_6 - x_8)^2 + (x_7 - x_9)^2 &= \ell_{23}(t)^2, \\ (x_7 - x_1)^2 + (x_8 - x_2)^2 + (x_9 - x_3)^2 &= \ell_{31}(t)^2, \end{aligned}$$

where $\ell_{12}(t)$, $\ell_{23}(t)$, and $\ell_{31}(t)$ are given. The manifold $\mathscr{M}$ in this case is a 6-dimensional manifold embedded in $\mathbf{R}^9$ whose shape changes with time.

Example 5.6. Recall the bead on the rotating hoop of Exercise 1.3 on page 6. With the obvious choice of the xyz coordinates, the position vector of the bead is

$$\boldsymbol{r} = \langle R\sin\varphi\cos\Omega t, R\sin\varphi\sin\Omega t, R\cos\varphi\rangle.$$

The manifold $\mathscr{M}$ in this case is a the spinning hoop itself. Geometrically it is a one-dimensional spinning object (a circle) embedded in $\mathbf{R}^3$. It is given by the pair of equations

$$x\sin\Omega t = y\cos\Omega t,$$
$$x^2 + y^2 + z^2 = a^2.$$

The first equation is that of plane that contains the z axis and spins about it with an angular velocity of Ω. The second equation is that of a sphere of radius a centered at the origin. The intersection of the two objects is the spinning hoop.

In general, a system of N particles subject to M constraint equations of the form

$$\varphi_i(\boldsymbol{x}, t) = 0, \quad i = 1, 2, \ldots, M, \tag{5.6}$$

where $\varphi_i : \mathbf{R}^{3N} \times \mathbf{R} \to \mathbf{R}$, $i = 1, 2, \ldots, M$. These define a $(3N - M)$-dimensional manifold $\mathscr{M}$ embedded in R^{3N}. The system's possible orbits are confined to lie in that manifold. Generally $\mathscr{M}$ may move/deform with time, as it was the case in Examples 5.5 and 5.6. However, if the equations (5.6) are independent of time, as it was the case in Examples 5.3 and 5.4, then $\mathscr{M}$ remains unchanged during the motion. That corresponds to a set of constraints of the form

$$\varphi_i(\boldsymbol{x}) = 0, \quad i = 1, 2, \ldots, M, \tag{5.6'}$$

Constraints of type (5.6) are not the most general. Some very interesting mechanical systems impose constraints on the *velocity*, $\dot{\boldsymbol{x}}$, as in $\varphi_i(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = 0$. The rolling of a coin on the floor, for instance, has a constraint that depends on $\dot{\boldsymbol{x}}$, therefore (5.6) is inadequate for that purpose.

Definition 5.7. *Constraints of the type* (5.6) *are called* holonomic. *All other types of constraints are called* nonholonomic.

Definition 5.8. *A mechanical system whose only constraints are of the holonomic type is called a* holonomic system.

We will begin our study of Lagrangian dynamics with holonomic systems. Nonholonomic constraints will be brought up in the later chapters.

Remark 5.2. You may be interested to know that holonomic constraints of type (5.6) are called *rheonomic* while those of type (5.6') are called *scleronomic*. I prefer to call them with the more user-friendly terms "time-dependent" and "time-independent" instead.

Remark 5.3. The term *holonomic* was introduced by Hertz in [10]:

> §123. A material system between whose possible positions all conceivable continuous motions are also possible motions is called a holonomous system.
>
> The term means that such a system obeys integral (ὅλος) laws (νόμος), whereas material systems in general obey only differential conditions.

Admittedly that definition is rather vague, but its meaning is clarified further down:

> §132. When from the differential equations of a material system an equal number of finite equations between the coordinates of the system can be deduced, the system is holonomous.

By "finite equations" he means algebraic, as opposed to differential, equations.

5.3 ▪ Generalized coordinates

In a holonomic system of N particles subject to M holonomic constraints, the $3N$ Cartesian components of the position vector $\boldsymbol{x}$ are not quite suitable for the analysis of motion —they cannot serve as independent variables since they are interrelated through the M constraint equations (5.6). A much better approach is to parametrize the $n = (3N - M)$-dimensional configuration manifold $\mathscr{M}$ through a suitably chosen n independent variables $q_1, q_2, \dots, q_n$, called the system's *generalized coordinates*. The parametrization is certainly not unique, however in practice there often is an "obvious" choice. We write $\boldsymbol{q}$ when we wish to refer to the n variables $q_1, q_2, \dots, q_n$ collectively.

The parameters $\boldsymbol{q}$ form a (generally curvilinear) coordinate system on $\mathscr{M}$. Since the motion's orbit lies in $\mathscr{M}$, the system's state as a function of time may be expressed in terms of $\boldsymbol{q}(t)$. *The purpose of analytical mechanics is to express Newton's equations of motion* (5.4) *in terms of the generalized coordinates* $\boldsymbol{q}$.

Remark 5.4. A familiar example curvilinear coordinates, albeit not directly related to mechanics, is the system of addressing locations on the surface of the Earth through their longitude λ and latitude φ. In this context, $\mathscr{M}$ is the Earth's surface, and λ and φ are the coordinates q_1 and q_2.

Any $\boldsymbol{q}$ identifies a point on the manifold $\mathscr{M}$. Since $\mathscr{M}$ is embedded in $\mathbf{R}^{3N}$, it also identifies a point $\boldsymbol{x} \in \mathbf{R}^{3N}$. That is, the system's configuration vector $\boldsymbol{x}$ is a function of $\boldsymbol{q}$. We write this is a $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{q}, t)$, or in components:

$$x_i = x_i(\boldsymbol{q}, t), \quad i = 1, 2, \dots, 3N. \tag{5.7}$$

The t in this equations accounts for the possible motion/deformation of the manifold related to time-dependent constraints (5.6). In the case of time-independent constraints (5.6'), $\mathscr{M}$ is independent of time, and (5.7) reduces to

$$x_i = x_i(\boldsymbol{q}), \quad i = 1, 2, \dots, 3N. \tag{5.7'}$$

Differentiating the $\boldsymbol{q}$ to $\boldsymbol{x}$ mapping of (5.7) with respect to time, we obtain an expression for the velocities in terms of generalized coordinates:

$$\dot{x}_i = \sum_k \frac{\partial x_i(\boldsymbol{q}, t)}{\partial q_k} \dot{q}_k + \frac{\partial x_i(\boldsymbol{q}, t)}{\partial t}. \tag{5.8}$$

[The last term will be absent in the case of (5.7').] Let us observe that although the position x_i is a function of $\boldsymbol{q}$ and t only, the velocity $\dot{x}_i$ is a function of $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and t. Let's record this here for future reference:

$$\dot{x}_i = \dot{x}_i(\boldsymbol{q}, \dot{\boldsymbol{q}}, t), \quad i = 1, 2, \dots 3N. \tag{5.9}$$

The following theorem establishes a couple of very useful mathematical identities:

Theorem 5.9. *Let x_i and $\dot{x}_i$ be as in* (5.7) *and* (5.9)*. Then for any $i = 1,2,\dots,3N$ and $j = 1,2,\dots,M$, we have:*

$$\frac{\partial \dot{x}_i(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial \dot{q}_j} = \frac{\partial x_i(\boldsymbol{q},t)}{\partial q_j}. \tag{5.10}$$

$$\frac{\partial \dot{x}_i(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial q_j} = \frac{d}{dt}\Big(\frac{\partial x_i(\boldsymbol{q},t)}{\partial q_j}\Big) \tag{5.11}$$

Proof. The assertion (5.10) is an immediate consequence of (5.8). As to (5.11), it's a matter of differentiating (5.8) with respect to q_j and then exchanging the differentiation order in the resulting second order partial derivatives:

$$\begin{aligned}\frac{\partial \dot{x}_i(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial q_j} &= \sum_k \frac{\partial^2 x_i(\boldsymbol{q},t)}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 x_i(\boldsymbol{q},t)}{\partial q_j \partial t} \\ &= \sum_k \frac{\partial}{\partial q_k}\Big(\frac{\partial x_i(\boldsymbol{q},t)}{\partial q_j}\Big)\dot{q}_k + \frac{\partial}{\partial t}\Big(\frac{\partial x_i(\boldsymbol{q},t)}{\partial q_j}\Big) \\ &= \frac{d}{dt}\Big(\frac{\partial x_i(\boldsymbol{q},t)}{\partial q_j}\Big).\end{aligned}$$

□

5.4 ▪ Virtual displacements, virtual work, and generalized force

Figure 5.2 depicts a representation of the orbit of a system of N particles on a manifold $\mathcal{M}$ embedded in $\mathbf{R}^{3N}$. Pick an arbitrary point, let's say $\hat{\boldsymbol{x}}$, of the orbit and then consider the tangent at that point to the manifold. We explore that tangent through infinitesimal excursions away from $\hat{\boldsymbol{x}}$. Such excursions are called *virtual displacements* and commonly written as $\delta \boldsymbol{x}$. I should emphasize that we are viewing the whole picture as a fossil frozen in time. The excursions have nothing to do with the system's motion which will continue along the predetermined orbit once we unfreeze the time. The "$\delta \boldsymbol{x}$" notation is used to distinguish between virtual displacements and the actual differential of the motion $d\boldsymbol{x}$.

The obvious way of producing a virtual displacement is through incrementing the generalized coordinates $\boldsymbol{q}$. A change in $\boldsymbol{q}$ amounts to a displacement within the manifold $\mathcal{M}$. Therefore the differential $\delta \boldsymbol{q}$ is a displacement within $\mathcal{M}$'s tangent. In view of (5.7), we have:

$$\delta x_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\delta q_j. \tag{5.12}$$

Let $\boldsymbol{f}$ be the force vector, see (5.3b), at the point $\hat{\boldsymbol{x}}$. Under a virtual displacement $\delta \boldsymbol{x}$, the force performs a work δW, called *virtual work*, given by

$$\delta W = \boldsymbol{f}\cdot\delta\boldsymbol{x} = \sum_{i=1}^{3N} f_i \delta x_i = \sum_{i=1}^{3N} f_i\Big(\sum_{j=1}^{n}\frac{\partial x_i}{\partial q_j}\delta q_j\Big) = \sum_{j=1}^{n}\Big(\sum_{i=1}^{3N} f_i\frac{\partial x_i}{\partial q_j}\Big)\delta q_j = \sum_{j=1}^{n} Q_j \delta q_j.$$

Letting

$$Q_j = \sum_{i=1}^{3N} f_i \frac{\partial x_i}{\partial q_j}, \quad j = 1,2,\dots,n, \tag{5.13}$$

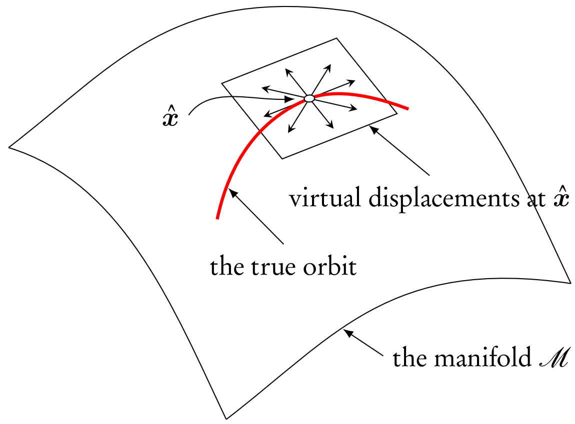


Figure 5.2: The system's orbit lies on a manifold $\mathscr{M}$ determined by the holonomic constraints. Virtual displacements at $\hat{\boldsymbol{x}}$ are tangent to the manifold.

the virtual work is now expressed as

$$\delta W = \boldsymbol{f} \cdot \delta \boldsymbol{x} = \boldsymbol{Q} \cdot \delta \boldsymbol{q}.$$

The vector $\boldsymbol{Q}$ is called the *generalized force* at $\hat{\boldsymbol{x}}$. The component Q_j is called the component of the generalized force along the generalized coordinate q_j.

Example 5.10. Consider the simple pendulum of Figure 1.1. The position vector $\boldsymbol{r} = \langle \ell \sin\varphi, \ell \cos\varphi \rangle$, therefore the vector $\boldsymbol{x}$ (see (5.3a)) is

$$\begin{aligned} \boldsymbol{x} &= \langle x_1, x_2 \rangle \\ &= \langle \ell \sin\varphi, \ell \cos\varphi \rangle, \end{aligned}$$

and the constraint is $x_1^2 + x_2^2 - \ell^2$, therefore the configuration manifold $\mathscr{M}$ coincides with the circle swept by the pendulum's bob, embedded in the configuration space $\mathbf{R}^2$. The angle φ plays the role of the generalized coordinate in this case; any value of φ identifies a point on $\mathscr{M}$. Let us write φ and Q_φ instead of q_1 and Q_1 for clarity. The force vector is $\langle 0, mg \rangle$. We compute the generalized force by applying (5.13):

$$\begin{aligned} Q_\varphi &= f_1 \frac{\partial x_1}{\partial \varphi} + f_2 \frac{\partial x_2}{\partial \varphi} \\ &= 0 \times (\ell \cos\varphi) + mg \times (-\ell \sin\varphi) = -mg\ell \sin\varphi. \end{aligned}$$

Observe that Q_φ turns out to be equal to the moment of the weight vector $\boldsymbol{f}$ about the pendulum's pivot.

Example 5.11. Consider the double-pendulum of Figure 1.2. The position vectors of its two masses are given by

$$\boldsymbol{r}_1 = \langle \ell_1 \sin\varphi, \ell_1 \cos\varphi \rangle, \qquad \boldsymbol{r}_2 = \boldsymbol{r}_1 + \langle \ell_2 \sin\psi, \ell_2 \cos\psi \rangle,$$

therefore the vectors $\boldsymbol{x}$ and $\boldsymbol{f}$ (see (5.3)) are

$$\begin{aligned}
\boldsymbol{x} &= \langle x_1, x_2, x_3, x_4\rangle \\
&= \langle \ell_1 \sin\varphi, \ell_1\cos\varphi \ell_1\sin\varphi + \ell_2\sin\psi, \ell_1\cos\varphi + \ell_2\cos\psi\rangle, \\
\boldsymbol{f} &= \langle f_1, f_2, f_3, f_4\rangle \\
&= \langle 0, m_1 g, 0, m_2 g\rangle.
\end{aligned}$$

The configuration space is $\mathbf{R}^4$ in this case. The two constraints

$$(x_1 - x_2)^2 = \ell_1^2, \qquad (x_3 - x_4)^2 = \ell_2^2$$

result in a two-dimensional configuration manifold $\mathscr{M}$ embedded in $\mathbf{R}^4$. The angles φ and ψ serve as generalized coordinates on $\mathscr{M}$. Let us write φ and ψ for the generalized coordinates instead of the generic q_1 and q_2, for clarity. We write Q_φ and Q_ψ for the corresponding generalized forces. forces instead of Q_1 and Q_2 By applying (5.13) we get

$$\begin{aligned}
Q_\varphi &= f_1\frac{\partial x_1}{\partial\varphi} + f_2\frac{\partial x_2}{\partial\varphi} + f_3\frac{\partial x_3}{\partial\varphi} + f_4\frac{\partial x_4}{\partial\varphi} = -m_1 g\ell_1\sin\varphi - m_2 g\ell_2\sin\varphi, \\
Q_\psi &= f_1\frac{\partial x_1}{\partial\psi} + f_2\frac{\partial x_2}{\partial\psi} + f_3\frac{\partial x_3}{\partial\psi} + f_4\frac{\partial x_4}{\partial\psi} = -m_2 g\ell_2\sin\psi.
\end{aligned}$$

5.5 ▪ External versus reaction forces

In equation (5.1) the force $\boldsymbol{f}_k$ applied to particle k is the *resultant of all forces* acting on that particle. For instance, in the triangular system of Figure 5.1(b), forces applied to m_1 consist of $m_1\boldsymbol{g} + \boldsymbol{f}_{12} + \boldsymbol{f}_{13}$. The fist term is the gravitational force applied to m_1, that is its weight, which is known. We call it an *external force*. The other two are generated dynamically within the rods, and are unknowns to be determined. We call then *internal forces* or more frequently, *constraint reactions* because they arise due to the unchanging lengths of the rods.

The constraint reactions get eliminated in the Lagrangian formulation as we shall see. Their elimination reduces the problem's unknowns, and hence simplifies the equations of motion significantly. In anticipation of that development, we write the total force $\boldsymbol{f}_k$ in (5.1) as $\boldsymbol{f}_k + \boldsymbol{f}'_k$, where, with some abuse of notations, we have recycled the notation $\boldsymbol{f}_k$ to signify the external forces only, and $\boldsymbol{f}'$ the internal forces, applied to the particle k. After flattening the vectors in accordance with (5.3), equation (5.4) takes on the form

$$m_i\ddot{x}_i = f_i + f'_i, \quad i = 1, 2, \ldots, 3N. \tag{5.14}$$

The argument that leads to the elimination of the constraint reactions proceeds as follows. The orbit of (5.14) lies in the constraint manifold $\mathscr{M}$ in $\mathbf{R}^{3N}$. External forces applied to the particles push and pull the point $\boldsymbol{x}$ in a direction tangent to $\mathscr{M}$. But what keeps $\boldsymbol{x}$ from flying away from $\mathscr{M}$? The manifold holds it back, that's what! If, for example, $\boldsymbol{x}$ speeds over a round protrusion on the manifold, centrifugal forces will tend to pull it away from the manifold. The manifold, however, exerts just the right amount of opposite force, the *constraint reaction*, which holds $\boldsymbol{x}$ attached to $\mathscr{M}$. In the physical space, that is $\mathbf{R}^3$, the manifold's reactions manifests itself as the forces that develop in the system's interconnecting links, such as $\boldsymbol{f}_{12}$ and $\boldsymbol{f}_{13}$ noted above.

The crucial observation that leads to the elimination of the constraint reactions from the equations of motion is that *the constraint reaction is orthogonal to the constraint manifold*. If it weren't, then it would have a component tangent to the manifold, which will then perform work during the motion. But such a behavior is uncharacteristic of a passive constraint surface, so we disallow it.

Since a virtual displacement $\delta \boldsymbol{x}$ is tangent to the constraint manifold (see Figure 5.2), the orthogonality of the reaction force $\boldsymbol{f}'$ to $\mathscr{M}$ is expressed naturally as $\boldsymbol{f}' \cdot \delta \boldsymbol{x} = 0$ for all virtual displacements $\delta \boldsymbol{x}$, or in expanded form

$$\sum_{i=1}^{3N} f_i' \delta x_i = 0 \quad \text{for all virtual displacements } \delta \boldsymbol{x}.$$

We note that, however, that due to (5.12)

$$\sum_{i=1}^{3N} f_i' \delta x_i = \sum_{i=1}^{3N} f_i' \left(\sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j} \delta q_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{3N} f_i' \frac{\partial x_i}{\partial q_j} \right) \delta q_j,$$

therefore

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{3N} f_i' \frac{\partial x_i}{\partial q_j} \right) \delta q_j = 0 \quad \text{for all virtual displacements } \delta \boldsymbol{q},$$

from which it follows that

$$\sum_{i=1}^{3N} f_i' \frac{\partial x_i}{\partial q_j} = 0, \quad j = 1, 2, \ldots, n. \tag{5.15}$$

5.6 ▪ The equations of motion for a holonomic system

At this point, the motion of a system consisting of N point masses and M holonomic constraints has been encapsulated into the $3N + M$ equations (5.14) and (5.6) in the unknowns x_i and f_i'. It is the goal of this section to re-express the equations of motions as a system of only $n = 3N - M$ differential equations for the n generalized coordinates $\boldsymbol{q}$ as the unknowns. We begin with multiplying the equation (5.14) by $\partial x_i / \partial q_j$ and summing over i:

$$\sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} = \sum_{i=1}^{3N} f_i \frac{\partial x_i}{\partial q_j} + \sum_{i=1}^{3N} f_i' \frac{\partial x_i}{\partial q_j}.$$

The second summation on the right-hand side is zero due to (5.15). The first summation on the right-hand side is the generalized force Q_j; see (5.13). Therefore obtain

$$\sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} = Q_j, \quad j = 1, 2, \ldots, n. \tag{5.16}$$

To simplify the left-hand side, we begin with a preliminary preparation. The kinetic energy of the system is

$$\tilde{T}(\dot{\boldsymbol{x}}) = \sum_{k=1}^{3N} \frac{1}{2} m_k \dot{x}_k^2.$$

I have written $\tilde{T}$ rather than the usual T for a reason which will become obvious shortly. Now observe that for any i

$$\frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_i} \sum_{k=1}^{3N} \frac{1}{2} m_k \dot{x}_k^2 = m_i \dot{x}_i,$$

therefore

$$\frac{d}{dt}\frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i} = m_i \ddot{x}_i.$$

Then, the left-hand side of (5.16) may be calculated as

$$\begin{aligned}\sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} &= \sum_{i=1}^{3N} \frac{d}{dt}\left[\frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i}\right]\frac{\partial x_i}{\partial q_j} \\ &= \sum_{i=1}^{3N} \frac{d}{dt}\left[\frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i}\frac{\partial x_i}{\partial q_j}\right] - \sum_{i=1}^{3N} \frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i}\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right),\end{aligned}$$

where in the last step we have used the differentiation formula $u'v = (uv)' - uv'$. Now apply (5.10) to the first summation on the right-hand side, and apply (5.11) to the second summation, to get

$$\begin{aligned}\sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} &= \sum_{i=1}^{3N} \frac{d}{dt}\left[\frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right] - \sum_{i=1}^{3N} \frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_j} \\ &= \frac{d}{dt}\left[\sum_{i=1}^{3N} \frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right] - \sum_{i=1}^{3N} \frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_j}.\end{aligned}$$

Let us recall that the Cartesian velocity components $\dot{\boldsymbol{x}}$ and the generalized velocity components $\dot{\boldsymbol{q}}$ are related through (5.8). Therefore the kinetic energy, which we have taken to be a function of $\dot{\boldsymbol{x}}$, may equally well be expressed as a function of $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, and t. We write the latter as T to distinguish it from the former $\tilde{T}$:

$$\tilde{T}(\dot{\boldsymbol{x}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t),$$

and then note that by the chain rule

$$\frac{\partial T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial \dot{q}_j} = \sum_{i=1}^{3N} \frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_j} \quad \text{and} \quad \frac{\partial T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial q_j} = \sum_{i=1}^{3N} \frac{\partial \tilde{T}(\dot{\boldsymbol{x}})}{\partial \dot{x}_i}\frac{\partial \dot{x}_i}{\partial q_j}.$$

Consequently

$$\sum_{i=1}^{3N} m_i \ddot{x}_i \frac{\partial x_i}{\partial q_j} = \frac{d}{dt}\left[\frac{\partial T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial \dot{q}_j}\right] - \frac{\partial T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial q_j},$$

and therefore (5.16) takes the form

$$\frac{d}{dt}\left[\frac{\partial T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial \dot{q}_j}\right] - \frac{\partial T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial q_j} = \mathrm{Q}_j, \quad j = 1, 2, \ldots, n. \tag{5.17}$$

These n second order differential equations in the n unknowns $q_1, q_2, \ldots, q_n$ are called *Lagrange's equation of motion for a holonomic systems*.

In particular, if the external forces $\boldsymbol{Q}$ are derived from a potential, that is, if there exists a scalar function function $V(\boldsymbol{q}, t)$ so that $\mathrm{Q}_j = -\partial V / \partial q_j$, then Lagrange's equation of motion take on the form

$$\frac{d}{dt}\left[\frac{\partial T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial \dot{q}_j}\right] - \frac{\partial T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)}{\partial q_j} = -\frac{\partial V(\boldsymbol{q}, t)}{\partial q_j}, \quad j = 1, 2, \ldots, n.$$

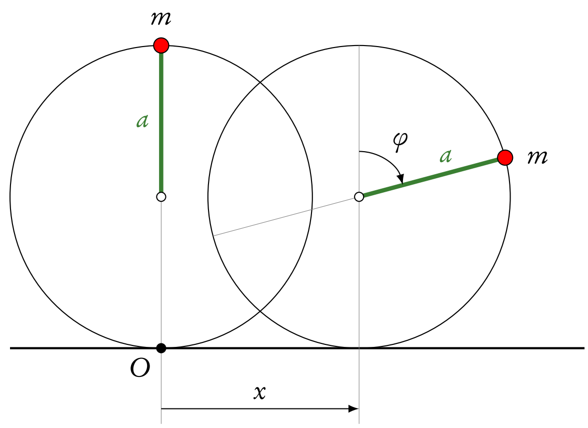


Figure 5.3: The massless hoop rolls against a horizontal line while remaining in a vertical plane. A point of mass m is affixed to the hoop. The angle φ or the distance x may be used as generalized coordinates in the configuration space (Exercises 5.2 and 5.3).

In that case we define the system's *Lagrangian* as

$$L(\boldsymbol{q},\dot{\boldsymbol{q}},t)=T(\boldsymbol{q},\dot{\boldsymbol{q}},t)-V(\boldsymbol{q},t)$$

and observe that since V does not depend on $\boldsymbol{q}$, we have $\partial L/\partial \dot{q}_j = \partial T/\partial \dot{q}$, therefore the equations of motion collapse to

$$\frac{d}{dt}\left[\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial \dot{q}_j}\right]-\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial q_j}=0, \quad j=1,2,\ldots,n. \tag{5.18}$$

Exercises

5.1. Find the gneralized forces Q_φ and Q_θ in the spherical pendulum of Figure 1.4 (page 7).

5.2. A massless hoop of radius a rolls without slipping on a horizontal line, while remaining in a vertical plane. A particle of mass m is firmly attached to the hoop, as seen in Figure 5.3. Use the angle φ of the mass's radius relative to the vertical as generalized coordinate. Find the generalized force Q_φ.

5.3. In the previous problem use the distance x travelled by the contact point (see the figure) as generalized coordinate. Find the generalized force Q_x.

5.4. Suppose the external forces f_i in (5.14) are derived from a potential, that is, there exists a scalar-valued function $\tilde{V}(\boldsymbol{x},t)$ such that $f_i=-\partial\tilde{V}(\boldsymbol{x},t)/\partial x_i$ for $i=1,2,\ldots,3N$. Let $V(\boldsymbol{q},t)=\tilde{V}(\boldsymbol{x},t)$ be the representation of the potential as a function of generalized coordinates. Show that the generalized forces Q_j are derived from the potential V, that is, $Q_j=\partial V(\boldsymbol{q},t)/\partial q_j$, for $j=1,2,\ldots,n$.

Chapter 6

Constraint reactions

In Chapter 5 we considered a mechanical system consisting of N point masses subject to M holonomic constraints (5.6), and derived the equation of motion (5.17), where the generalized forces $\{Q_j\}_{j=1}^{n}$ are related to the externally applied forces $\{f_i\}_{i=1}^{3N}$ through (5.13). Here $n = 3N - M$.

In the derivation of the equations we assumed that the generalized coordinates were independent of each other. This was used in the derivation of equation (5.15) where we assumed thqat the virtual displacements $\delta \boldsymbol{q}$ were arbitrary. Among other this, this resulted in the elimination of the internal reaction forces f_i' from the equations of motion.

Suppose, however, that we *are* interested in finding out the reaction forces. It is the goal of this chapter to explain how. The key idea is to forgo the assumption of independence of the generalized coordinates.

Specifically, we assume that the n generalized coordinates $q_1, q_2, \ldots, q_n$ are greater than the minimum necessary to specify the system's configuration. This implies that one or more relationships exists among the q_j's. Suppose that they are m such relationships:

$$
\begin{aligned}
&a_{11}\, dq_1 + a_{12}\, dq_2 + \cdots a_{1n}\, dq_n + a_{1t}\, dt = 0,\\
&a_{21}\, dq_1 + a_{22}\, dq_2 + \cdots a_{2n}\, dq_n + a_{2t}\, dt = 0,\\
&\cdots\\
&a_{m1}\, dq_1 + a_{m2}\, dq_2 + \cdots a_{mn}\, dq_n + a_{mt}\, dt = 0,
\end{aligned}
\tag{6.1}
$$

where each a_{ij} and a_{it} can be a given function of $\boldsymbol{q}$ and t. For convenience, we write these set of constraints in the compact form

$$
\sum_{k=1}^{n} a_{lk}(q,t)\, dq_k + a_{lt}(q,t)\, dt = 0, \quad l = 1, 2, \ldots, m, \tag{6.2}
$$

Remark 6.1. By dividing (6.2) through by dt, we see that

$$
\sum_{k=1}^{n} a_{lk}(q,t)\dot{q}_k + a_{lt}(q,t) = 0, \quad l = 1, 2, \ldots, m, \tag{6.3}
$$

that is, the equations impose restrictions on the system's velocities.

Repeating **(TODO)** the calculations of Chapter 5, we arrive at the following equations of motion:

$$\frac{d}{dt}\left[\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial \dot{q}_j}\right]-\frac{\partial L(\boldsymbol{q},\dot{\boldsymbol{q}},t)}{\partial q_j}=\sum_{k=1}^{m}a_{kj}\lambda_k, \quad j=1,2,\ldots,n. \tag{6.4}$$

These, together with (6.3), form a system of $n+m$ equations in the $m+n$ unknowns $q_1,q_2,\ldots,q_n$, $\lambda_1,\lambda_2,\ldots,\lambda_m$. The summation that appeart on the right-hand side gives the generalized reaction forces

$$Q'_j=\sum_{k=1}^{m}a_{kj}\lambda_k. \tag{6.5}$$

Once the coefficients λ_k have been computed, we may use (6.5) in conjunction with

$$Q'_j=\sum_{i=1}^{3N}f'_i\frac{\partial x_i}{\partial q_j} \tag{6.6}$$

to determine the raction force compnents f'_i.

Example 6.1. Let us revisit the simple pendulum of Figure 1.1 in page 2, and calculate the force within its connecting rod.

In Section 1.3 we derived the equation of motion (1.5) in terms of the angle φ which served as the generalized coordinate. The position vector of the pendulum's bob relative to the suspension point was expressed as $\boldsymbol{r}=\ell\boldsymbol{e}_r$ in that context, where ℓ is the length of the pendulum's rod.

Here we change the setting of the problem as follows. We express the configuration of pendulum in terms of not one, but *two* genealized coordinates: φ, which is the pendulum's angle as before; and ρ which is the rod's length and which is viewed as a variable. We impose the constraint $\rho=\ell$ retroactively to recover the physical model.

Referring to Figure 1.1 we have $\boldsymbol{r}=\rho\boldsymbol{e}_r$, therefore the bob's velocity is given by

$$\boldsymbol{v}=\dot{\boldsymbol{r}}=\dot{\rho}\boldsymbol{e}_r+\rho\dot{\boldsymbol{e}}_r=\dot{\rho}\boldsymbol{e}_r-\rho\dot{\varphi}\dot{\boldsymbol{e}}_\varphi,$$

where we have made use of (1.1). The kinetic and potential energies are

$$T=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2), \quad V=-mg\rho\cos\varphi,$$

which leads to the lagrangian

$$L=T-V=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\varphi}^2)+mg\rho\cos\varphi.$$

We calculate

$$\frac{\partial L}{\partial\dot{\rho}}=m\dot{\rho}, \quad \frac{\partial L}{\partial\rho}=m\rho\dot{\varphi}^2+mg\cos\varphi \quad \frac{\partial L}{\partial\dot{\varphi}}=m\rho^2\dot{\varphi}, \quad \frac{\partial L}{\partial\varphi}=-mg\rho\sin\varphi.$$

The constraint of inextensibility $\rho=\ell$ implies that $d\rho=0$, which we write as $(1)d\rho+(0)d\varphi+(0)dt=0$ to conform to the genral template (6.1). It follows that $a_{11}=1$, $a_{12}=0$, $a_{1t}=0$, therefore the equations (6.4) take the form

$$\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{\rho}}\Big)-\frac{\partial L}{\partial\rho}=a_{11}\lambda_1$$
$$\frac{d}{dt}\Big(\frac{\partial L}{\partial\dot{\varphi}}\Big)-\frac{\partial L}{\partial\varphi}=a_{12}\lambda_1,$$

that is

$$(m\dot{\rho})^{\cdot} - (m\rho\dot{\varphi}^2 + mg\cos\varphi) = \lambda_1,$$
$$(m\rho^2\dot{\varphi})^{\cdot} + mg\rho\sin\varphi = 0,$$

or in expanded form:

$$m\ddot{\rho} - (m\rho\dot{\varphi}^2 + mg\cos\varphi) = \lambda_1,$$
$$m\rho^2\ddot{\varphi} + 2m\rho\dot{\rho}\dot{\varphi} + mg\rho\sin\varphi = 0.$$

Applying the constraint $\rho = \ell$ reduces these to

$$-m\ell\dot{\varphi}^2 - mg\cos\varphi = \lambda_1,$$
$$m\ell^2\ddot{\varphi} + mg\ell\sin\varphi = 0.$$

The second equation is the usual equation of motion of a simple pendulum. In principle, we may plug its solution, $\varphi(t)$, into the first equation to find λ_1, but we don't do it that way. Instead, we compute the generalized reaction forces Q'_r and Q'_φ from (6.5):

$$Q'_r = (1)\lambda_1 = -m\ell\dot{\varphi}^2 - mg\cos\varphi, \qquad Q'_\varphi = (0)\lambda_1 = 0.$$

Then, we apply (6.6) to translate these into the physical components of the forces. Toward that end, let us observe that $\boldsymbol{r} = \rho\boldsymbol{e}_r$, therefore

$$\frac{\partial \boldsymbol{r}}{\partial \rho} = \boldsymbol{e}_r, \qquad \frac{\partial \boldsymbol{r}}{\partial \varphi} = \rho\frac{\partial \boldsymbol{e}_r}{\partial \varphi} = \rho\boldsymbol{e}_\varphi,$$

therefore (6.6) reads

$$Q'_r = \boldsymbol{f}' \cdot \frac{\partial \boldsymbol{r}}{\partial \rho} = \boldsymbol{f}' \cdot \boldsymbol{e}_r, \qquad Q'_\varphi = \boldsymbol{f}' \cdot \frac{\partial \boldsymbol{r}}{\partial \varphi} = \boldsymbol{f}' \cdot \rho\boldsymbol{e}_\varphi,$$

whence

$$\boldsymbol{f}' \cdot \boldsymbol{e}_r = -(m\ell\dot{\varphi}^2 + mg\cos\varphi) \qquad \boldsymbol{f}' \cdot \boldsymbol{e}_\varphi = 0.$$

Since $\boldsymbol{f}' = (\boldsymbol{f}' \cdot \boldsymbol{e}_r)\boldsymbol{e}_r = (\boldsymbol{f}' \cdot \boldsymbol{e}_\varphi)\boldsymbol{e}_\varphi$, we conclude that

$$\boldsymbol{f}' = -(m\ell\dot{\varphi}^2 + mg\cos\varphi)\boldsymbol{e}_r.$$

This tells us that the constraint reaction force $\boldsymbol{f}'$ lies in the direction of the pendulum's rod, and its magnitude equals the sum of the centrifugal force $m\ell\dot{\varphi}^2$ and the compoment $mg\cos\varphi$ of the bob's weight along the rod.

Example 6.2. Consider a the rolling hoop of Exercise (2) of Chapter 5. Find the force at contact point between the hoop and the plane.

Referring to Figure 5.3, we use the hoop's rotation angle φ and the horizontal translation x of its center as a generalized coordinates. The no-slip condition imposes the constraint $x = a\varphi$. We will use φ and x as overdetermined generalized coordinates subject to $x = a\varphi$, that is $dx - a\,d\varphi = 0$, thefore accoring to the template (6.1), $a_{11} = 1$, $a_{12} = -a$.

The position vector of the mass m is $\boldsymbol{r} = \langle x+a\sin\varphi, a+a\cos\varphi\rangle$, therefore the velocity is $\boldsymbol{v} = \dot{\boldsymbol{r}} = \langle \dot{x} + a\dot{\varphi}\cos\varphi, -a\dot{\varphi}\sin\varphi\rangle$. It follows that $\|\boldsymbol{v}\|^2 = \dot{x}^2 + 2a\dot{x}\dot{\varphi}\cos\varphi + a^2\dot{\varphi}^2$, therefore the kinetic and potential energies are

$$T = \frac{1}{2}m(\dot{x}^2 + 2a\dot{x}\dot{\varphi}\cos\varphi + a^2\dot{\varphi}^2), \quad V = mga(1+\cos\varphi).$$

Therefore

$$L = \frac{1}{2}m(\dot{x}^2 + 2a\dot{x}\dot{\varphi}\cos\varphi + a^2\dot{\varphi}^2) - mga(1+\cos\varphi).$$

and

$$\frac{\partial L}{\partial \dot{x}} = m(\dot{x} + a\dot{\varphi}\cos\varphi), \quad \frac{\partial L}{\partial x} = 0,$$
$$\frac{\partial L}{\partial \dot{\varphi}} = m(a\dot{x}\cos\varphi + a^2\dot{\varphi}) \quad \frac{\partial L}{\partial \varphi} = m(-a\dot{x}\dot{\varphi}\sin\varphi + ga\sin\varphi).$$

Then (6.4) takes the form

$$\begin{aligned} &m(\dot{x} + a\dot{\varphi}\cos\varphi)^{\cdot} = a_{11}\lambda_1 = \lambda_1, \\ &m(a\dot{x}\cos\varphi + a^2\dot{\varphi})^{\cdot} - m(-a\dot{x}\dot{\varphi}\sin\varphi + ga\sin\varphi) = a_{12}\lambda_1 = -a\lambda_1. \end{aligned}$$

Now substitute $x = a\varphi$ and simplify:

$$\begin{aligned} -ma(1+\cos\varphi)\ddot{\varphi} + ma\dot{\varphi}^2\sin\varphi &= \lambda_1, \\ ma^2(1+\cos\varphi)\ddot{\varphi} + mga\sin\varphi &= -a\lambda_1. \end{aligned} \tag{6.7}$$

To eliminate λ_1, divide the second equation through by a and add the result to the first equation. We get

$$2ma(1+\cos\varphi)\ddot{\varphi} - ma\Big(\frac{g}{a} + \dot{\varphi}^2\Big)\sin\varphi = 0.$$

This is the hoop's equation of motion. To compute the constraint force, solve this for $\ddot{\varphi}$:

$$\ddot{\varphi} = \frac{a\big(\frac{g}{a} + \dot{\varphi}^2\big)\sin\varphi}{2a(1+\cos\varphi)}$$

and substitute the result in the first of (6.7). We get:

$$\lambda_1 = \frac{1}{2}ma\Big(\dot{\varphi}^2 - \frac{g}{a}\Big)\sin\varphi. \tag{6.8}$$

Now we compute the generalized forces Q'_x and Q'_φ from (6.5):

$$Q'_x = a_{11}\lambda_1 = \lambda_1, \quad Q'_\varphi = a_{12}\lambda_1 = -a\lambda_1.$$

Then, we apply (6.6) to translate these into the physical components of the forces. Toward that end, let us observe that

$$\frac{\partial \boldsymbol{r}}{\partial x} = \langle 1, 0\rangle, \quad \frac{\partial \boldsymbol{r}}{\partial \varphi} = \langle a\cos\varphi, -a\sin\varphi\rangle,$$

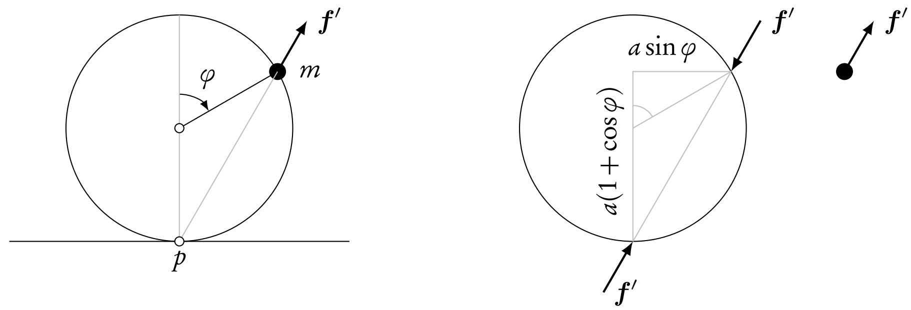


Figure 6.1: whatever

therefore, letting $\boldsymbol{f} = \langle f_x, f_y \rangle$, we get

$$Q'_x = \boldsymbol{f} \cdot \frac{\partial \boldsymbol{r}}{\partial x} = f_x \quad Q'_\varphi = \boldsymbol{f} \cdot \frac{\partial \boldsymbol{r}}{\partial \varphi} = a f_x \cos\varphi - a f_y \sin\varphi.$$

It follows that

$$f_x = \lambda_1, \quad a f_x \cos\varphi - a f_y \sin\varphi = -a\lambda_1.$$

We solve this as a system of two equations in the two unknowns f_x and f_y:

$$f_x = \lambda_1, \quad f_y = \frac{1+\cos\varphi}{\sin\varphi}\lambda_1.$$

Upon substitution for λ_1 from (6.8) we conclude that

$$f_x = \frac{1}{2}ma\big(\dot\varphi^2 - \frac{g}{a}\big)\sin\varphi,$$
$$f_y = \frac{1}{2}ma\big(\dot\varphi^2 - \frac{g}{a}\big)(1+\cos\varphi),$$

whence

$$\boldsymbol{f} = \frac{1}{2}ma\big(\dot\varphi^2 - \frac{g}{a}\big)\langle \sin\varphi, 1+\cos\varphi \rangle.$$

This result has a significant mechanical/geometric interpretation. Refer to Figure 6.1. The vector $a\langle \sin\varphi, 1+\cos\varphi\rangle$ extends from the contact point p to the mass m, therefore the constraint force $\boldsymbol{f}'$ is parallel to that vector.

Exercises

6.1. The hoop of Example 6.2 rolls, without slipping, down an incline which makes an angle α with respect to the horizontal. See Figure 6.2. Use the angle φ as the generalized coordinate. Derive the equation of motion. *Hint:* (a) Express the vectors $\boldsymbol{i}'$ and $\boldsymbol{j}'$ in terms of $\boldsymbol{i}$, $\boldsymbol{j}$, and the angle α; (b) Express the position vector $\boldsymbol{r}$ of the mass m in terms of the basis vectors $\boldsymbol{i}'$ and $\boldsymbol{j}'$; (c) Apply the results of (a) and (b) to express $\boldsymbol{r}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

6.2. In the the previous exercise, find the contact force between the hoop and incline.

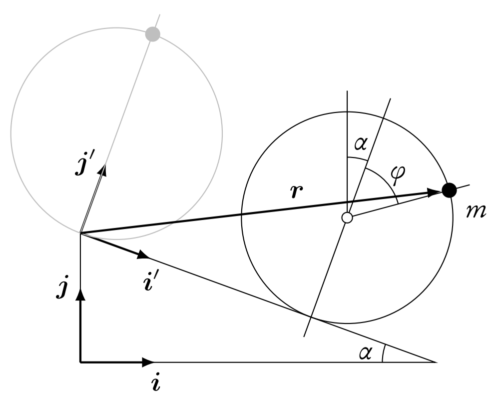


Figure 6.2: Hoop rolling on an incline. The hoop's initial position is shown in gray (Exercise 6.1).

Chapter 7

Angular velocity

When you hurl a rock in the air, or toss a Frisbee, the object spins in general as it moves. Associated with the motion is a vector $\boldsymbol{\omega}(t)$, called the object's *angular velocity vector*, (or just *angular velocity* for short,) whose direction and magnitude at any instant of time t indicate orientation and the rate of rotation. It is the goal of this section to make the definition of the angular velocity precise.

Toward that end, let $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ be an orthonormal set of vectors which is firmly affixed to the spinning object, therefore moves with it. The choice of the letter $\boldsymbol{b}$ is this notation is to remind us that we are dealing with a *body coordinate system*.[9]

In elementary linear algebra we learn that the component of an arbitrary vector $\boldsymbol{r}$ in the direction of the unit vector $\boldsymbol{b}_j$ is $\boldsymbol{r}\cdot\boldsymbol{b}_j$. Therefore the vector $\boldsymbol{r}$ may be expressed in the basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ as $\boldsymbol{r} = \sum_{j=1}^{3}(\boldsymbol{r}\cdot\boldsymbol{b}_j)\boldsymbol{b}_j$.[10] Therefore, the rate of change, $\dot{\boldsymbol{b}}_i$, of the vector $\boldsymbol{b}_i$ in the basis $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ may be expressed as

$$\dot{\boldsymbol{b}}_i = \sum_{j=1}^{3}(\dot{\boldsymbol{b}}_i\cdot\boldsymbol{b}_j)\boldsymbol{b}_j.$$

Let $a_{ij} = \dot{\boldsymbol{b}}_i\cdot\boldsymbol{b}_j$ be the matrix of the coefficients. We claim that the matrix, let's call it A, is skew-symmetric. Indeed, we have $\boldsymbol{b}_i\cdot\boldsymbol{b}_j = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$ Upon differentiating this we get $\dot{\boldsymbol{b}}_i\cdot\boldsymbol{b}_j + \dot{\boldsymbol{b}}_j\cdot\boldsymbol{b}_i = 0$, whence $a_{ij} + a_{ji} = 0$, which proves A is skew-symmetric.

The general form of a 3×3 skew-symmetric matrix is

$$A = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}. \tag{7.1}$$

[9] As the body rotates, the vectors $\{\boldsymbol{b}_j\}_{j=1}^3$ rotate with it, therefore they vary with time from the point of view of a stationary observer. On occasion we write $\{\boldsymbol{b}_j(t)\}_{j=1}^3$ to make explicit the dependence of these vectors on the time t.

[10] See equation (8.7) on page 49 for a simple proof.

Therefore

$$\dot{\boldsymbol{b}}_1 = \omega_3 \boldsymbol{b}_2 - \omega_2 \boldsymbol{b}_3, \tag{7.2a}$$
$$\dot{\boldsymbol{b}}_2 = \omega_1 \boldsymbol{b}_3 - \omega_3 \boldsymbol{b}_1, \tag{7.2b}$$
$$\dot{\boldsymbol{b}}_3 = \omega_2 \boldsymbol{b}_1 - \omega_1 \boldsymbol{b}_2. \tag{7.2c}$$

The coefficients ω_1, ω_2, ω_3 measure the rate of change of the triad $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$, and consequently, the rate of change of orientation of the body to which the triad is attached. The angular velocity vector, defined as

$$\boldsymbol{\omega} = \omega_1 \boldsymbol{b}_1 + \omega_2 \boldsymbol{b}_2 + \omega_3 \boldsymbol{b}_3 \tag{7.3}$$

encapsulates that rate of change.

Remark 7.1. Actually calling $\boldsymbol{\omega}$ a *vector* is premature. A vector, as defined precisely in Section 8.1, is a physical object in the sense that it is independent of any coordinate system which may be used in defining it. Here $\boldsymbol{\omega}$ has been defined in terms of its components on the $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ triad. But the triad is not an integral part of the rotating body. Does $\boldsymbol{\omega}$ survive if the triad goes away? Does $\boldsymbol{\omega}$ *remain the same* if that triad is replaced by another?

It turns out that the answer to both of those question is in the positive. Indeed, $\boldsymbol{\omega}$ *is not* an artifact of the arbitrarily chosen triad. It is an intrinsic property of the body's motion. The presence of a coordinate triad affixed to the body is immaterial. The proof of this claim is not too hard but it takes us deeper into tensor analysis, so I will skip it for now.

Remark 7.2. We leave it as an exercise to show that differentiating (7.3) and applying the equations (7.2), resutls in

$$\boldsymbol{\omega} = \dot{\omega}_1 \boldsymbol{b}_1 + \dot{\omega}_2 \boldsymbol{b}_2 + \dot{\omega}_3 \boldsymbol{b}_3. \tag{7.4}$$

Remark 7.3. On the one hand, in (7.1) we have $a_{23} = \omega_1$. On the other hand, we have the $a_{ij} = \dot{\boldsymbol{b}}_i \cdot \boldsymbol{b}_j$ by definition. It follows that $\omega_1 = \dot{\boldsymbol{b}}_2 \cdot \boldsymbol{b}_3$. Similar expressions are obtained for ω_2 and ω_3. Let's make a record of this:

$$\omega_1 = \dot{\boldsymbol{b}}_2 \cdot \boldsymbol{b}_3, \qquad \omega_2 = \dot{\boldsymbol{b}}_3 \cdot \boldsymbol{b}_1, \qquad \omega_3 = \dot{\boldsymbol{b}}_1 \cdot \boldsymbol{b}_2. \tag{7.5}$$

Remark 7.4. From the definition of $\boldsymbol{\omega}$ in (7.3) it follows that

$$\boldsymbol{\omega} \times \boldsymbol{b}_1 = (\omega_1 \boldsymbol{b}_1 + \omega_2 \boldsymbol{b}_2 + \omega_3 \boldsymbol{b}_3) \times \boldsymbol{b}_1 = -\omega_2 \boldsymbol{b}_3 + \omega_3 \boldsymbol{b}_2.$$

Therefore the equations (7.2) may be written as

$$\dot{\boldsymbol{b}}_1 = \boldsymbol{\omega} \times \boldsymbol{b}_1, \qquad \dot{\boldsymbol{b}}_2 = \boldsymbol{\omega} \times \boldsymbol{b}_2, \qquad \dot{\boldsymbol{b}}_3 = \boldsymbol{\omega} \times \boldsymbol{b}_3. \tag{7.6}$$

Remark 7.5. Consider points O and P in the body, and let $\boldsymbol{r} = \overrightarrow{OP}$. Then $\boldsymbol{r} = \alpha_1 \boldsymbol{b}_1 + \alpha_2 \boldsymbol{b}_2 + \alpha_3 \boldsymbol{b}_3$, where α_1, α_2, α_3 are constants. The velocity $\boldsymbol{v}$ of P relative to O is given by $\boldsymbol{v} = \dot{\boldsymbol{r}}$. Let us observe that

$$\begin{aligned} \boldsymbol{v} = \dot{\boldsymbol{r}} &= \alpha_1 \dot{\boldsymbol{b}}_1 + \alpha_2 \dot{\boldsymbol{b}}_2 + \alpha_3 \dot{\boldsymbol{b}}_3 \\ &= \alpha_1(\boldsymbol{\omega} \times \boldsymbol{b}_1) + \alpha_2(\boldsymbol{\omega} \times \boldsymbol{b}_2) + \alpha_3(\boldsymbol{\omega} \times \boldsymbol{b}_3) = \boldsymbol{\omega} \times (\alpha_1 \boldsymbol{b}_1 + \alpha_2 \boldsymbol{b}_2 + \alpha_3 \boldsymbol{b}_3) = \boldsymbol{\omega} \times \boldsymbol{r}. \end{aligned}$$

The formula

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{7.7}$$

is used throughout the rest of this book.

Exercises

7.1. Derive (7.4).

Chapter 8

The moment of inertia tensor

8.1 ▪ A brief introduction to tensor algebra

In much of our working and thinking, we tend to blur the distinction between a vector as a pointed arrow in space ↗, and an n-tuple of numbers $\langle x_1, x_2, \ldots, x_n \rangle$. Yet, the two concepts are drastically different. The former would have been quite a natural object to Euclid in 300 BC, but the latter would been very foreign to him—for at the time they lay two millennia in the future.

The blurring between the two ways of looking at vectors is harmless much of the time, but there are places where a strict distinction between the two views is crucially important. *Tensor algebra* makes a bridge between the two and goes substantially beyond. Tensor algebra, along with *tensor analysis*, are indispensable tools in differential geometry, continuum mechanics, and general relativity, and perhaps other disciplines as well. For a general exposition of tensor algebra and tensor analysis, see [3].

Tensor algebra is not a sine qua non of rigid body mechanics, therefore only rarely it is brought into play. There is, however, the ubiquitous use of the term *moment of inertia tensor*, which hints tantalizingly to a connection to tensors behind the scenes, however the connection is only rarely brought out. It is the purpose of this section and the next to introduce the minimal tensor algebra which explains the "tensor" in the "moment of inertia tensor".

8.1.1 ▪ Tensor algebra

Vectors, dot product, cross product. In the three-dimensional Euclidean space fix a point called *the origin*, and consider the set $\mathscr{V}$ of all vectors (in the sense of pointed arrows) whose tails are attached to the origin. We define the sum $\boldsymbol{x}+\boldsymbol{y}$ of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathscr{V}$ through the parallelogram rule, that is, we form a parallelogram based on the vectors, and consider the parallelogram's diagonal as their sum as illustrated in Figure 8.1. Multiplying a vector by a number stretches/shrinks the vector's length by the magnitude of that number. If the number is negative, the resulting vectors flips, that is, it points in the opposite of the original's direction. See Figure 8.1 for an illustration.

The length of the vector $\boldsymbol{x}$ is written $||\boldsymbol{x}||$. The *dot product* $\boldsymbol{x}\cdot\boldsymbol{y}$ of a pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathscr{V}$ is a number defined by

$$\boldsymbol{x}\cdot\boldsymbol{y} = ||\boldsymbol{x}||\,||\boldsymbol{y}||\cos\theta,$$

where θ is the angle between the vectors; see Figure 8.2. Let us note that (a) the dot

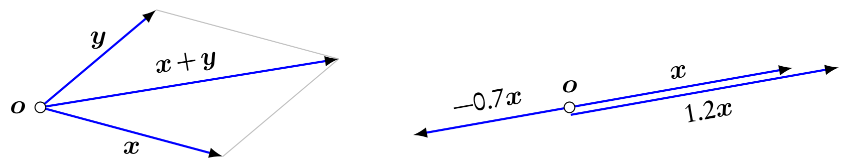


Figure 8.1: On the left: The sum $\boldsymbol{x}+\boldsymbol{y}$ of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is formed through the parallelogram rule. On the right: Multiplication by a number stretches/shrinks/flips a vector. The vector $1.2\boldsymbol{x}$ is drawn in a slightly displaced position relative to the origin $\boldsymbol{o}$ for the sake of visualization.

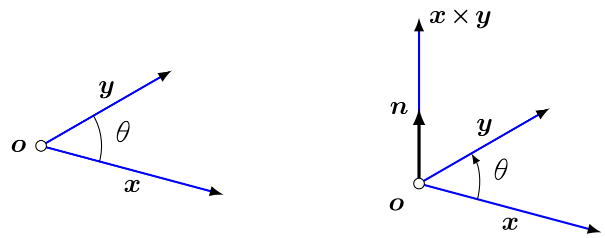


Figure 8.2: On the left: The dot product of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is the number $\boldsymbol{x}\cdot\boldsymbol{y} = ||\boldsymbol{x}||\,||\boldsymbol{y}||\cos\theta$. On the right: The cross product of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is the vector $\boldsymbol{x}\times\boldsymbol{y} = \big(||\boldsymbol{x}||\,||\boldsymbol{y}||\sin\theta\big)\boldsymbol{n}$, where $\boldsymbol{n}$ is a unit vector which is perpendicular to the plane of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, and is oriented according to the right-hand rule.

product is *commutative*, that is $\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{y}\cdot\boldsymbol{x}$; (b) if the vectors are orthogonal to each other, that is, $\theta = \pi/2$, then $\boldsymbol{x}\cdot\boldsymbol{y} = 0$; and (c) a vector's length may be expressed as a dot product: $||x||^2 = \boldsymbol{x}\cdot\boldsymbol{x}$.

The *cross product* of $\boldsymbol{x}\times\boldsymbol{y}$ of a pair of vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathscr{V}$ is the vector

$$\boldsymbol{x}\times\boldsymbol{y} = \big(||\boldsymbol{x}||\,||\boldsymbol{y}||\sin\theta\big)\boldsymbol{n},$$

where $0 \le \theta \le \pi$ is the angle between the vectors, and $\boldsymbol{n}$ is a unit vector which is perpendicular to the plane of the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, and is oriented according to the right-hand rule. The latter means that if you align your right hand's thumb with $\boldsymbol{n}$, then a rotation by an angle θ in the direction pointed at by your fingers will take the vector $\boldsymbol{x}$ to $\boldsymbol{y}$. It follows that the cross product is *anticommutative*, which means that $\boldsymbol{x}\times\boldsymbol{y} = -\boldsymbol{y}\times\boldsymbol{x}$.

Tensors. A function $L : \mathscr{V} \to \mathscr{V}$ is said to be *linear* if

$$\begin{aligned} L(\boldsymbol{x}+\boldsymbol{y}) &= L(\boldsymbol{x}) + L(\boldsymbol{y}) && \text{for all } \boldsymbol{x},\boldsymbol{y} \in \mathscr{V}, \\ L(\alpha\boldsymbol{x}) &= \alpha L(\boldsymbol{x}) && \text{for all } \boldsymbol{x} \in \mathscr{V},\ \alpha \in \mathbf{R}. \end{aligned}$$

A linear function from $\mathscr{V}$ to $\mathscr{V}$ is called a *tensor on* $\mathscr{V}$, or just *tensor*, for short. It is customary to write $L\boldsymbol{x}$ instead of $L(\boldsymbol{x})$ when L is a tensor, as we will do from now on. The *identity tensor*, I, is the tensor with the property

$$I\boldsymbol{x} = \boldsymbol{x} \quad \text{for all } \boldsymbol{x} \in \mathscr{V}. \tag{8.1}$$

The sum $L_1 + L_2$ of tensors L_1 and L_2 is a tensor defined by

$$(L_1 + L_2)\boldsymbol{x} = L_1\boldsymbol{x} + L_2\boldsymbol{x} \quad \text{for all } \boldsymbol{x} \in \mathscr{V}. \tag{8.2}$$

The product αL of a tensor L and a number $\alpha \in \mathbf{R}$ is a tensor defined by

$$(\alpha L)x = \alpha(L\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \mathscr{V}. \tag{8.3}$$

The set of tensors on $\mathscr{V}$, equipped with the operations defined in (8.2) and (8.3), is a vector space in the sense of an *abstract vector space* (not to be confused with vectors of the pointed-arrow kind).

The dyadic product. The *dyadic product* $\boldsymbol{a} \otimes \boldsymbol{b}$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathscr{V}$ is a tensor on $\mathscr{V}$ defined by

$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{x} = (\boldsymbol{b} \cdot \boldsymbol{x})\boldsymbol{a} \quad \text{for all } \boldsymbol{x} \in \mathscr{V}. \tag{8.4}$$

Lemma 8.1. *For any pair of vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathscr{V}$ we have*

$$||\boldsymbol{a} \times \boldsymbol{b}||^2 = \boldsymbol{a} \cdot \big(||\boldsymbol{b}||^2 I - \boldsymbol{b} \otimes \boldsymbol{b}\big)\boldsymbol{a}, \tag{8.5}$$

where I is the identity tensor.

Proof. Let θ be the angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. We have:

$$\begin{aligned}
||\boldsymbol{a} \times \boldsymbol{b}||^2 &= ||\boldsymbol{a}||^2 ||\boldsymbol{b}||^2 \sin^2\theta \\
&= ||\boldsymbol{a}||^2 ||\boldsymbol{b}||^2 (1 - \cos^2\theta) \\
&= ||\boldsymbol{a}||^2 ||\boldsymbol{b}||^2 - ||\boldsymbol{a}||^2 ||\boldsymbol{b}||^2 \cos^2\theta \\
&= ||\boldsymbol{b}||^2 (\boldsymbol{a} \cdot \boldsymbol{a}) - (\boldsymbol{a} \cdot \boldsymbol{b})^2 \\
&= \boldsymbol{a} \cdot \big[||\boldsymbol{b}||^2 \boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{b}\big] \\
&= \boldsymbol{a} \cdot \big[||\boldsymbol{b}||^2 \boldsymbol{a} - (\boldsymbol{b} \otimes \boldsymbol{b})\boldsymbol{a}\big] \\
&= \boldsymbol{a} \cdot (||\boldsymbol{b}||^2 I - \boldsymbol{b} \otimes \boldsymbol{b})\boldsymbol{a}.
\end{aligned}$$

□

8.1.2 ▪ Connection with R^3 and 3×3 matrices

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be an *orthonormal set* in $\mathscr{V}$. This means that the three vectors are mutually perpendicular, and each is of length one. Thus

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \tag{8.6}$$

The symbol δ_{ij} defined above is called the *Kronecker delta*.

In Exercise 8.1 you will show that an orthonormal set is linearly independent. If follows that the orthonormal set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ forms a basis for the three-dimensional vector space $\mathscr{V}$. Consequently, any $\boldsymbol{x} \in \mathscr{V}$ may be expressed as a linear combination of the form

$$\boldsymbol{x} = c_1\boldsymbol{e}_1 + c_2\boldsymbol{e}_2 + c_3\boldsymbol{e}_3.$$

Dot-multiplying through by $\boldsymbol{e}_1$ and taking into account the orthonormality of basis vectors, we get $\boldsymbol{x} \cdot \boldsymbol{e}_1 = c_1$. Repeating with $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ we conclude that $c_i = \boldsymbol{x} \cdot \boldsymbol{e}_i$, $i = 1, 2, 3$. Thus we have established the identity

$$\boldsymbol{x} = (\boldsymbol{x} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + (\boldsymbol{x} \cdot \boldsymbol{e}_2)\boldsymbol{e}_2 + (\boldsymbol{x} \cdot \boldsymbol{e}_3)\boldsymbol{e}_3, \quad \text{for all } \boldsymbol{x} \in \mathscr{V}. \tag{8.7}$$

For each i, the coefficient $\boldsymbol{x}\cdot\boldsymbol{e}_i$ is called the *component* of the vector $\boldsymbol{x}$ along $\boldsymbol{e}_i$.

Remark 8.1. Let us state emphatically that the components $\boldsymbol{x}\cdot\boldsymbol{e}_i$ *are not* properties of the vector $\boldsymbol{x}$ at all! If you replace one basis with another which is rotated in an arbitrary manner relative to the first, then the components of $\boldsymbol{x}$ will be different in general, while $\boldsymbol{x}$ has not been touched. Even if you remove the basis altogether, the vector $\boldsymbol{x}$ will happily continue to exist. Having said all that, it is sometimes useful to work with $\langle c_1, c_2, c_3\rangle \in \mathbf{R}^3$ as sort of an "avatar" of $\boldsymbol{x}\in\mathcal{V}$, as long you remain cognizant of what it is.

Remark 8.2. Applying the defintion of the diatic product (8.4), the identity (8.7) may be written in the equivalent form

$$\boldsymbol{x} = (\boldsymbol{e}_1\otimes\boldsymbol{e}_1)\boldsymbol{x} + (\boldsymbol{e}_2\otimes\boldsymbol{e}_2)\boldsymbol{x} + (\boldsymbol{e}_3\otimes\boldsymbol{e}_3)\boldsymbol{x},$$

which has at least two implications. First, upon factoring the $\boldsymbol{x}$ on the right-hand side, we see that

$$I = \boldsymbol{e}_1\otimes\boldsymbol{e}_1 + \boldsymbol{e}_2\otimes\boldsymbol{e}_2 + \boldsymbol{e}_3\otimes\boldsymbol{e}_3.$$

Second, $\boldsymbol{e}_i\otimes\boldsymbol{e}_i$ acting on any vector $\boldsymbol{x}$ produces the projection of $\boldsymbol{x}$ in the direction $\boldsymbol{e}_i$.

We noted earlier that the set of tensors on $\mathcal{V}$, equipped with the operations defined in (8.2) and (8.3), is a vector space of its own. The following lemma shows how to construct a basis for that vector space.

Lemma 8.2. *Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be an orthonormal basis in $\mathcal{V}$. Then $\{\boldsymbol{e}_i\otimes\boldsymbol{e}_j\}_{i,j=1}^3$ is a basis for the space of tensors on $\mathcal{V}$.*

Proof. We will show that any tensor L is a linear combination of the nine dyadic products $\{\boldsymbol{e}_i\otimes\boldsymbol{e}_j\}_{i,j=1}^3$. Toward that end, pick an arbitrary $\boldsymbol{x}\in\mathcal{V}$,

$$\boldsymbol{x} = \sum_{i=1}^3 (\boldsymbol{x}\cdot\boldsymbol{e}_i)\boldsymbol{e}_i,$$

and let $\boldsymbol{y} = L\boldsymbol{x}$. By the linearity of L we have

$$\boldsymbol{y} = L\boldsymbol{x} = \sum_{i=1}^3 (\boldsymbol{x}\cdot\boldsymbol{e}_i)L\boldsymbol{e}_i.$$

it follows that

$$\boldsymbol{y}\cdot\boldsymbol{e}_j = \sum_{i=1}^3 (\boldsymbol{x}\cdot\boldsymbol{e}_i)(L\boldsymbol{e}_i\cdot\boldsymbol{e}_j) = \sum_{i=1}^3 (L\boldsymbol{e}_i\cdot\boldsymbol{e}_j)(\boldsymbol{x}\cdot\boldsymbol{e}_i),$$

and consequently

$$\begin{aligned}\boldsymbol{y} &= \sum_{j=1}^3 (\boldsymbol{y}\cdot\boldsymbol{e}_j)\boldsymbol{e}_j \\ &= \sum_{j=1}^3\sum_{i=1}^3 (L\boldsymbol{e}_i\cdot\boldsymbol{e}_j)(\boldsymbol{x}\cdot\boldsymbol{e}_i)\boldsymbol{e}_j \\ &= \sum_{j=1}^3\sum_{i=1}^3 (L\boldsymbol{e}_i\cdot\boldsymbol{e}_j)(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)\boldsymbol{x},\end{aligned}$$

where we have made use of the definition (8.4). Since $\boldsymbol{y} = L\boldsymbol{x}$, this tells us that

$$L\boldsymbol{x} = \sum_{i=1}^{3}\sum_{j=1}^{3}(L\boldsymbol{e}_i \cdot \boldsymbol{e}_j)(\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\boldsymbol{x} \quad \text{for all } \boldsymbol{x} \in \mathscr{V},$$

therefore

$$L = \sum_{i=1}^{3}\sum_{j=1}^{3}(L\boldsymbol{e}_i \cdot \boldsymbol{e}_j)(\boldsymbol{e}_i \otimes \boldsymbol{e}_j), \tag{8.8}$$

which shows that L is a linear combination of the nine dyadic products $\{\boldsymbol{e}_i \otimes \boldsymbol{e}_j\}_{i,j=1}^{3}$. In other words, the set of vectors $\{\boldsymbol{e}_i \otimes \boldsymbol{e}_j\}_{i,j=1}^{3}$ spans the set of all tensors. To show that the set is a basis, it remains to show that it is linearly independent. You will do that in Exercise 8.2. □

Remark 8.3. The coefficients in (8.8) are called the *components* of L in the basis $\{\boldsymbol{e}_i \otimes \boldsymbol{e}_j\}_{i,j=1}^{3}$. Letting $\ell_{ij} = L\boldsymbol{e}_i \cdot \boldsymbol{e}_j$, we may write (8.8) in the abbreviated form

$$L = \sum_{i=1}^{3}\sum_{j=1}^{3}\ell_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j.$$

At times it is useful to identify the tensor L with the 3×3 matrix with components ℓ_{ij} but the caveats of Remark 8.1 apply equally well here. The components ℓ_{ij} are artifacts of the choice of the basis vectors. The tensor is an intrinsic property of the system and will continue to exist even when the basis vectors are obliterated.

Remark 8.4. In view of the one-to-one correspondence between 3×3 matrices and tensors on $\mathscr{V}$ noted above, every property or theorem in matrix algebra finds a counterpart in tensor analysis. For instance, a nonzero vector $\boldsymbol{x}$ is called and *eigenvector* of the tensor L if $L\boldsymbol{x} = \lambda\boldsymbol{x}$ for some $\lambda \in \mathbf{R}$.[11] The coefficient λ is the *eigenvalue* corresponding to that eigenvector.

8.1.3 ▪ Symmetric tensors

A tensor L is said to be *symmetric* if

$$\boldsymbol{a} \cdot (L\boldsymbol{b}) = (L\boldsymbol{a}) \cdot \boldsymbol{b} \quad \text{for all } \boldsymbol{a}, \boldsymbol{b} \in \mathscr{V}.$$

It is easy to show that if L is symmetric, then the matrix ℓ_{ij} of its components (see Remark 8.3) is symmetric, that is $\ell_{ij} = \ell_{ji}$ for $i, j = 1, 2, 3$.

We know from matrix analysis that the eigenvalues of a symmetric matrix are real, and its eigenvector may selected as an orthonormal set. This, along with what is known as *diagonalization of matrices* in linear algebra, lead to the *spectral decomposition theorem* in tensor algebra:

Theorem 8.3 (Spectral decomposition, cf. [3, p. 137]**).** *The eigenvalues λ_1, λ_2, λ_3 of a symmetric tensor L are real, and the corresponding eigenvectors, $\boldsymbol{g}_1$, $\boldsymbol{g}_2$, $\boldsymbol{g}_3$ may be selected as an orthonormal set. Furthermore,*

$$L = \lambda_1\boldsymbol{g}_1 \otimes \boldsymbol{g}_1 + \lambda_2\boldsymbol{g}_2 \otimes \boldsymbol{g}_2 + \lambda_3\boldsymbol{g}_3 \otimes \boldsymbol{g}_3. \tag{8.9}$$

[11] say something about complex fields

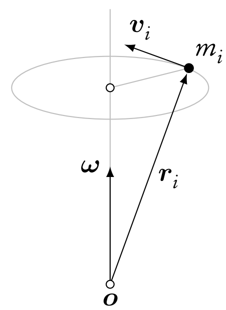


Figure 8.3: whatever

Remark 8.5. According to (8.8), a tensor is a linear combination of the *nine* basis elements $\{e_i \otimes e_j\}_{i,j=1}^3$ in general. In contrast, (8.9) presents L as a linear combination of *only three* special elements $\{g_j \otimes g_j\}_{j=1}^3$ constructed from L's eigenvectors.

8.2 ▪ The moment of inertia tensor

Consider a set of particles of masses m_i, $i = 1,2,\ldots,N$, which are thoroughly interconnected through massless rigid rods, so that the entire assembly forms a rigid, object. Suppose that the object rotates with angular velocity ω about a fixed axis passing through the origin. Figure 8.3 depicts the vector ω, and a representative particle of mass m_i and position vector Let r_i. Then the particle's velocity is $v_i = \omega \times r_i$, therefore the kinetic energy of the N-particle system is given by

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \|v_i\|^2 = \sum_{i=1}^{N} \frac{1}{2} m_i \|\omega \times r_i\|^2,$$

which according to Lemma 8.1 is equivalent to

$$\begin{aligned} T &= \sum_{i=1}^{N} \frac{1}{2} m_i \omega \cdot \big(\|r_i\|^2 I - r_i \otimes r_i\big)\omega \\ &= \frac{1}{2}\omega \cdot \left(\sum_{i=1}^{N} m_i \big(\|r_i\|^2 I - r_i \otimes r_i\big)\right)\omega. \end{aligned}$$

We introduce the tensor

$$\mathscr{I} = \sum_{i=1}^{N} m_i \big(\|r_i\|^2 I - r_i \otimes r_i\big), \tag{8.10}$$

whereby the kinetic energy takes the simple form

$$T = \frac{1}{2}\omega \cdot \mathscr{I}\omega. \tag{8.11}$$

The tensor $\mathscr{I}$ is called the *moment of inertia tensor* of the N-particle system. Let us note that $\mathscr{I}$ is independent of ω, so the orientation of the axis about which the system rotates, or the speed of rotation, is immaterial. However, $\mathscr{I}$ does depend on the choice of the origin of the vectors—changing the origin will affect the position vectors r_i, therefore the the tensor $\mathscr{I}$. The change, however, obeys a simple translation rule which we will develop in the next section. For now let us observe another aspect of (8.10). The fact that the system under consideration consists of N rigidly connected point masses is hardly of particular importance. A general rigid solid may be approximated by a union a large number of tiny parts, as one does in the theory of integration, and then pass to the limit as the number of the parts goes to infinity, and the sizes of the individual parts go to zero, while maintaining a fix mass for the aggregate.

To be specific, let $\mathscr{B}$ be the solid object, dm be the differential mass of the "part" of $\mathscr{B}$ indicated by the position vector $\boldsymbol{r}$ relative to some origin $\boldsymbol{o}$. Then the obvious extension of (8.10) takes the following form for the solid's moment of inertia:

$$\mathscr{I} = \int_{\mathscr{B}} \big(\|\boldsymbol{r}\|^2 I - \boldsymbol{r} \otimes \boldsymbol{r} \big)\, dm. \tag{8.12}$$

If $\rho(\boldsymbol{r})$ is the density of the body at the position $\boldsymbol{r}$, then $dm = \rho\, dV$, where V is the volume element, and the formula above takes the form

$$\mathscr{I} = \int_{\mathscr{B}} \rho(\boldsymbol{r}) \big(\|\boldsymbol{r}\|^2 I - \boldsymbol{r} \otimes \boldsymbol{r} \big)\, dV. \tag{8.13}$$

8.3 ▪ Translation of the origin

As noted above, the moment of inertia tensor of a rigid body depends on the choice of the origin of the vectors. To see how a translation of the origin affects the tensor, let $\mathscr{I}_{\boldsymbol{o}}$ and $\mathscr{I}_{\boldsymbol{o}'}$ be the moment of inertia tensors of a rigid body $\mathscr{B}$ relative to two origins $\boldsymbol{o}$ and $\boldsymbol{o}'$, respectively. According to (8.12) we have:

$$\mathscr{I}_{\boldsymbol{o}} = \int_{\mathscr{B}} \big(\|\boldsymbol{r}\|^2 I - \boldsymbol{r} \otimes \boldsymbol{r} \big)\, dm, \qquad \mathscr{I}_{\boldsymbol{o}'} = \int_{\mathscr{B}} \big(\|\boldsymbol{r}'\|^2 I - \boldsymbol{r}' \otimes \boldsymbol{r}' \big)\, dm,$$

where $\boldsymbol{r}$ and $\boldsymbol{r}'$ are the position vectors of a generic point $\boldsymbol{p} \in \mathscr{B}$ relative to $\boldsymbol{o}$ and $\boldsymbol{o}'$, as seen in Figure 8.4.

Theorem 8.4. *Let $\mathbf{c} \in \mathscr{B}$ the the body's center of mass, and let us write $\boldsymbol{r}'_{\boldsymbol{c}}$ for the position vector of $\mathbf{c}$ relative to $\boldsymbol{o}'$. Furthermore, let $\boldsymbol{\tau} = \boldsymbol{o}' - \boldsymbol{o}$. Then we have:*

$$\mathscr{I}_{\boldsymbol{o}} = \mathscr{I}_{\boldsymbol{o}'} + m \Big[\big(2\boldsymbol{r}'_{\boldsymbol{c}} \cdot \boldsymbol{\tau} + \|\boldsymbol{\tau}\|^2 \big) I - \boldsymbol{r}'_{\boldsymbol{c}} \otimes \boldsymbol{\tau} - \boldsymbol{\tau} \otimes \boldsymbol{r}'_{\boldsymbol{c}} - \boldsymbol{\tau} \otimes \boldsymbol{\tau} \Big], \tag{8.14}$$

where m is the body's mass.

Proof. Referring to Figure 8.4 we have $\boldsymbol{r} = \boldsymbol{r}' + \tau$, therefore

$$\begin{aligned}
\mathscr{I}_{\boldsymbol{o}} &= \int_{\mathscr{B}} \big(\|\boldsymbol{r}\|^2 I - \boldsymbol{r} \otimes \boldsymbol{r} \big)\, dm \\
&= \int_{\mathscr{B}} \big(\|\boldsymbol{r}' + \boldsymbol{\tau}\|^2 I - (\boldsymbol{r}' + \boldsymbol{\tau}) \otimes (\boldsymbol{r}' + \boldsymbol{\tau}) \big)\, dm \\
&= \int_{\mathscr{B}} \Big[\big(\|\boldsymbol{r}'\|^2 + 2\boldsymbol{r}' \cdot \boldsymbol{\tau} + \|\boldsymbol{\tau}\|^2 \big) I - \boldsymbol{r}' \otimes \boldsymbol{r}' - \boldsymbol{r}' \otimes \boldsymbol{\tau} - \boldsymbol{\tau} \otimes \boldsymbol{r}' - \boldsymbol{\tau} \otimes \boldsymbol{\tau} \Big]\, dm \\
&= \int_{\mathscr{B}} \big(\|\boldsymbol{r}'\|^2 I - \boldsymbol{r}' \otimes \boldsymbol{r}' \big)\, dm + \int_{\mathscr{B}} \Big[\big(2\boldsymbol{r}' \cdot \boldsymbol{\tau} + \|\boldsymbol{\tau}\|^2 \big) I - \boldsymbol{r}' \otimes \boldsymbol{\tau} - \boldsymbol{\tau} \otimes \boldsymbol{r}' - \boldsymbol{\tau} \otimes \boldsymbol{\tau} \Big]\, dm.
\end{aligned}$$

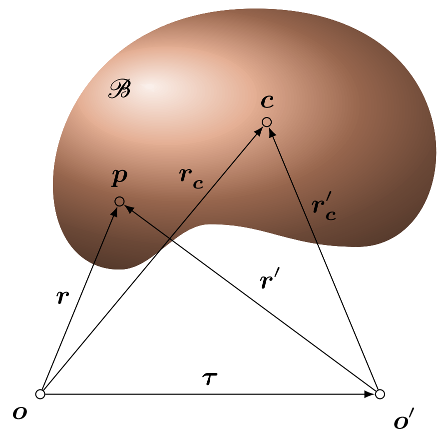


Figure 8.4: The moment of inertia tensor of a rigid solid $\mathscr{B}$ depends on the choice of the origin. The origins o' and o' in this illustration are related through $o' - o = \tau$. The solid's center of mass is c.

The first intergral on the right-hand side equals $\mathscr{I}_{o'}$. The second integral may be simplified by noting that

$$m = \int_{\mathscr{B}} dm, \qquad r'_c = \frac{1}{m}\int_{\mathscr{B}} r'\, dm,$$

and consequently $\int_{\mathscr{B}} r'\, dm = m r'_c$, □

Corollary 8.5. *Let* $c \in \mathscr{B}$ *be the body's center of mass as before, and let* $\mathscr{I}_c$ *be the moment of inertia tensor relative to* c. *Then the moment of inertia tensor* $\mathscr{I}_o$ *of* $\mathscr{B}$ *relative to any origin* o *is given by*

$$\mathscr{I}_o = \mathscr{I}_c + m\big[\|r_c\|^2 I - r_c \otimes r_c\big]. \tag{8.15}$$

Proof. Apply (8.14) with o' set to c. Then $r'_c = \mathbf{0}$, and the formula reduces to

$$\mathscr{I}_o = \mathscr{I}_c + m\big[\|\tau\|^2) I - \tau \otimes \tau\big].$$

Then the observation that $\tau = o' - o = c - o = r_c$ completes the proof. □

Remark 8.6. The moment of intertia tensor of a rigid system of N point masses is given in (8.10). In particualr, the moment of intertia tensor of a single point of mass m at a position r relative to an origin o is

$$m\big[\|r\|^2 I - r \otimes r\big].$$

Therefore the translation formula (8.15) may be interpreted as saying that the moment of interia of a body relative of any point o equals the moment of inertia relative to its center of mass, plus the moment of inertia relative to o of a fictitious point of mass m situated at the center of mass.

Remark 8.7. The moment of inertia tensor $\mathscr{I}_c$ of a ridig body relative to its center of mass is *an intrinsic property* of the body, just like its total mass or its center of mass are.

The total mass is a scalar, the center of mass is expressed through its position vector, and the moment of inertia $\mathscr{I}_{\boldsymbol{c}}$ is a tensor.

8.4 ▪ The principal moments of inertia

In the exercises you will verity that a moment of inertia tensor is a symmetric tensor, that is:

$$\boldsymbol{m}\cdot\mathscr{I}\boldsymbol{n}=\boldsymbol{n}\cdot\mathscr{I}\boldsymbol{m},\quad \text{for all } \boldsymbol{m},\boldsymbol{n}\in\mathscr{V}.$$

Then according to the Spectral Decomposition Theorem (page 51) $\mathscr{I}$ admits a representation of the form (8.9). In that context, the eigenvalues λ_1, λ_2, λ_3 of $\mathscr{I}$ are called the object's *principal moments of inertia*, and the eigenvectors $\boldsymbol{g}_1$, $\boldsymbol{g}_2$, $\boldsymbol{g}_3$ are called the *principal axes* of the moment of inertia.

Tables of the principal moments of inertia of many common geometric solids are available in books and websites. Wikipedia has a page for it at:

`http://en.wikipedia.org/wiki/List_of_moments_of_inertia`

Example 8.6. Consider a solid circular cylinder of radius r, length ℓ, and total mass m. Assume that the mass is distributed uniformly, that is, the density is constant. The principal axes of the moment of inertia tensor $\mathscr{I}_{\boldsymbol{c}}$ relative to the center of mass are: $\boldsymbol{g}_3$ along the cylinder's axis; $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ arbitrary unit vectors so that $\{\boldsymbol{g}_1,\boldsymbol{g}_2,\boldsymbol{g}_3\}$ forms an orthonormal set. The corresponding principal moments of inertia are $\lambda_1=\lambda_2=\frac{1}{12}m(3r^2+\ell^2)$, $\lambda_3=\frac{1}{2}mr^2$. Therefore

$$\mathscr{I}_{\boldsymbol{c}}=\frac{1}{12}m(3r^2+\ell^2)\boldsymbol{g}_1\otimes\boldsymbol{g}_1+\frac{1}{12}m(3r^2+\ell^2)\boldsymbol{g}_2\otimes\boldsymbol{g}_2+\frac{1}{2}mr^2\boldsymbol{g}_3\otimes\boldsymbol{g}_3. \tag{8.16}$$

The limiting case of $r=0$ is interesting in its own right. Such an object is called a *slender rod* of length ℓ and mass m. In that case we have:

$$\mathscr{I}_{\boldsymbol{c}}=\frac{1}{12}m\ell^2\boldsymbol{g}_1\otimes\boldsymbol{g}_1+\frac{1}{12}m\ell^2\boldsymbol{g}_2\otimes\boldsymbol{g}_2. \tag{8.17}$$

The principal moments of inertia $\mathscr{I}$ of a slender rod relative to one of its endpoints may be calculated from the translation formula (8.15).

Exercises

8.1. Show that an orthonormal set of vectors is linearly independent.

8.2. Complete the proof of Lemma 8.2 by showing that the set $\{\boldsymbol{e}_i\otimes\boldsymbol{e}_j\}_{i,j=1}^3$ is linearly independent. *Hint:* It suffices to show that $\sum_{i=1}^3\sum_{j=1}^3 c_{ij}(\boldsymbol{e}_i\otimes\boldsymbol{e}_j)=0$ implies that every c_{ij} is zero. Begin by applying each side of that equality to $\boldsymbol{e}_k$.

8.3. Show that if the tensor L is symmetric, then the matrix ℓ_{ij} of its components (see Remark 8.3) is symmetric, that is $\ell_{ij}=\ell_{ji}$ for $i,j=1,2,3$.

8.4. Use (8.17) in conjunction with the translation formula (8.15) to calculate the moment of inertia tensor of a slender rod relative to one of its endpoints.

8.5. Look up the principal moments of inertia of a circular hoop in Wikipedia relative to the hoop's center. Then apply the translation formula (8.15) to calculate the moment of inertia tensor of the hoop relative to a point on its rim.

Chapter 9

Rigid body dynamics through the Gibbs-Appell formulation

9.1 ▪ A formula for the Gibbs function

In this chapter we investigate the motion of a right body $\mathscr{B}$ of mass m and moment of inertia tensor $\mathscr{I}$. Let $\boldsymbol{r}_c$ be the position vector of the body's center of mass.

Let P be a generic infinitesimal part of mass dm of $\mathscr{B}$, and let $\boldsymbol{r}$ be the position vector of P relative to C. Then, by the definition of center of mass, we have

$$\int_{\mathscr{B}} \boldsymbol{r}\, dm = \boldsymbol{0}. \tag{9.1}$$

We assume that an external force $\boldsymbol{f}\, dm$ acts on the part P. The resultant of forces applied to the body, and their moments about the center of mass, are

$$\boldsymbol{F} \equiv \int_{\mathscr{B}} \boldsymbol{f}\, dm, \qquad \boldsymbol{M} \equiv \int_{\mathscr{B}} \boldsymbol{r} \times \boldsymbol{f}\, dm. \tag{9.2}$$

The contribution of the part P to the overall Gibbs function is $\frac{1}{2}\|\dot{\boldsymbol{v}}\|^2\, dm - \boldsymbol{f}\cdot\dot{\boldsymbol{v}}\, dm$, and therefore the overall Gibbs function is

$$\mathfrak{G} = \int_{\mathscr{B}} \frac{1}{2}\|\dot{\boldsymbol{v}}\|^2\, dm - \int_{\mathscr{B}} \boldsymbol{f}\cdot\dot{\boldsymbol{v}}\, dm. \tag{9.3}$$

We now proceed to expand, evaluate, and simplify this expression.

Let $\boldsymbol{\omega}$ be the body's angular velocity. Then $\dot{\boldsymbol{r}} = \boldsymbol{\omega}\times\boldsymbol{r}$, and therefore the velocity of the point P is $\boldsymbol{v} = \dot{\boldsymbol{r}}_c + \dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_c + \boldsymbol{\omega}\times\boldsymbol{r}$, and therefore the acceleration of P is

$$\dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}}_c + \dot{\boldsymbol{\omega}}\times\boldsymbol{r} + \boldsymbol{\omega}\times\dot{\boldsymbol{r}}.$$

Substituting this in (9.3) and expanding, we get

$$\begin{aligned}
\mathfrak{G} = {} & \frac{1}{2}\int_{\mathscr{B}} \|\ddot{\boldsymbol{r}}_c\|^2\, dm + \frac{1}{2}\int_{\mathscr{B}} \|\dot{\boldsymbol{\omega}}\times\boldsymbol{r}\|^2\, dm + \frac{1}{2}\int_{\mathscr{B}} \|\boldsymbol{\omega}\times\dot{\boldsymbol{r}}\|^2\, dm \\
& + \int_{\mathscr{B}} \ddot{\boldsymbol{r}}_c\cdot(\dot{\boldsymbol{\omega}}\times\boldsymbol{r})\, dm + \int_{\mathscr{B}} \ddot{\boldsymbol{r}}_c\cdot(\boldsymbol{\omega}\times\dot{\boldsymbol{r}})\, dm + \int_{\mathscr{B}} (\dot{\boldsymbol{\omega}}\times\boldsymbol{r})\cdot(\boldsymbol{\omega}\times\dot{\boldsymbol{r}})\, dm \\
& - \int_{\mathscr{B}} \boldsymbol{f}\cdot\ddot{\boldsymbol{r}}_c\, dm - \int_{\mathscr{B}} \boldsymbol{f}\cdot(\dot{\boldsymbol{\omega}}\times\boldsymbol{r})\, dm - \int_{\mathscr{B}} \boldsymbol{f}\cdot(\boldsymbol{\omega}\times\dot{\boldsymbol{r}})\, dm.
\end{aligned}$$

Now let's analyze each of the nine integrals on the right-hand side.

1. The integrand $||\ddot{\boldsymbol{r}}_c||^2$ in the first integral is independent of the position vector $\boldsymbol{r}$, and therefore the integral evaluates to

$$\frac{1}{2}\int_{\mathscr{B}} ||\ddot{\boldsymbol{r}}_c||^2\,dm = \frac{1}{2}m||\ddot{\boldsymbol{r}}_c||^2.$$

2. The integrand $||\dot{\boldsymbol{\omega}} \times \boldsymbol{r}||^2$ of the second integral may be expanded through the identity (8.5) into the form $||\dot{\boldsymbol{\omega}} \times \boldsymbol{r}||^2 = \dot{\boldsymbol{\omega}} \cdot \big(||\boldsymbol{r}||^2 I - \boldsymbol{r} \otimes \boldsymbol{r}\big)\dot{\boldsymbol{\omega}}$. The integral of parenthesized expression is the moment of inertia tensor, $\mathscr{I}$, and therefore

$$\frac{1}{2}\int_{\mathscr{B}} ||\dot{\boldsymbol{\omega}} \times \boldsymbol{r}||^2\,dm = \frac{1}{2}\dot{\boldsymbol{\omega}} \cdot \mathscr{I}\dot{\boldsymbol{\omega}}.$$

3. In the third integral we substitute $\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}$, and thus obtain

$$\frac{1}{2}\int_{\mathscr{B}} ||\boldsymbol{\omega} \times \dot{\boldsymbol{r}}||^2\,dm = \frac{1}{2}\int_{\mathscr{B}} ||\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})||^2\,dm$$

   This expression contains no acceleration terms, and therefore it will drop out when applying the Gibbs-Appell equations. Consequently we may safely ignore it. This is indicated by the ellipses in the final result below.

4. The integrand of the fourth integral consists of a triple vector product. We rotate the elements to get the $\boldsymbol{r}$ factor out, as in $\ddot{\boldsymbol{r}}_c \cdot (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) = \boldsymbol{r} \cdot (\ddot{\boldsymbol{r}}_c \cdot \dot{\boldsymbol{\omega}})$. The factor $(\ddot{\boldsymbol{r}}_c \cdot \dot{\boldsymbol{\omega}})$ is independent of $\boldsymbol{r}$, and therefore

$$\int_{\mathscr{B}} \ddot{\boldsymbol{r}}_c \cdot (\dot{\boldsymbol{\omega}} \times \boldsymbol{r})\,dm = \Big(\int_{\mathscr{B}} \boldsymbol{r}\,dm\Big) \cdot (\ddot{\boldsymbol{r}}_c \cdot \dot{\boldsymbol{\omega}}),$$

   which, according to (9.1), evaluates to zero.

5. The integrand of the fifth integral consists of a triple vector product. We rotate the elements to get the $\dot{\boldsymbol{r}}$ factor out, then replace $\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}$, as in

$$\begin{aligned}\int_{\mathscr{B}} \ddot{\boldsymbol{r}}_c \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}})\,dm &= \int_{\mathscr{B}} \dot{\boldsymbol{r}} \cdot (\ddot{\boldsymbol{r}}_c \cdot \boldsymbol{\omega})\,dm \\ &= \int_{\mathscr{B}} (\boldsymbol{\omega} \times \boldsymbol{r}) \cdot (\ddot{\boldsymbol{r}}_c \cdot \boldsymbol{\omega})\,dm = \Big(\boldsymbol{\omega} \times \int_{\mathscr{B}} \boldsymbol{r}\,dm\Big) \cdot (\ddot{\boldsymbol{r}}_c \cdot \boldsymbol{\omega}),\end{aligned}$$

   which, according to (9.1), evaluates to zero.

6. To evaluate the sixth integral, we begin with simplifying the integrand.

$$\begin{aligned}(\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}) &= (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \cdot \big[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\big] \\ &= (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \cdot \big[(\boldsymbol{\omega} \cdot \boldsymbol{r})\boldsymbol{\omega} - ||\boldsymbol{\omega}||^2\boldsymbol{r}\big] = (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \cdot \big[(\boldsymbol{\omega} \cdot \boldsymbol{r})\boldsymbol{\omega}\big] \\ &= (\boldsymbol{\omega} \cdot \boldsymbol{r})\big[(\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \cdot \boldsymbol{\omega}\big] = (\boldsymbol{\omega} \cdot \boldsymbol{r})\big[(\boldsymbol{\omega} \times \dot{\boldsymbol{\omega}}) \cdot \boldsymbol{r}\big] \\ &= (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \big[-(\boldsymbol{\omega} \cdot \boldsymbol{r})\boldsymbol{r}\big] = (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \big[-(\boldsymbol{r} \otimes \boldsymbol{r})\boldsymbol{\omega}\big] \\ &= (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \big[||\boldsymbol{r}||^2\boldsymbol{\omega} - (\boldsymbol{r} \otimes \boldsymbol{r})\boldsymbol{\omega}\big] \\ &= (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \big[||\boldsymbol{r}||^2 I - (\boldsymbol{r} \otimes \boldsymbol{r})\big]\boldsymbol{\omega}.\end{aligned}$$

Consequently,

$$\int_{\mathscr{B}} (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}})\, dm = (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \mathscr{I} \boldsymbol{\omega} = (\boldsymbol{\omega} \times \mathscr{I} \boldsymbol{\omega}) \cdot \dot{\boldsymbol{\omega}}.$$

7. In the seventh integral, the factor $\ddot{\boldsymbol{r}}_c$ is independent of the position vector $\boldsymbol{r}$, and therefore, according to (9.2) we have

$$\int_{\mathscr{B}} \boldsymbol{f} \cdot \ddot{\boldsymbol{r}}_c \, dm = \Big(\int_{\mathscr{B}} \boldsymbol{f} \cdot dm \Big) \ddot{\boldsymbol{r}}_c = \boldsymbol{F} \cdot \ddot{\boldsymbol{r}}_c.$$

8. We rotate the elements of the the triple vector product in the integrand of the eighth integral, integrate, and apply (9.2) to obtain

$$\int_{\mathscr{B}} \boldsymbol{f} \cdot (\dot{\boldsymbol{\omega}} \times \boldsymbol{r})\, dm = \int_{\mathscr{B}} \dot{\boldsymbol{\omega}} \cdot (\boldsymbol{r} \times \boldsymbol{f})\, dm = \dot{\boldsymbol{\omega}} \cdot \int_{\mathscr{B}} \boldsymbol{r} \times \boldsymbol{f} \, dm = \boldsymbol{M} \cdot \dot{\boldsymbol{\omega}}.$$

9. In the integrand of the ninth integral we have $\boldsymbol{\omega} \times \dot{\boldsymbol{r}} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$, which contains no acceleration terms and therefore may be ignored.

Putting all the pieces together we arrive at

$$\mathfrak{G} = \frac{1}{2} m \|\ddot{\boldsymbol{r}}_c\|^2 + \frac{1}{2} \dot{\boldsymbol{\omega}} \cdot \mathscr{I} \dot{\boldsymbol{\omega}} + (\boldsymbol{\omega} \times \mathscr{I} \boldsymbol{\omega}) \cdot \dot{\boldsymbol{\omega}} - \boldsymbol{F} \cdot \ddot{\boldsymbol{r}}_c - \boldsymbol{M} \cdot \dot{\boldsymbol{\omega}} + \cdots, \tag{9.4}$$

where the ellipses indicate terms that do not contain accelerations.

9.2 ▪ Example: Euler's equations of motion

If the resultant applied force $\boldsymbol{F}$ is zero, then the body's center of mass is not accelerated, that is, $\ddot{\boldsymbol{r}}_c = \mathbf{0}$, and therefore then the Gibbs-Appell equations $\frac{\partial \mathfrak{G}}{\partial \dot{\boldsymbol{\omega}}} = 0$ reduce to

$$\mathscr{I} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathscr{I} \boldsymbol{\omega} = \boldsymbol{M}.$$

These are known as *Euler's equations of the motion* of a rigid body. Letting I_1, I_2, I_3 be the body's principal moments of inertia, and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\boldsymbol{M} = (M_1, M_2, M_3)$ be the components of the angular velocity and the applied momentum along the principal axes of inertia, this takes the form

$$\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \times \left[\begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \right] = \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix},$$

which simplifies to

$$\begin{aligned} I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3 &= M_1, \\ I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_3 \omega_1 &= M_2, \\ I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1 \omega_1 &= M_3. \end{aligned}$$

9.3 ▪ Rotation about a fixed point

In Section 9.1 we developed a formula for the Gibbs function of the general motion of a rigid body. In this section we develop the corresponding formula when an arbitrary point of the rigid body is fixed in lab frame of reference, as in a spinning top.

Let's write O for the fixed point, and let $\boldsymbol{r}$ be the position vector, relative to O, of an arbitrary particle P of the body. As before, let $\boldsymbol{\omega}$ be the body's angular velocity. Then $\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}$, and therefore the velocity of the point P is $\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_c + \boldsymbol{\omega} \times \boldsymbol{r}$, and therefore the acceleration of P is

$$\dot{\boldsymbol{v}} = \dot{\boldsymbol{\omega}} \times \boldsymbol{r} + \boldsymbol{\omega} \times \dot{\boldsymbol{r}}.$$

Substituting this in (9.3) and expanding, we get

$$\begin{aligned}\mathfrak{G} = \frac{1}{2}\int_{\mathscr{B}} \|\dot{\boldsymbol{\omega}} \times \boldsymbol{r}\|^2\, dm + \frac{1}{2}\int_{\mathscr{B}} \|\boldsymbol{\omega} \times \dot{\boldsymbol{r}}\|^2\, dm + \qquad \int_{\mathscr{B}} (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}})\, dm \\ - \int_{\mathscr{B}} \boldsymbol{f} \cdot (\dot{\boldsymbol{\omega}} \times \boldsymbol{r})\, dm - \int_{\mathscr{B}} \boldsymbol{f} \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}})\, dm.\end{aligned}$$

Now let's analyze each of the nine integrals on the right-hand side.

1. The integrand $\|\dot{\boldsymbol{\omega}} \times \boldsymbol{r}\|^2$ of the first integral may be expanded through the identity (8.5) into the form $\|\dot{\boldsymbol{\omega}} \times \boldsymbol{r}\|^2 = \dot{\boldsymbol{\omega}} \cdot \big(\|\boldsymbol{r}\|^2 I - \boldsymbol{r} \otimes \boldsymbol{r}\big)\dot{\boldsymbol{\omega}}$. The integral of parenthesized expression is the moment of inertia tensor, $\mathscr{I}$, *with respect to the fixed point* O, and therefore

$$\frac{1}{2}\int_{\mathscr{B}} \|\dot{\boldsymbol{\omega}} \times \boldsymbol{r}\|^2\, dm = \frac{1}{2}\dot{\boldsymbol{\omega}} \cdot \mathscr{I}\dot{\boldsymbol{\omega}}.$$

2. In the second integral we substitute $\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r}$, and thus obtain

$$\frac{1}{2}\int_{\mathscr{B}} \|\boldsymbol{\omega} \times \dot{\boldsymbol{r}}\|^2\, dm = \frac{1}{2}\int_{\mathscr{B}} \|\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\|^2\, dm$$

   This expression contains no acceleration terms, and therefore it will drop out when applying the Gibbs-Appell equations. Consequently we may safely ignore it. This is indicated by the ellipses in the final result below.

3. To evaluate the third integral, we apply the identity

$$(\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}}) = (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \big[\|\boldsymbol{r}\|^2 I - (\boldsymbol{r} \otimes \boldsymbol{r})\big]\boldsymbol{\omega}$$

   derived in Section 9.1. Then

$$\int_{\mathscr{B}} (\dot{\boldsymbol{\omega}} \times \boldsymbol{r}) \cdot (\boldsymbol{\omega} \times \dot{\boldsymbol{r}})\, dm = (\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \mathscr{I}\boldsymbol{\omega} = (\boldsymbol{\omega} \times \mathscr{I}\boldsymbol{\omega}) \cdot \dot{\boldsymbol{\omega}},$$

   where $\mathscr{I}$ is the moment of inertia tensor *with respect to the fixed point* O.

4. We rotate the elements of the the triple vector product in the integrand of the fourth integral and integrate apply (9.2) to obtain

$$\int_{\mathscr{B}} \boldsymbol{f} \cdot (\dot{\boldsymbol{\omega}} \times \boldsymbol{r})\, dm = \int_{\mathscr{B}} \dot{\boldsymbol{\omega}} \cdot (\boldsymbol{r} \times \boldsymbol{f})\, dm = \dot{\boldsymbol{\omega}} \cdot \int_{\mathscr{B}} \boldsymbol{r} \times \boldsymbol{f}\, dm = \boldsymbol{M} \cdot \dot{\boldsymbol{\omega}},$$

   where M is the moment of the applied forces *with respect to the fixed point* O.

5. In the integrand of the fifth integral we have $\boldsymbol{\omega} \times \dot{\boldsymbol{r}} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})$, which contains no acceleration terms and therefore may be ignored.

Putting all the pieces together we arrive at

$$\mathfrak{G} = \frac{1}{2}\dot{\boldsymbol{\omega}} \cdot \mathscr{I}\dot{\boldsymbol{\omega}} + (\boldsymbol{\omega} \times \mathscr{I}\boldsymbol{\omega}) \cdot \dot{\boldsymbol{\omega}} - \boldsymbol{M} \cdot \dot{\boldsymbol{\omega}} + \cdots, \tag{9.5}$$

where the ellipses indicate terms that do not contain accelerations.

Remark 9.1. If the applied force density $\boldsymbol{f}$ is constant, then there is a simple formula for the moment M. Specifically, we have

$$M = \int_{\mathscr{B}} \boldsymbol{r} \times \boldsymbol{f}\, dm = \Big(\int_B \boldsymbol{r}\, dm \Big) \times \boldsymbol{f} = m\boldsymbol{r}_c \times \boldsymbol{f} = \boldsymbol{r}_c \times \boldsymbol{F},$$

where $\boldsymbol{r}_c$ is the position vector of the body's center of mass relative to O, and $\boldsymbol{F}$ is the resultant of the external forces applied to the body.

Chapter 10

The Gibbs-Appell formulation of dynamics

The goal of this chapter is to derive the Gibbs-Appell equations of motion and then show a few applications. For now, I have two versions of the proof here, one based on the presentation in Lurie [12] and the other based on the presentation in Gantmacher [8]. The part based on Lurie is not quite finished; the one based on Gantmacher is pretty complete.

Due to the reliance on two different presentations, the notation in the illustrations/examples is very inconsistent. At one point I will rewrite this chapter by merging the two presentations into one, and introduce a consistent notation.

Some of the material from the previous chapters is repeated for the sake of making this chapter somewhat self-contained.

10.1 ▪ Gibbs-Appell according to Lurie [12]

The goal of this (lengthy) section is to derive the Gibbs-Appell equations of motion (10.27).

10.1.1 ▪ Acceleration in generalized coordinates

Consider the dynamics of N point masses, whose position vectors relative to an origin $\boldsymbol{o}$ are $\boldsymbol{r}_i$, $i = 1, \ldots, N$.

Suppose that the system's configuration may be specified through n *independent* generalized coordinates $\boldsymbol{q} = \langle q_1, \ldots, q_n \rangle$. Thus, $\boldsymbol{r}_i = \boldsymbol{r}_i(\boldsymbol{q}, t)$, and the velocities are given by

$$\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \sum_{s=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_s} \dot{q}_s + \frac{\partial \boldsymbol{r}_i}{\partial t}, \quad i = 1, 2, \ldots, N. \tag{10.1}$$

We differentiate the velocities to calculate the accelerations $\boldsymbol{w}_i$:

$$\boldsymbol{w}_i = \dot{\boldsymbol{v}}_i = \sum_{s=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_s} \ddot{q}_s + \sum_{k=1}^{n} \sum_{s=1}^{n} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_k \partial q_s} \dot{q}_k \dot{q}_s + 2 \sum_{s=1}^{n} \frac{\partial^2 \boldsymbol{r}_i}{\partial q_s \partial t} \dot{q}_s + \frac{\partial^2 \boldsymbol{r}_i}{\partial t^2}. \tag{10.2}$$

10.1.2 ▪ Ideal constraints and the fundamental equation of dynamics

Continuing the previous subsection's analysis of the motion of N particles, let $\boldsymbol{F}_i + \boldsymbol{R}_i$ be the totality of forces acting on the particle i, where $\boldsymbol{F}_i$ is the resultant of the (known) externally applied forces and $\boldsymbol{R}_i$ is the resultant of the (unknown) internal forces/reactions.

Newton's law of motion states that

$$m_i \boldsymbol{w}_i = \boldsymbol{F}_i + \boldsymbol{R}_i, \quad i = 1, \ldots, N. \tag{10.3}$$

We *assume* that the system's constraints are *ideal*, that is, they do not perform work in any virtual displacement. (See page 271 of Lurie's book for a discussion.) Thus, for all virtual displacements $\delta \boldsymbol{r}_i$ we have

$$\sum_{i=1}^{N} \boldsymbol{R}_i \cdot \delta \boldsymbol{r}_i = 0. \tag{10.4}$$

As a consequence, multiplying (10.3) by $\delta \boldsymbol{r}_i$ and summing over i eliminates the the reaction forces $\boldsymbol{R}_i$:

$$\sum_{i=1}^{N} m_i \boldsymbol{w}_i \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i. \tag{10.5}$$

Lagrange called this the *fundamental equation of dynamics*.

10.1.3 ▪ Virtual work and generalized force

The virtual displacement $\delta \boldsymbol{r}_i$ of particle i is related to the virtual displacement $\delta \boldsymbol{q}$ of the generalized coordinates through

$$\delta \boldsymbol{r}_i = \sum_{s=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_s} \delta q_s.$$

The expression on the right-hand side of (10.5) is the virtual work performed by all external forces $\boldsymbol{F}_i$ under the virtual displacements $\delta \boldsymbol{r}_i$. It may be expressed as

$$\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \left(\sum_{s=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_s} \delta q_s \right) = \sum_{s=1}^{n} \left(\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_s} \right) \delta q_s = \sum_{s=1}^{n} Q_s \, \delta q_s,$$

where we have let

$$Q_s = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_s}. \tag{10.6}$$

Q_s is called the *generalized force corresponding to the generalized coordinate* q_s.

TODO: It can be shown that

$$\frac{\partial \boldsymbol{r}_i}{\partial q_s} = \frac{\partial \dot{\boldsymbol{r}}_i}{\partial \dot{q}_s} = \frac{\partial \ddot{\boldsymbol{r}}_i}{\partial \ddot{q}_s}.$$

In formulating the Gibbs-Appell equations, it works better if we replace (10.6) with

$$Q_s = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \ddot{\boldsymbol{r}}_i}{\partial \ddot{q}_s}. \tag{10.7}$$

10.1.4 ▪ Constraints

Suppose that the n generalized coordinates are related through l (generally nonholonomic) constraints

$$\sum_{s=1}^{n} a_{ks}(\boldsymbol{q},t)\dot{q}_s + a_k(\boldsymbol{q},t) = 0, \quad k = 1,\dots,l, \tag{10.8}$$

or equivalently,

$$\sum_{s=1}^{n} a_{ks}(\boldsymbol{q},t)\,dq_s + a_k(\boldsymbol{q},t)\,dt = 0. \quad k = 1,\dots,l. \tag{10.9}$$

Consequently, virtual displacements $\delta\boldsymbol{q}$ satisfy

$$\sum_{s=1}^{n} a_{ks}(\boldsymbol{q},t)\,\delta q_s = 0, \quad k = 1,\dots,l. \tag{10.10}$$

We assume that the $l \times n$ coefficient matrix a_{ks} is full-rank, therefore (10.8) may be solved for l of the generalized velocities in terms of the rest. Thus:

$$\dot{q}_r = \sum_{s=1}^{n-l} b_{r,l+s}(\boldsymbol{q},t)\dot{q}_{l+s} + b_r(\boldsymbol{q},t), \quad r = 1,\dots,l. \tag{10.11}$$

Similarly, (10.10) may be solved for l of the virtual displacements in terms of the rest:

$$\delta q_r = \sum_{s=1}^{n-l} b_{r,l+s}(\boldsymbol{q},t)\,\delta q_{l+s}, \quad r = 1,\dots,l. \tag{10.12}$$

Differentiating (10.11) we obtain an expression for the accelerations:

$$\ddot{q}_r = \sum_{s=1}^{n-l} b_{r,l+s}(\boldsymbol{q},t)\ddot{q}_{l+s} + \sum_{s=1}^{n-l} \dot{b}_{r,l+s}(\boldsymbol{q},t)\dot{q}_{l+s} + \sum_{s=1}^{n-l} \dot{b}_r(\boldsymbol{q},t), \quad r = 1,\dots,l.$$

For reasons which will become clear shortly, the terms which involve no generalized accelerations in our calculations are irrelevant, therefore, to simplify the notation, we write the above as

$$\ddot{q}_r = \sum_{s=1}^{n-l} b_{r,l+s}(\boldsymbol{q},t)\ddot{q}_{l+s} + \cdots. \tag{10.13}$$

From here on, the ellipsis "$\cdots$" in an equation indicates additive terms that involve no generalized accelerations.

Now we apply (10.13) to eliminate the accelerations $\ddot{q}_r$, $r = 1,\dots,l$ from (10.2). We have:

$$\begin{aligned}
\boldsymbol{w}_i &= \sum_{s=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_s}\ddot{q}_s + \cdots \\
&= \sum_{r=1}^{l} \frac{\partial \boldsymbol{r}_i}{\partial q_r}\ddot{q}_r + \sum_{s=1}^{n-l} \frac{\partial \boldsymbol{r}_i}{\partial q_{l+s}}\ddot{q}_{l+s} + \cdots \\
&= \sum_{r=1}^{l} \frac{\partial \boldsymbol{r}_i}{\partial q_r}\left(\sum_{s=1}^{n-l} b_{r,l+s}(\boldsymbol{q},t)\ddot{q}_{l+s} + \cdots\right) + \sum_{s=1}^{n-l} \frac{\partial \boldsymbol{r}_i}{\partial q_{l+s}}\ddot{q}_{l+s} + \cdots \\
&= \sum_{s=1}^{n-l}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_{l+s}} + \sum_{r=1}^{l} \frac{\partial \boldsymbol{r}_i}{\partial q_r} b_{r,l+s}(\boldsymbol{q},t)\right)\ddot{q}_{l+s} + \cdots.
\end{aligned}$$

Upon introducing the notation

$$\boldsymbol{c}_{i,l+s} = \frac{\partial \boldsymbol{r}_i}{\partial q_{l+s}} + \sum_{r=1}^{l} \frac{\partial \boldsymbol{r}_i}{\partial q_r} b_{r,l+s}(\boldsymbol{q},t), \quad i=1,\ldots,N, \quad s=1,\ldots,n-l, \tag{10.14}$$

the acceleration takes the form

$$\boldsymbol{w}_i = \sum_{s=1}^{n-l} \boldsymbol{c}_{i,l+s}\ddot{q}_{l+s} + \cdots, \quad i=1,\ldots,N.$$

Since the terms hidden under the ellipsis do not involve the generalized accelerations, we conclude that

$$\frac{\partial \boldsymbol{w}_i}{\partial \ddot{q}_{l+s}} = \boldsymbol{c}_{i,l+s}. \tag{10.15}$$

10.1.5 ▪ Virtual displacements

According to (10.12), a virtual displacement $\delta \boldsymbol{r}_i$ is given by

$$\begin{aligned}
\delta \boldsymbol{r}_i &= \sum_{k=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_k}\,\delta q_k = \sum_{r=1}^{l} \frac{\partial \boldsymbol{r}_i}{\partial q_r}\,\delta q_r + \sum_{s=1}^{n-l} \frac{\partial \boldsymbol{r}_i}{\partial q_{l+s}}\,\delta q_{l+s} \\
&= \sum_{r=1}^{l} \frac{\partial \boldsymbol{r}_i}{\partial q_r}\left(\sum_{s=1}^{n-l} b_{r,l+s}(\boldsymbol{q},t)\,\delta q_{l+s}\right) + \sum_{s=1}^{n-l} \frac{\partial \boldsymbol{r}_i}{\partial q_{l+s}}\,\delta q_{l+s} \\
&= \sum_{s=1}^{n-l}\left(\frac{\partial \boldsymbol{r}_i}{\partial q_{l+s}} + \sum_{s=1}^{n-l} \frac{\partial \boldsymbol{r}_i}{\partial q_r} b_{r,l+s}(\boldsymbol{q},t)\right)\delta q_{l+s} \\
&= \sum_{s=1}^{n-l} \boldsymbol{c}_{i,l+s}\,\delta q_{l+s},
\end{aligned} \tag{10.16}$$

where $\boldsymbol{c}_{i,l+s}$ is as in (10.14). Then, in view of (10.15) we conclude that

$$\delta \boldsymbol{r}_i = \sum_{s=1}^{n-l} \frac{\partial \boldsymbol{w}_i}{\partial \ddot{q}_{l+s}}\,\delta q_{l+s}, \quad i=1,\ldots,N. \tag{10.17}$$

10.1.6 ▪ Back to the fundamental equation: Part 1

We use (10.17) to reformulate the fundamental equation of dynamics (10.5):

$$\begin{aligned}
\sum_{i=1}^{N} m_i \boldsymbol{w}_i \cdot \delta \boldsymbol{r}_i &= \sum_{i=1}^{N} m_i \boldsymbol{w}_i \cdot \sum_{s=1}^{n-l} \frac{\partial \boldsymbol{w}_i}{\partial \ddot{q}_{l+s}}\,\delta q_{l+s}, \\
&= \sum_{s=1}^{n-l}\left(\sum_{i=1}^{N} m_i \boldsymbol{w}_i \cdot \frac{\partial \boldsymbol{w}_i}{\partial \ddot{q}_{l+s}}\right)\delta q_{l+s}.
\end{aligned}$$

This motivates the introduction of a quantity $\mathfrak{G}$ known as the *Gibbs function* or the *energy of acceleration*:

$$\mathfrak{G} = \frac{1}{2}\sum_{i=1}^{N} m_i \boldsymbol{w}_i \cdot \boldsymbol{w}_i,$$

We observe that

$$\frac{\partial \mathfrak{G}}{\partial \ddot{q}_{l+s}} = \sum_{i=1}^{N} m_i \boldsymbol{w}_i \cdot \frac{\partial \boldsymbol{w}_i}{\partial \ddot{q}_{l+s}},$$

whence

$$\sum_{i=1}^{N} m_i \boldsymbol{w}_i \cdot \delta \boldsymbol{r}_i = \sum_{s=1}^{n-l} \frac{\partial \mathfrak{G}}{\partial \ddot{q}_{l+s}} \delta q_{l+s}. \tag{10.18}$$

Let us note that we consider $\mathfrak{G}$ as a function of the form $\mathfrak{G} = \mathfrak{G}(\boldsymbol{q}, \dot{q}_{l+1}, \ldots, \dot{q}_n, t)$, therefore the nonholonomic constraints (10.8) are automatically accounted for.

10.1.7 ▪ Back to the fundamental equation: Part 2

In the previous section we reformulated the left-hand side of the fundamental equation of dynamics (10.5). In this section we reformulate its right-hand side. According to (10.16) we have:

$$\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \sum_{s=1}^{n-l} \boldsymbol{c}_{i,l+s}\, \delta q_{l+s} = \sum_{s=1}^{n-l} \left(\sum_{i=1}^{N} \boldsymbol{F}_i \cdot\, \boldsymbol{c}_{i,l+s} \right) \delta q_{l+s} = \sum_{s=1}^{n-l} \tilde{Q}_{l+s}\, \delta q_{l+s}, \tag{10.19}$$

where we have let

$$\tilde{Q}_{l+s} = \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \boldsymbol{c}_{i,l+s}, \quad s = 1, \ldots, n-l.$$

Upon substituting for $\boldsymbol{c}_{i,l+s}$ from its definition in (10.14), we see that

$$\begin{aligned} \tilde{Q}_{l+s} &= \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \left(\frac{\partial \boldsymbol{r}_i}{\partial q_{l+s}} + \sum_{r=1}^{l} \frac{\partial \boldsymbol{r}_i}{\partial q_r} b_{r,l+s}(\boldsymbol{q}, t), \right) \\ &= \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_{l+s}} + \sum_{r=1}^{l} \left(\sum_{i=1}^{N} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_r} \right) b_{r,l+s}(\boldsymbol{q}, t), \end{aligned}$$

which, according to (10.6) reduces to

$$\tilde{Q}_{l+s} = Q_{l+s} + \sum_{r=1}^{l} Q_r\, b_{r,l+s}(\boldsymbol{q}, t), \quad s = 1, \ldots, n-l.$$

10.1.8 ▪ The Gibbs-Appell equations of motion

Substituting (10.18) and (10.19) in the fundamental equation of dynamics (10.5), we arrive at

$$\sum_{s=1}^{n-l} \frac{\partial \mathfrak{G}}{\partial \ddot{q}_{l+s}} \delta q_{l+s} = \sum_{s=1}^{n-l} \tilde{Q}_{l+s}\, \delta q_{l+s},$$

that is,

$$\sum_{s=1}^{n-l} \left(\frac{\partial \mathfrak{G}}{\partial \ddot{q}_{l+s}} \delta q_{l+s} - \tilde{Q}_{l+s} \right) \delta q_{l+s}.$$

Since the variations δq_{l+s} are independent, we conclude that

$$\frac{\partial \mathfrak{G}}{\partial \ddot{q}_{l+s}} = \tilde{Q}_{l+s}, \quad s = 1, \ldots, n-l. \tag{10.20}$$

These are the *Gibbs-Appell equation of motion*.

The $n-l$ second order differential equations (10.20), along with the l constraint equations (10.8) form a system of n differential equation in the n unknowns $q_1(t), \dots, q_n(t)$.

10.1.9 ▪ Quasi-velocities

In a mechanical system with generalized coordinates $\boldsymbol{q} = \langle q_1, \dots, q_n \rangle$, expressions of the type

$$\omega_s = \sum_{k=1}^{n} a_{sk}(\boldsymbol{q}, t)\dot{q}_k + a_s(\boldsymbol{q}, t), \quad s = 1, \dots, n \tag{10.21}$$

are called *quasi-velocities*. At times it is simpler to model a system in terms of suitably defined quasi-velocities $\omega_1, \dots, \omega_n$ rather than generalized coordinates $q_1, \dots, q_n$. If the number of naturally occurring quasi-velocities is $n' < n$, then we extend their number to n by defining the rest through $\omega_s = \dot{q}_s$, $s = n'+1, \dots, n$.

We assume that the quasi-velocities are defined so that the coefficient matrix a_{sk} is nonsingular. It follows that (10.21) may be solved for $\dot{q}$'s in terms of ω's:

$$\dot{q}_r = \sum_{s=1}^{n} b_{rs}(\boldsymbol{q}, t)\omega_s + b_r(\boldsymbol{q}, t), \quad s = 1, \dots, n.$$

Additionally, with each equation in (10.21) we associate a differential as follows

$$d\pi_s = \sum_{k=1}^{n} a_{sk}(\boldsymbol{q}, t)\, dq_k + a_s(\boldsymbol{q}, t)\, dt, \quad s = 1, \dots, n. \tag{10.22}$$

Comparing (10.21) and (10.22) we see that

$$d\pi_s = \omega_s\, dt. \tag{10.23}$$

Remark: There is no reason to expect the expression on the right to be an exact differential, therefore although $d\pi_s$ makes sense as defined, it should not be assumed that it is the differential of a function π_s.

For any arbitrary function $\varphi(\boldsymbol{q}, t)$ we have:

$$\begin{aligned}
d\varphi &= \sum_{r=1}^{n} \frac{\partial \varphi}{\partial q_r}\, dq_r + \frac{\partial \varphi}{\partial t}\, dt \\
&= \sum_{r=1}^{n} \frac{\partial \varphi}{\partial q_r}\, \dot{q}_r\, dt + \frac{\partial \varphi}{\partial t}\, dt \\
&= \sum_{r=1}^{n} \frac{\partial \varphi}{\partial q_r} \left(\sum_{s=1}^{n} b_{rs}(\boldsymbol{q}, t)\omega_s + b_r(\boldsymbol{q}, t) \right) dt + \frac{\partial \varphi}{\partial t}\, dt \\
&= \sum_{s=1}^{n} \left(\sum_{r=1}^{n} b_{rs}(\boldsymbol{q}, t) \frac{\partial \varphi}{\partial q_r} \right) d\pi_s + \sum_{r=1}^{n} b_r(\boldsymbol{q}, t) \frac{\partial \varphi}{\partial q_r}\, dt + \frac{\partial \varphi}{\partial t}\, dt.
\end{aligned}$$

We introduce the purely symbolic notation

$$\frac{\partial \varphi}{\partial \pi_s} = \sum_{r=1}^{n} b_{rs}(\boldsymbol{q}, t) \frac{\partial \varphi}{\partial q_r}, \tag{10.24}$$

whereupon the previous calculation leads to

$$d\varphi = \sum_{s=1}^{n} \frac{\partial \varphi}{\partial \pi_s} d\pi_s + \left(\sum_{r=1}^{n} b_r(\boldsymbol{q},t) \frac{\partial \varphi}{\partial q_r} + \frac{\partial \varphi}{\partial t} \right) dt. \tag{10.25}$$

Applying (10.24) to $\boldsymbol{r}_i$ we get:

$$\frac{\partial \boldsymbol{r}_i}{\partial \pi_s} = \sum_{r=1}^{n} b_{rs}(\boldsymbol{q},t) \frac{\partial \boldsymbol{r}_i}{\partial q_r},$$

and therefore we obtain an expression for the velocity vector in terms of the quasi-coordinates:

$$\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \text{TODO} \ldots$$

10.1.10 ▪ Appell's equations of motion in terms of quasi-velocities

The main ingredients of Appell's equations of motion (10.20) are the generalized accelerations $\ddot{q}_s$. Those equations, however, are rarely used in that form. We will find that expressing the equations in terms of *quasi-velocities*, defined below, leads to major simplification.

Thus, given the l constraints (10.8), define the quasi-velocities

$$\omega_k = \sum_{s=1}^{n} a_{ks}(\boldsymbol{q},t)\dot{q}_s + a_k(\boldsymbol{q},t), \quad k = 1,\ldots,l, \tag{10.26}$$

therefore the constraint equations take the form

$$\omega_k = 0, \quad k = 1,\ldots,l.$$

We solve (10.26) for the $\dot{\boldsymbol{q}}$ in terms of the quasi-velocities. Since the first l of the quasi-velocities are zero, each $\dot{q}_r$ is a function of ω_{l+s}, $s = 1,\ldots,n-l$. Then we express the Gibbs function $\mathfrak{G}$ as a function of

$$q_1,\ldots,q_n, \quad \omega_{l+1},\ldots,\omega_n \quad \dot{\omega}_{l+1},\ldots,\dot{\omega}_n.$$

The rest of the derivation is horrendous, so we skip forward…

Then, it turns out that the Gibbs-Appell equation of motion take the form

$$\frac{\partial \mathfrak{G}}{\partial \dot{\omega}_{l+s}} = Q_{l+s}, \quad s = 1,\ldots,l. \tag{10.27}$$

Despite the complexity of the derivation, equations (10.27) are much easier to apply than the original (10.20). In particular, these result in first order differential equations in quasi-velocities, as apposed to the second order differential equations which we were getting before.

10.2 ▪ Gibbs-Appell according to Gantmacher [8]

The goal of this (lengthy) section is to derive the Gibbs-Appell equations of motion (10.47)

10.2.1 ▪ Pseudocoordinates

Consider the motion of N particles of masses m_ν and position vectors $\boldsymbol{r}_\nu$ subject to d algebraic and g differential constraints. Utilizing the d algebraic constraints, we inroduce $m = 3N - d$ generalized *idependent* coordinates $\boldsymbol{q} = \langle q_1, \ldots, q_m \rangle$, and express the system's configures in them as

$$\boldsymbol{r}_\nu = \boldsymbol{r}_\nu(\boldsymbol{q}, t), \quad \nu = 1, \ldots, N. \tag{10.28}$$

From this it follows that

$$\dot{\boldsymbol{r}}_\nu = \sum_{i=1}^{m} \frac{\partial \boldsymbol{r}_\nu}{\partial q_i} \dot{q}_i + \frac{\partial \boldsymbol{r}_\nu}{\partial t}, \quad \nu = 1, \ldots, N, \tag{10.29}$$

and

$$\delta \boldsymbol{r}_\nu = \sum_{i=1}^{m} \frac{\partial \boldsymbol{r}_\nu}{\partial q_i} \delta q_i, \quad \nu = 1, \ldots, N. \tag{10.30}$$

The vectors $\boldsymbol{r}_\nu$ and $\dot{\boldsymbol{r}}_\nu$ should satisfy the g differential constraints:

$$\sum_{\nu=1}^{m} \boldsymbol{l}_{\beta\nu}(\boldsymbol{r}_\nu, t) \cdot \dot{\boldsymbol{r}}_\nu + D_\beta(\boldsymbol{r}_\nu, t) = 0, \quad \beta = 1, \ldots, g. \tag{10.31}$$

Substituting for $\boldsymbol{r}_\nu$ and $\dot{\boldsymbol{r}}_\nu$ from (10.28) and (10.29), the constraint equations take the form

$$\sum_{i=1}^{m} A_{\beta i}(\boldsymbol{q}, t)\dot{q}_i + A_\beta(\boldsymbol{q}, t) = 0, \quad \beta = 1, \ldots, g. \tag{10.32}$$

Thus, the generalized coordinates $q_1, \ldots, q_m$ can take on arbitrary values, but the generalized vecocities $\dot{q}_1, \ldots, \dot{q}_m$ are constrained by (10.32).

Assuming that the g constraints in (10.32) are indepdendent, that is, the matrix $A_{\beta i}$ has full rank, we may solve for g of the velocities $\dot{q}_1, \ldots, \dot{q}_m$ in terms of the remaining $n = m - g = 3N - d - g$, the number n being the system's degrees of freedom. Thus, the velocities $\dot{q}_1, \ldots, \dot{q}_n$ may take on arbitrary values, while the remaining g velocities are determined from (10.32).

In practice, instead of the n generalized velocities noted above, it is often preferable to use n linear combination of the generalized velocities, such as

$$\dot{\pi}_s = \sum_{i=1}^{m} f_{si} \dot{q}_i, \quad i = 1, \ldots, n. \tag{10.33}$$

The quantities $\dot{\pi}_s$ defined here are called *pseudovelocities*. The notation $\dot{\pi}_s$ is entirely pro forma—there is no requirement that the right-hand side of (10.33) be and exact derivative, therefore although the notation $\dot{\pi}_s$ is well-defined, there is no function π_s which it is the derivative of. Nevertheless, we will have occasions to refer to the un-dotted symbol π_s which has no predefind meaning, but we will define it in an appropriate and constent way. The symbol π_s is called a *pseudocoordinate*.

We impose an invertibility requirement on the definitions (10.33) as follows. Consider the system of $g + n = m$ equation obtained as the union of the equations (10.32) and (10.33), and view it as a system of m linear equaitons in the m unknowns $\dot{q}_1, \ldots, \dot{q}_m$. We require that system to be invertible. Thus, the m generalized velocities may be expressed in terms of the n pseudovelocities:

$$\dot{q}_i = \sum_{s=1}^{n} h_{is}(\boldsymbol{q}, t)\dot{\pi}_s + h_i(\boldsymbol{q}, t), \quad i = 1, \ldots, m. \tag{10.34}$$

Thus, any set of m generalized velocities that satify the motion constraints (10.32) define a corresponding set of pseudovelocities $\dot{\pi}_s$, and conversely, any set of n arbitrary pseudovelocities define a set of generalized velocities that satify the motion constraints. The important point here is that there is no constraint on the pseudovelocities.

Let us note that due to the constrain (10.32) on generalized velocities, the generalized displacement are subject to the constraints

$$\sum_{i=1}^{m} A_{\beta i}\,\delta q_i = 0, \quad \beta = 1,\dots,g. \tag{10.35}$$

Then in view of the equations (10.33) also introduce the notation

$$\delta\pi_s = \sum_{i=1}^{m} A_{\beta i} f_{si}\,\delta q_i. \quad s = 1,\dots,n. \tag{10.36}$$

From what it has been said, the union of the equations (10.35) and (10.36) is invertivle and the inverse has the form

$$\delta q_i = \sum_{s=1}^{n} h_{is}\,\delta\pi_s, \quad i = 1,\dots,m. \tag{10.37}$$

Thus, as argumed before, the expressions $\delta\pi_s$ may take on arbitrary values. The corresponding q_i obtained from (10.37) will automatically satisfy the constraints (10.35).

10.2.2 ▪ Work and generalized forces

Let us compute the work done by the external forces in a virtual displacement:

$$\delta W = \sum_{\nu=1}^{N} \boldsymbol{F}_\nu\cdot\delta\boldsymbol{r}_\nu = \sum_{\nu=1}^{N} \boldsymbol{F}_\nu\cdot\sum_{i=1}^{m}\frac{\partial\boldsymbol{r}_\nu}{\partial q_i}\,\delta q_i = \sum_{i=1}^{m}\Big(\sum_{\nu=1}^{N}\boldsymbol{F}_\nu\cdot\frac{\partial\boldsymbol{r}_\nu}{\partial q_i}\Big)\delta q_i = \sum_{i=1}^{m} Q_i\,\delta q_i, \tag{10.38}$$

where

$$Q_i = \sum_{\nu=1}^{N}\boldsymbol{F}_\nu\cdot\frac{\partial\boldsymbol{r}_\nu}{\partial q_i}, \quad i = 1,\dots,m. \tag{10.39}$$

Thus, we have obtained an expression for the virtual work in terms of the virtual displacements δq_i. Utilizing (10.37), we may express the virtual work in terms of the $\delta\pi_s$:

$$\delta W = \sum_{i=1}^{m} Q_i \sum_{s=1}^{n} h_{is}\,\delta\pi_s, = \sum_{s=1}^{n}\Big(\sum_{i=1}^{m} h_{is} Q_i\Big)\delta\pi_s, = \sum_{s=1}^{n}\Pi_s\,\delta\pi_s, \tag{10.40}$$

where we have let

$$\Pi_s = \sum_{i=1}^{m} h_{is} Q_i, \quad s = 1,\dots,n. \tag{10.41}$$

The Π_s defined above are called the *generalized forces corresponding to the pseudocoordiante* π_s, $s = 1,\dots,n$.

10.2.3 ▪ Newton's equations in pseudocoordinates

Equations (10.29) express the particle velocities in terms of the generalized velocities. Substituting for the latter from (10.34) we obtain the particle velocities in terms of the pseudovelocities:

$$\dot{\boldsymbol{r}}_\nu = \sum_{s=1}^{n} \boldsymbol{e}_{\nu s}(\boldsymbol{q},t)\dot{\pi}_s + \boldsymbol{e}_\nu(\boldsymbol{q},t), \quad \nu = 1,\dots,N. \tag{10.42}$$

Then it follows that

$$\delta \boldsymbol{r}_\nu = \sum_{s=1}^{n} \boldsymbol{e}_{\nu s}(\boldsymbol{q},t)\delta\pi_s, \quad \nu = 1,\dots,N, \tag{10.43}$$

and

$$\ddot{\boldsymbol{r}}_\nu = \sum_{s=1}^{n} \boldsymbol{e}_{\nu s}(\boldsymbol{q},t)\ddot{\pi}_s + \cdots, \quad \nu = 1,\dots,N.$$

where the ellipsis indicate terms that are free of the *pseudoaccelerations* $\ddot{\pi}_s$, $s = 1,\dots,n$. For future reference, let us note that

$$\frac{\partial \ddot{\boldsymbol{r}}_\nu}{\partial \ddot{\pi}_s} = \boldsymbol{e}_{\nu s}. \tag{10.44}$$

Consider the fundamental equations of dynamics:

$$\delta W - \sum_{\nu=1}^{N} m_\nu \ddot{\boldsymbol{r}}_\nu \cdot \delta \boldsymbol{r}_\nu = \boldsymbol{0}.$$

Substituting from (10.40) and (10.43) this takes the form

$$\sum_{s=1}^{n} \Pi_s\,\delta\pi_s - \sum_{\nu=1}^{N} m_\nu \ddot{\boldsymbol{r}}_\nu \cdot \Big(\sum_{s=1}^{n} \boldsymbol{e}_{\nu s}(\boldsymbol{q},t)\,\delta\pi_s \Big) = 0,$$

that is

$$\sum_{s=1}^{n} \Pi_s\,\delta\pi_s - \sum_{s=1}^{n} \Big(\sum_{\nu=1}^{N} m_\nu \ddot{\boldsymbol{r}}_\nu \cdot \boldsymbol{e}_{\nu s}(\boldsymbol{q},t) \Big) \delta\pi_s = 0,$$

and finally

$$\sum_{s=1}^{n} \Big[\Pi_s - \sum_{s=1}^{n} \Big(\sum_{\nu=1}^{N} m_\nu \ddot{\boldsymbol{r}}_\nu \cdot \boldsymbol{e}_{\nu s}(\boldsymbol{q},t) \Big) \Big] \delta\pi_s = 0,$$

Since $\delta\pi_s$ are unconstrained, we conclude that

$$\sum_{\nu=1}^{N} m_\nu \ddot{\boldsymbol{r}}_\nu \cdot \boldsymbol{e}_{\nu s}(\boldsymbol{q},t) = \Pi_s, \quad s = 1,\dots,n. \tag{10.45}$$

10.2.4 ▪ The energy of the acceleration

We introduce the "*energy of acceleration*" which is defined as

$$U = U(\boldsymbol{q},\dot{\boldsymbol{\pi}},\ddot{\boldsymbol{\pi}}) = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \|\ddot{\boldsymbol{r}}_\nu\|^2. \tag{10.46}$$

Then, in view of (10.44), we see that

$$\frac{\partial U}{\partial \ddot{\pi}_s} = \sum_{\nu=1}^{N} m_\nu \ddot{\boldsymbol{r}}_\nu \cdot \frac{\partial \ddot{\boldsymbol{r}}_\nu}{\partial \ddot{\pi}_s} = \sum_{\nu=1}^{N} m_\nu \ddot{\boldsymbol{r}}_\nu \cdot \boldsymbol{e}_{\nu s}, \quad s = 1, \dots, n,$$

which, in conjunction with (10.45) leads to

$$\frac{\partial U}{\partial \ddot{\pi}_s} = \Pi_s, \quad s = 1, \dots, n, \tag{10.47}$$

This are *Appell's equations of motion*.

Upon a close examination, equaions (10.47) are a system of n first order differential equations in the n unknowns $\dot{\pi}_s$, $s = 1, \dots, n$. Note that there are no such things and π_s, so the equations are indeed first order in $\dot{\pi}_s$.

The equations are not complete, however, since they also involve the generalized coordinates q_i, $i = 1, \dots, n$. To complete the system, we append to it the g equations (10.32) and (10.33), as in:

$$\begin{aligned} &\frac{\partial U}{\partial \ddot{\pi}_s} = \Pi_s, && s = 1, \dots, n, \\ &\sum_{i=1}^{m} A_{\beta i}(\boldsymbol{q}, t)\dot{q}_i + A_\beta(\boldsymbol{q}, t) = 0, && \beta = 1, \dots, g, \\ &\dot{\pi}_s = \sum_{i=1}^{m} f_{si}\dot{q}_i, && i = 1, \dots, n. \end{aligned}$$

This combined set consists of $2n + g$ equations. Let us note that $n + g = m$, theefore we have a system of $m + n$ first order differential equations in the $m + n$ unknwns $\{q_i\}_{i=1}^m$ and $\{\dot{\pi}_s\}_{s=1}^n$.

10.3 ▪ A modification noted by Desloge

The generalized forces Π_s in the Gibbs-Appel equations of motion (10.47) are computed from (10.41), where, in turn, the Q_i are computed from (10.39). Desloge [5, 6] notes that if we define

$$R = \sum_{\nu=1}^{N} \boldsymbol{F}_\nu \cdot \ddot{\boldsymbol{r}}_\nu, \tag{10.48}$$

and express the result in terms of $\boldsymbol{q}$, $\dot{\boldsymbol{\pi}}$, and $\ddot{\boldsymbol{\pi}}$, then Π_s may be computed more easily through the formula

$$\Pi_s = \frac{\partial R}{\partial \ddot{\pi}_s}, \quad s = 1, \dots, n. \tag{10.49}$$

Consequently, (10.47) takes on the form

$$\frac{\partial U}{\partial \ddot{\pi}_s} = \frac{\partial R}{\partial \ddot{\pi}_s} \quad s = 1, \dots, n,$$

which motivates the introduction of the *Gibbs function*

$$\mathfrak{G} = U - R, \tag{10.50}$$

whereby the equations (10.47) are expressed as

$$\frac{\partial \mathfrak{G}}{\partial \ddot{\pi}_s} = 0, \quad s = 1, \ldots, n. \tag{10.51}$$

I haven't checked the details of the reasoning here, but in a couple of applications which I calculated, (10.47) and (10.51) produce identical results.

10.4 ▪ The simple pendulum via Gibbs-Appell

Here we will apply the Gibbs-Appell's equations (10.20) to derive the familiar equation of motion of a simple pendulum. This is certainly an overkill; deriving the equations of motion of a holonomic system such as a simple pendulum is done much more easily through the Lagrangian approach. The power of the Gibbs-Appell approach manifests itself when applied to nonholonomic systems, as we will see in the next section.

But for now, let's look at the simple pendulum as illustrated in Figure 1.1 on page 2. We take the angle φ as the problem's generalized coordinate, express the position vector $\boldsymbol{r}$ of the pendulum's bob in terms of φ, then calculate the velocity and the acceleration:

$$\begin{aligned}
\boldsymbol{r} &= \ell\langle \sin\varphi, \cos\varphi\rangle, \\
\boldsymbol{v} = \dot{\boldsymbol{r}} &= \ell\langle \dot\varphi\cos\varphi, -\dot\varphi\sin\varphi\rangle, \\
\boldsymbol{w} = \dot{\boldsymbol{v}} &= \ell\langle \ddot\varphi\cos\varphi - \dot\varphi^2\sin\varphi, -\ddot\varphi\sin\varphi - \dot\varphi^2\cos\varphi\rangle,
\end{aligned}$$

whence $\|\boldsymbol{w}\|^2 = \ell^2(\ddot\varphi^2 + \dot\varphi^4)$. Considering that the force vector is $\boldsymbol{f} = \langle 0, mg\rangle$, we are lead to the Gibbs function

$$\begin{aligned}
\mathfrak{G} &= \frac{1}{2}m\|\boldsymbol{w}\|^2 - \boldsymbol{f}\cdot\boldsymbol{w} \\
&= \frac{1}{2}\ell^2(\ddot\varphi^2 + \dot\varphi^4) + mg\ell(\ddot\varphi\sin\varphi + \dot\varphi^2\cos\varphi).
\end{aligned}$$

Since the Gibbs-Appell equations of motion (10.20) reduce to $\partial\mathfrak{G}/\partial\ddot\varphi = 0$ in our case, we conclude that

$$\ell^2\ddot\varphi + mg\ell\sin\varphi = 0,$$

which is the familiar equation of motion of a simple pendulum.

10.5 ▪ The Čaplygin sleigh

The Čaplygin sleigh consists of two point masses m_1 and m_2 connected through a rigid massless rod of length ℓ. The two masses slide on a horizontal surface. The mass m_2 can slide freely on the surface with no resistance at all. Mass m_1 rides on a sharp blade, as in an ice hockey skate, which allows motion only in the direction of the rod. We wish to describe the dynamics of the sleigh. Figure 10.1 depicts the sleigh. The position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ of the masses m_1 and m_2 may be specified through the generalized coordinates $x(t)$, $y(t)$, and $\varphi(t)$, shown on the figure. We have:

$$\boldsymbol{r}_1 = \langle x, y\rangle, \quad \boldsymbol{r}_2 = \boldsymbol{r}_1 + \langle \ell\cos\varphi, \ell\sin\varphi\rangle.$$

Since the mass m_1 can slide only along the direction of the rod, the velocity $\dot{\boldsymbol{r}}_1 = \langle \dot x, \dot y\rangle$ is parallel to the vector $\langle\cos\varphi, \sin\varphi\rangle$, therefore $\dot x/\cos\varphi = \dot y/\sin\varphi$, that is,

$$\dot x\sin\varphi - \dot y\cos\varphi = 0. \tag{10.52}$$

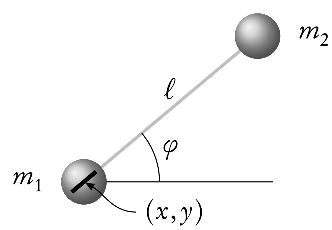


Figure 10.1: The Čaplygin sleigh's configuration may be described in terms of the generalized coordinates x, y, and φ.

This is the concrete version of the general constraint (10.8).

To calculate the Gibbs function, with begin with

$$\boldsymbol{v}_1 = \dot{\boldsymbol{r}}_1 = \langle \dot{x}, \dot{y} \rangle, \qquad \boldsymbol{v}_2 = \dot{\boldsymbol{r}}_2 = \langle \dot{x} + \ell\dot{\varphi}\sin\varphi, \dot{y} + \ell\dot{\varphi}\cos\varphi \rangle.$$

and then

$$\begin{aligned} \boldsymbol{w}_1 &= \dot{\boldsymbol{v}}_1 = \langle \ddot{x}, \ddot{y} \rangle, \\ \boldsymbol{w}_2 &= \dot{\boldsymbol{v}}_2 = \langle \ddot{x} - \ell\dot{\varphi}^2\cos\varphi - \ell\ddot{\varphi}\sin\varphi, \ddot{y} - \ell\dot{\varphi}^2\sin\varphi + \ell\ddot{\varphi}\cos\varphi \rangle. \end{aligned}$$

Thus, we arrive at

$$\begin{aligned} \mathfrak{G} = {} & \frac{1}{2}(m_1+m_2)(\ddot{x}^2+\ddot{y}^2) + \frac{1}{2}m_2\ell^2\ddot{\varphi}^2 \\ & - m_2\ell(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi)\ddot{x} + m_2\ell(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi)\ddot{y} + \frac{1}{2}m_2\ell^2\dot{\varphi}^4. \end{aligned}$$

Now, following the idea in (10.11) of eliminating redundant generalized velocities, we solve (10.52) for $\dot{y}$, and eliminate it from the equations. Substituting $\dot{y} = \dot{x}\tan\varphi$ in the above expression for $\mathfrak{G}$, we obtain

$$\begin{aligned} \mathfrak{G} = {} & \frac{m_1+m_2}{2\cos^2\varphi}\ddot{x}^2 + \frac{1}{\cos^3\varphi}\big[(m_1+m_2)\dot{x}\sin\varphi - m_2\ell\dot{\varphi}\cos^2\varphi\big]\dot{\varphi}\ddot{x} \\ & + \frac{1}{2}m_2\ell\ddot{\varphi}^2 + \frac{1}{\cos\varphi}m_2\ell\dot{x}\dot{\varphi}\ddot{\varphi} + \cdots, \end{aligned} \tag{10.53}$$

where, as before, the ellipsis stands for terms that involve no accelerations.

There are no externally applied forces on the sleigh, therefore the equations of motion (10.20) reduce to

$$\frac{\partial\mathfrak{G}}{\partial\ddot{x}} = 0, \qquad \frac{\partial\mathfrak{G}}{\partial\ddot{\varphi}} = 0,$$

that is,

$$\frac{m_1+m_2}{\cos^2\varphi}\ddot{x} + \frac{1}{\cos^3\varphi}\big[(m_1+m_2)\dot{x}\sin\varphi - m_2\ell\dot{\varphi}\cos^2\varphi\big]\dot{\varphi} = 0, \tag{10.54a}$$

$$m_2\ell\ddot{\varphi} + \frac{1}{\cos\varphi}m_2\ell\dot{x}\dot{\varphi} = 0. \tag{10.54b}$$

We may solve this system of two second order differential equations for the two unknowns $x(t)$ and $\varphi(t)$, subject to initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, $\varphi(0) = \varphi_0$, and $\dot\varphi(0) = \dot\varphi_0$. We may compute $\dot y$ retroactively from the constraint equation $\dot y(t) = \dot x(t)\tan\varphi(t)$, and then integrate it to find $y(t)$ subject to the initial condition $y(0) = y_0$.

In practice, when solving the system on a computer, it is easier to adjoin the constraint equation (10.52) to the two equations, as in

$$\frac{m_1+m_2}{\cos^2\varphi}\ddot x + \frac{1}{\cos^3\varphi}\big[(m_1+m_2)\dot x\sin\varphi - m_2\ell\dot\varphi\cos^2\varphi\big]\dot\varphi = 0,$$
$$m_2\ell\ddot\varphi + \frac{1}{\cos\varphi}m_2\ell\dot x\dot\varphi = 0,$$
$$\dot x\sin\varphi - \dot y\cos\varphi = 0,$$

and solve the whole thing in one fell swoop for the three unknowns $x(t)$, $y(t)$, and $\varphi(t)$ by applying the initial conditions

$$x(0) = x_0, \quad \dot x(0) = \dot x_0, \quad \varphi(0) = \varphi_0, \quad \dot\varphi(0) = \dot\varphi_0, \quad y(0) = y_0.$$

Note that there is no initial condition on $\dot y(0)$ since the $\dot x_0$ and φ_0 determine $\dot y(0)$ through the constraint equation (10.52).

10.6 ▪ The Čaplygin sleigh revisited

In Section 10.5 we obtained the differential equations of motion of the Čaplygin sleigh by applying the Gibbs-Appell equations (10.20). Here we solve the same problem by applying the equations (10.27). Toward that end, let us introduce the generalized velocities v and ω defined through

$$v = \frac{\dot x}{\cos\varphi}, \qquad \omega = \dot\varphi.$$

Thus, ω is the rod's angular velocity. To understand v, note that $\dot x = v\cos\varphi$, therefore v is the speed (not velocity!) of the mass m_1.

Now, substitute $\dot\varphi = \omega$ and $\dot x = v\cos\varphi$ in (10.53) and simplify:

$$\mathfrak{G} = \frac{1}{2}(m_1+m_2)\dot v^2 + \frac{1}{2}m_2\ell^2\dot\omega^2 + m_2\ell v\omega\dot\omega - m_2\ell\omega^2\dot v + \frac{1}{2}\big((m_1+m_2)v^2 + m_2\ell^2\omega^2\big)\omega^2.$$

Equations (10.27) in this case take the form

$$\frac{\partial\mathfrak{G}}{\partial\dot v} = 0, \qquad \frac{\partial\mathfrak{G}}{\partial\dot\omega} = 0,$$

and thus,

$$(m_1+m_2)\dot v - m_2\ell\omega^2 = 0, \tag{10.55a}$$
$$m_2\ell^2\dot\omega + m_2\ell v\omega = 0. \tag{10.55b}$$

These are much more pleasant-looking equations compared to (10.54) which we had obtained before. In fact, they are quite amenable to solving by hand. To wit, multiply the fist equation by v, the second by ω, and add. We get

$$(m_1+m_2)v\dot v + m_2\ell^2\omega\dot\omega = 0,$$

that is

$$\Big(\frac{1}{2}(m_1+m_2)v^2+\frac{1}{2}m_2\ell^2\omega^2\Big)^{\cdot}=0,$$

whence

$$E=\frac{1}{2}(m_1+m_2)v^2+\frac{1}{2}m_2\ell^2\omega^2 \tag{10.56}$$

is a constant of the motion. (In fact, it is the sleigh's kinetic energy.) The value of E is determined by the initial conditions $v(0)$ and $\omega(0)$.

Now, multiplying (10.55a) by $\ell/2$ and adding to (10.56), we get

$$\frac{1}{2}(m_1+m_2)\ell\dot{v}+\frac{1}{2}(m_1+m_2)v^2=E,$$

that is,

$$\ell\dot{v}+v^2=\alpha^2,\quad \text{where } \alpha^2=\frac{2E}{m_1+m_2}.$$

We solve this separable equation for v subject to the initial condition $v(0)=v_0$, and obtain

$$v(t)=\alpha\tanh\Big[\frac{\alpha}{\ell}t+\tanh^{-1}\frac{v_0}{\alpha}\Big]. \tag{10.57}$$

We see that $\lim_{t\to\infty}v(t)=\alpha$, that is, the sleigh's speed approaches α in the long run.

Now that we have $v(t)$, the angular velocity $\omega(t)$ may be computed easily. From the differential equation $\ell\dot{v}+v^2=\alpha^2$ we get $\dot{v}=\frac{1}{\ell}(\alpha^2-v^2)$. We substitute this expression for $\dot{v}$ in (10.55a), then solve for ω^2:

$$\omega(t)^2=\frac{m_1+m_2}{m_2\ell^2}\big(\alpha^2-v(t)^2\big), \tag{10.58}$$

where $v(t)$ is given in (10.57). Note, in particular, that $v(t)\to\alpha$ implies $\omega(t)\to 0$, that is, the sleigh's spin slows down to zero in the long run.

Remark 10.1. Let's observe that (a) From the definition $\dot{x}=v\cos\varphi$ of the speed v we see that a positive v implies motion in which m_1 trails m_2; and (b) Since tanh is positive when its argument is positive, the solution (10.57) indicates that regardless of the initial conditions, $v(t)$ will be positive for sufficiently large t. Putting these two observations together, we conclude that regardless of the sleigh's initial conditions, in the long run it will orient itself so that m_1 trails m_2 in its motion.

10.7 ▪ The problem from page 63 of Gantmacher

A "dumbbell" consists of a rigid weighless rod of length ℓ with point masses of m each attached to its ends. The rod is free to move in a vertical plane, other than the requirement that the velocity of its center should point along the rod itself. Find the equaitons of motion.

We introduce a Cartesian coordiante system xy, where x is horizontal and y points up. Let $x(t)$ and $y(t)$ be the coordinates of the rod's center, and let $\varphi(t)$ be the rod's angle relative to the x axis. We use x, y, and φ as the problem's generalized coordiantes. Then $\boldsymbol{r}_c=\langle x,y\rangle$ is the position vector of the rod's midpoint. The position vectors of the two masses are

$$\boldsymbol{r}_1=\boldsymbol{r}_c-\frac{\ell}{2}\langle\cos\varphi,\sin\varphi\rangle,\quad \boldsymbol{r}_2=\boldsymbol{r}_c+\frac{\ell}{2}\langle\cos\varphi,\sin\varphi\rangle.$$

Now we calculate the velocities of the masses:

$$\dot{\boldsymbol{r}}_1 = \dot{\boldsymbol{r}}_c - \frac{\ell}{2}\dot{\varphi}\langle -\sin\varphi, \cos\varphi\rangle, \quad \dot{\boldsymbol{r}}_2 = \dot{\boldsymbol{r}}_c + \frac{\ell}{2}\dot{\varphi}\langle -\sin\varphi, \cos\varphi\rangle, \tag{10.59}$$

and their accelerations:

$$\ddot{\boldsymbol{r}}_1 = \ddot{\boldsymbol{r}}_c - \frac{\ell}{2}\ddot{\varphi}\langle -\sin\varphi, \cos\varphi\rangle - \frac{\ell}{2}\dot{\varphi}^2\langle -\cos\varphi, -\sin\varphi\rangle,$$
$$\ddot{\boldsymbol{r}}_2 = \ddot{\boldsymbol{r}}_c + \frac{\ell}{2}\ddot{\varphi}\langle -\sin\varphi, \cos\varphi\rangle + \frac{\ell}{2}\dot{\varphi}^2\langle -\cos\varphi, -\sin\varphi\rangle.$$

It follows that

$$S = \frac{1}{2}m\|\ddot{\boldsymbol{r}}_1\|^2 + \frac{1}{2}m\|\ddot{\boldsymbol{r}}_2\|^2 = m\|\ddot{\boldsymbol{r}}_c\|^2 + \frac{m\ell}{2}(\ddot{\varphi}^2 + \dot{\varphi}^4) = m(\ddot{x}^2 + \ddot{y}^2) + \frac{m\ell}{2}(\ddot{\varphi}^2 + \dot{\varphi}^4) \tag{10.60}$$

The formulation above indicates that we have made the tacit choice of using x, y and φ as generalized coordiantes for this problem. Although these coordiantes are independent of each other algebraically, they are related to each other through their derivatives, since according to the problem's statement, the velocity of the rod's midpoint is constrainted to lie along the rod's direction. This says that the velocity vector $\boldsymbol{r}_c = \langle \dot{x}, \dot{y}\rangle$ makes an angle of of φ with the x axis, as the rod does, therefore $\dot{y}/\dot{x} = \tan\varphi$, or equivalently,

$$\dot{x}\sin\varphi = \dot{y}\cos\varphi. \tag{10.61}$$

This is the problem's nonholonomic constraint.

To relate this to the general theorey developed in the previous sections, the problem has $m = 3$ generalized coordinates x, y and φ; and it has $g = 1$ nonholonomic constraint (10.61). Therefore its degrees of freedom are $n = m - g = 2$. Again, acoording to the preceding general theory, we should express the generalized velocities $\dot{x}$, $\dot{y}$ and $\dot{\varphi}$, and the constraint (10.61), in terms of $n = 2$ independent and unconstrained pseudovelocities $\dot{\pi}_1$ and $\dot{\pi}_1$.

There is no unique choice of the pseudovelocities $\dot{\pi}_1$ and $\dot{\pi}_1$. All is needed the requirement that the pseudovelocities (10.33) togehter with the constraint equations (10.32) be solvable for the generalized veclocities (10.34). Table 10.1 shows a few possible choices for the problem at hand. In Choice (1) we take pseudovelocities $\dot{\pi}_1$ and $\dot{\pi}_1$ to be the actual generalized velocities $\dot{x}$ and $\dot{\varphi}$. The generalized velocity $\dot{y}$ then is determiend through the constraint (10.61).

Choice (2) defines the pseudovelocity $\dot{\pi}_1$ to be the left-hand side of the constraint equation (10.61) which is a reasonable choice to make.

Choice (3) sets $\dot{\pi}_1$ to the seemingly odd expression $\dot{x}/\cos\varphi$. The resulting expressions for $\dot{x}$ and $\dot{y}$ are quite simple, in the sense that they are easy to differentiate; we are going to need $\ddot{x}$ and $\ddot{y}$ to calculate the energy of acceleration. The derivatives of $\dot{x}$ and $\dot{y}$ will be messier expressions with the Choices (1) and (2).

On a closer scrutinty, the setting of $\dot{\pi}_1 = \dot{x}/\cos\varphi$ is not that odd after all since it has a pleasant physical interpretation. Let v be the (scalar) speed of the rod's midpoint. Since the velocity points along the rod's direction, then it should be clear that $\dot{x} = v\cos\varphi$ and $\dot{y} = v\sin\varphi$. Comparing the $\dot{x}$ and $\dot{y}$ values in Choice (3), we see that $\dot{\pi}_1$ is the speed v.

We adopt the pseudovelocities of Choice (3) for the rest of this section. To ease the notational burden, however, we replace the ageneric symbols $\dot{\pi}_1$ and $\dot{\pi}_2$ with the more meaningful notation $v = \dot{\pi}_1$ and $\omega = \dot{\pi}_2$. In particular, according to Choice (3) we have

$$\dot{x} = v\cos\varphi, \quad \dot{y} = v\sin\varphi. \tag{10.62}$$

Choice (1)	$\begin{cases} \dot{\pi}_1 = \dot{x} \\ \dot{\pi}_2 = \dot{\varphi} \\ \dot{x}\sin\varphi = \dot{y}\cos\varphi \end{cases}$	$\iff$	$\begin{cases} \dot{x} = \dot{\pi}_1 \\ \dot{y} = \dot{\pi}_1 \tan\varphi \\ \dot{\varphi} = \dot{\pi}_2 \end{cases}$
Choice (2)	$\begin{cases} \dot{\pi}_1 = \dot{x}\sin\varphi \\ \dot{\pi}_2 = \dot{\varphi} \\ \dot{x}\sin\varphi = \dot{y}\cos\varphi \end{cases}$	$\iff$	$\begin{cases} \dot{x} = \dot{\pi}_1/\sin\varphi \\ \dot{y} = \dot{\pi}_1/\cos\varphi \\ \dot{\varphi} = \dot{\pi}_2 \end{cases}$
Choice (3)	$\begin{cases} \dot{\pi}_1 = \dot{x}/\cos\varphi \\ \dot{\pi}_2 = \dot{\varphi} \\ \dot{x}\sin\varphi = \dot{y}\cos\varphi \end{cases}$	$\iff$	$\begin{cases} \dot{x} = \dot{\pi}_1\cos\varphi \\ \dot{y} = \dot{\pi}_1\sin\varphi \\ \dot{\varphi} = \dot{\pi}_2 \end{cases}$

Table 10.1: There is no unique choice of pseudovelocities for a given problem. The choices given here are three out of infinite such possibilities for the dumbbell problem.

To express the energy of acceleration S in (10.60) in terms of the quasivelocities, we differentiate (10.62):

$$\ddot{x} = \dot{v}\cos\varphi - v\dot{\varphi}\sin\varphi = \dot{v}\cos\varphi - v\omega\sin\varphi, \tag{10.63a}$$

$$\ddot{y} = \dot{v}\sin\varphi + v\dot{\varphi}\cos\varphi = \dot{v}\sin\varphi + v\omega\cos\varphi, \tag{10.63b}$$

whence $\ddot{x}^2 + \ddot{y}^2 = \dot{v}^2 + v^2\omega^2$, and (10.60) changes to

$$S = m(\dot{v}^2 + v^2\omega^2) + \frac{m\ell}{2}(\dot{\omega}^2 + \omega^4) = m\dot{v}^2 + \frac{m\ell}{2}\dot{\omega}^2 + \cdots,$$

where the ellipses indicates terms which are free of the acceleration terms $\dot{v}$ and $\dot{\omega}$.

A force of $\boldsymbol{F} = \langle 0, -mg\rangle$ acts on each of the two masses. Therefore

$$\begin{aligned} \boldsymbol{F}\cdot\ddot{\boldsymbol{r}}_1 + \boldsymbol{F}\cdot\ddot{\boldsymbol{r}}_2 = \boldsymbol{F}\cdot(\ddot{\boldsymbol{r}}_1 + \ddot{\boldsymbol{r}}_2) = 2\boldsymbol{F}\cdot\ddot{\boldsymbol{r}}_c \\ = 2\langle 0, -mg\rangle \cdot \langle \ddot{x}, \ddot{y}\rangle = -2mg\ddot{y} = -2mg(\dot{v}\sin\varphi + v\omega\cos\varphi). \end{aligned}$$

In the last step we have substituted for $\ddot{y}$ from (10.63b).

We conclude that the Gibbs function is

$$G = m\dot{v}^2 + \frac{m\ell}{2}\dot{\omega}^2 + 2mg\dot{v}\sin\varphi + \cdots,$$

where, as usual, the ellipses indicates terms which are free of the accelerations.

Then the Appell equations $\partial G/\partial\dot{v} = 0$ and $\partial G/\partial\dot{\omega} = 0$ lead to

$$\dot{v} + g\sin\varphi = 0, \quad \dot{\omega} = 0.$$

To complete the system, we append the constraint equations (10.62). Therefore the complete set of differential equations of motion are

$$\begin{aligned} &\dot{x} = v\cos\varphi, \\ &\dot{y} = v\sin\varphi. \\ &\dot{\varphi} = \omega, \\ &\dot{v} + g\sin\varphi = 0, \\ &\dot{\omega} = 0. \end{aligned}$$

These form a set of five first order differential equations in the five unknowns x, y, φ, ω, and v.

This system is solvable in terms of elementary functions, as demostrated in Gantmacher. Plugging the raw system into Maple does not produce a good result immediately, but we can maneuver it toward the right solution as follows.

The equations above impliy that ω is a constant, and therefore $\varphi = \omega t + \varphi_0$. Therefore the system shrinks to

$$\begin{aligned} \dot{x} &= v\cos\varphi, \\ \dot{y} &= v\sin\varphi. \\ \dot{v} + g\sin\varphi &= 0, \end{aligned}$$

where $\varphi = \omega t + \varphi_0$. In solving this reduced system, Maple distinguishes between the cases where ω is zero and nonzero. If ω is nonzero, we get something like Gantmacher's. In particular, y consists of a linear term in t plus trigonometric function. If, however, ω is zero and $\varphi_0 = \pi/2$, we get a $-1/2gt^2$ term, as expected.

Note: In the above, I calclated the Gibbs function a la Desloge. We may calculate it a la Gantmacher if we wish. Here is how.

What I have written above as the quasivelocity v, Gantmacher calls it $\dot{\pi}$. The equation $\dot{y} = v\sin\varphi$ then takes the form $\dot{y} = \dot{\pi}\sin\varphi$, whereby we introduce the virtual displacement of the quasicoordinate π through $\delta y = (\sin\varphi)\,\delta\pi$.

In accordance with (10.59), the virtual displacements of the masses are

$$\delta \boldsymbol{r}_1 = \delta \boldsymbol{r}_c - \frac{\ell}{2}\langle -\sin\varphi, \cos\varphi\rangle\,\delta\varphi, \quad \delta \boldsymbol{r}_2 = \delta \boldsymbol{r}_c + \frac{\ell}{2}\langle -\sin\varphi, \cos\varphi\rangle\,\delta\varphi,$$

A force of $\boldsymbol{F} = \langle 0, -mg\rangle$ acts on each of the two masses therefore their virtual work is

$$\begin{aligned} \delta W = \boldsymbol{F}\cdot\delta\boldsymbol{r}_1 + \boldsymbol{F}\cdot\delta\boldsymbol{r}_2 = \boldsymbol{F}\cdot(\delta\boldsymbol{r}_1 + \delta\boldsymbol{r}_2) = 2\boldsymbol{F}\cdot\delta\boldsymbol{r}_c \\ = 2\langle 0, -mg\rangle\cdot\langle\delta x, \delta y\rangle = -2mg\,\delta y = -2mg\sin\varphi\,\delta\pi. \end{aligned} \tag{10.64}$$

However $\delta W = \Pi\,\delta\pi + \Phi\,\delta\varphi$, where Π and Φ are the generalized forces corresponding to the quasicoordinates π and φ. We conclude that $\Pi = -2mg\sin\varphi$ and $\Phi = 0$. The equations of motion are

$$\frac{\partial S}{\partial \ddot{\pi}} = \Pi, \quad \frac{\partial S}{\partial \ddot{\varphi}} = \Phi,$$

which agree with what we obtained earlier.

Chapter 11

Rigid body dynamics

The previous chapter's formulation of the Gibbs-Appell equations focused on the dynamics of point masses. Extending the results to rigid bodies is a matter of replacing the summations with integrals, which is easy in principle, but provides some challenges in practice. It is possible, with some work, to systematize the treatment and arrive at a general formula for the Gibbs function which applies to all rigid bodies. Lurie [12] and Desloge [5], for instance, have such formulas. Lurie's formula is packaged quite nicely. Desloge's result is a little more general but it is not packaged as nicely. Furthermore, Desloge's formula is actually incorrect(!) due to a false assumption made in its derivation. Desloge [6] points out his error and offers a corrected formula.

In the following sections I will offer a blending of Lurie's and Desloge's approaches which combines the nice packaging of the former with the generality of the latter.

11.1 ▪ Three frames of reference

It turns out that to derive the equation of motion of a rigid body, it is quite convenient to use not one, not two, but three(!) special purpose frames of reference simultaneously. In most applications a frame of reference takes the form of a right-handed orthonormal triad and a point called the frame's *origin*.

The stationary frame is defined by a triad $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ and the associated origin O. As the name implies, the stationary frame is fixed (not moving). Typically the $\boldsymbol{i}$ and $\boldsymbol{j}$ vectors lie in a horizontal plane and the vector $\boldsymbol{k}$ points upward, but that's not a requirement.

With the stationary frame we associate a Cartesian coordinate system whose x, y, and z axes are aligned with the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ vectors, respectively.

Ultimately, the purpose of rigid body dynamics is to express a body's motion relative to the stationary frame.

The body frame of reference is defined by a triad $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ and an associated origin o. The triad is firmly attached to the body, and therefore moves with it. There are no restrictions on the choice of the origin, nor on the triad's orientation. Whenever possible, however, we set the origin at the body's center of mass, and orient the triad along the body's principal axes of inertia, since that results in significant simplifications.

Since the triad $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ is affixed to the body, the triad's angular velocity is exactly the body's angular velocity $\boldsymbol{\omega}$.

The intermediate frame of reference is defined by a triad $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and an associated origin O'. The choice of this frame is entirely up to you. You choose it to fit the specific application at hand. The triad may rotate and the origin may move as you wish.

Remark 11.1. What's the purpose of the intermediate frame of reference? In formulating the equations of motion, some quantities are more easily expressed in once coordinate system than the other. For instance, the position of the body's center of mass is best expressed in terms of the stationary frame, while its spin is best express in terms of the body reference frame. Some other quantities may be difficult to express in either of those, but may be easy in terms of a special purpose intermediate frame of reference. All three frames are related through orthogonal transformations, therefore we may readily translate the information from one frame to another, as needed.

11.2 ▪ The energy of acceleration for a rigid body

Consider a rigid body $\mathscr{B}$ of mass m equipped with a body reference frame with an origin o and a orthonormal triad $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$, as in Section 11.1. Let $\boldsymbol{r}_o$ be the position vector of the point o relative to the stationary frame, that is $\boldsymbol{r}_o = \overrightarrow{Oo}$, and let $\mathscr{I}_o$ be body's moment of inertia relative to $\boldsymbol{o}$. Furthermore, let c be the body's center of mass, and let $\boldsymbol{\rho}_c = \overrightarrow{oc}$. Then we have[12]

$$\int_{\mathscr{B}} \frac{1}{2}\|\ddot{\boldsymbol{r}}\|^2\,dm = \frac{1}{2}m\|\ddot{\boldsymbol{r}}_o\|^2 + m(\ddot{\boldsymbol{r}}_o \times \dot{\boldsymbol{\omega}})\cdot\boldsymbol{\rho}_c + m(\ddot{\boldsymbol{r}}_o \times \boldsymbol{\omega})\cdot(\boldsymbol{\omega}\times\boldsymbol{\rho}_c) \\ + \frac{1}{2}\dot{\boldsymbol{\omega}}\cdot\mathscr{I}_o\dot{\boldsymbol{\omega}} + (\dot{\boldsymbol{\omega}}\times\boldsymbol{\omega})\cdot\mathscr{I}_o\boldsymbol{\omega} + \cdots, \quad (11.1)$$

where the ellipsis indicates terms that do not involve accelerations, and are, therefore, immaterial to the Gibbs-Appell equations of motion.

If the origin o of the body frame is chosen to coincide with the body's center of mass, then $\boldsymbol{\rho}_c = \boldsymbol{0}$ and (11.1) reduces to

$$\int_{\mathscr{B}} \frac{1}{2}\|\ddot{\boldsymbol{r}}\|^2\,dm = \frac{1}{2}m\|\ddot{\boldsymbol{r}}_c\|^2 + \frac{1}{2}\dot{\boldsymbol{\omega}}\cdot\mathscr{I}_c\dot{\boldsymbol{\omega}} + (\dot{\boldsymbol{\omega}}\times\boldsymbol{\omega})\cdot\mathscr{I}_c\boldsymbol{\omega} + \cdots, \quad (11.2)$$

where $\boldsymbol{r}_c$ is the position vector, relative to the stationary frame, of the body's center of mass, and $\mathscr{I}_c$ is the moment of inertia tensor relative to the center of mass.

11.3 ▪ The rolling coin

Here we analyze the somewhat challenging dynamics of a coin, modeled as a thin homogeneous disk of radius a, rolling without slipping on a horizontal floor; see Figure 11.1.

[12] This is a combination of equation (4.11.8) in Lurie [12] and (22) in Desloge [6]. I haven't personally verified them yet.

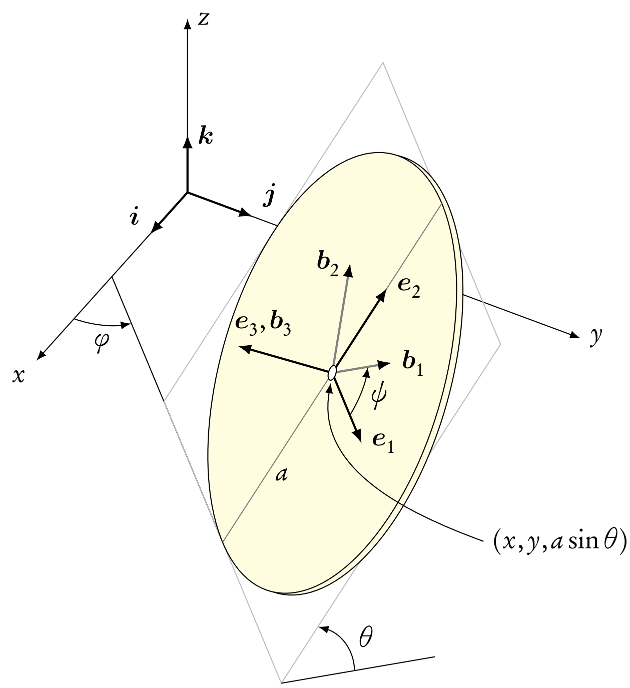


Figure 11.1: The rolling coin's configuration is specified through five generalized coordinates x, y, θ, φ, and ψ, as shown.. The no-slip conditions at the contact point with the floor imposes two nonholonomic constraints.

11.4 ▪ The three frames

Figure 11.1 depicts the coin along with the three frames of reference discussed in Section 11.1. Specifically, the $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ triad of the stationary frame has its origin on the floor and the $\boldsymbol{k}$ vector points upward; the $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ triad of the body frame has its origin at the disk's center and the vector $\boldsymbol{b}_3$ is perpendicular to the disk.

We choose an intermediate frame with its origin at the disk's center, and an associated orthonormal triad $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ oriented as follows. The vector $\boldsymbol{e}_3$ is perpendicular to the disk, and therefore coincides with $\boldsymbol{b}_3$. The vector $\boldsymbol{e}_1$ is horizontal. Then we set $\boldsymbol{e}_2 = \boldsymbol{e}_3 \times \boldsymbol{e}_1$.

The body frame is fixed to the coin, by definition, and rotates with it. The position of the body frame relative to the intermediate frame is described by a single parameter, the angle ψ, between the vectors $\boldsymbol{b}_1$ and $\boldsymbol{e}_1$ as indicated on the figure.

The coin's position relative to the stationary frame is specified through five generalized coordinates x, y, θ, φ, and ψ, where θ is the angle between the coin's plane and the floor; φ is the angle between the vectors $\boldsymbol{i}$ and $\boldsymbol{e}_1$ (it measures how the coin's plane is rotated relative to the $\boldsymbol{k}$ vector); and $(x, y, a\sin\theta)$ are the coordinates of the coin's center.

11.5 ▪ The angular velocity

From the geometry of Figure 11.1 it is evident that $\boldsymbol{e}_1 = \cos\varphi\,\boldsymbol{i} + \sin\varphi\,\boldsymbol{j}$. The unit vector $\boldsymbol{e}_2$ makes an angle θ with the floor and its horizontal projection is perpendicular to $\boldsymbol{e}_1$, therefore its horizontal and vertical projections are $(-\sin\varphi\,\boldsymbol{i} + \cos\varphi\,\boldsymbol{j})\cos\theta$ and $\sin\theta\,\boldsymbol{k}$. We conclude that $\boldsymbol{e}_2 = (-\sin\varphi\,\boldsymbol{i} + \cos\varphi\,\boldsymbol{j})\cos\theta + \sin\theta\,\boldsymbol{k}$. The vector $\boldsymbol{e}_3$ may be found with a similar geometric reasoning, or just by computing $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$. Here is a summary:

$$\boldsymbol{e}_1 = \cos\varphi\,\boldsymbol{i} + \sin\varphi\,\boldsymbol{j}, \tag{11.3a}$$
$$\boldsymbol{e}_2 = -\sin\varphi\cos\theta\,\boldsymbol{i} + \cos\varphi\cos\theta\,\boldsymbol{j} + \sin\theta\,\boldsymbol{k}, \tag{11.3b}$$
$$\boldsymbol{e}_3 = \sin\varphi\sin\theta\,\boldsymbol{i} - \cos\varphi\sin\theta\,\boldsymbol{j} + \cos\theta\,\boldsymbol{k}. \tag{11.3c}$$

We will need the time derivatives of these vectors shortly, so let's compute them right now. We have

$$\dot{\boldsymbol{e}}_1 = -\dot{\varphi}\sin\varphi\,\boldsymbol{i} + \dot{\varphi}\cos\varphi\,\boldsymbol{j}.$$

It follows that

$$\dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_1 = 0, \quad \dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 = \dot{\varphi}\cos\theta, \quad \dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_3 = -\dot{\varphi}\sin\theta,$$

therefore, according to (8.7), we have $\dot{\boldsymbol{e}}_1 = \dot{\varphi}\cos\theta\,\boldsymbol{e}_2 - \dot{\varphi}\sin\theta\,\boldsymbol{e}_3$. In a similar manner we compute $\dot{\boldsymbol{e}}_2$ and $\dot{\boldsymbol{e}}_2$ and express them in terms of the basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Here is what we get:

$$\dot{\boldsymbol{e}}_1 = \dot{\varphi}\cos\theta\,\boldsymbol{e}_2 - \dot{\varphi}\sin\theta\,\boldsymbol{e}_3, \tag{11.4a}$$
$$\dot{\boldsymbol{e}}_2 = -\dot{\varphi}\cos\theta\,\boldsymbol{e}_1 + \dot{\theta}\,\boldsymbol{e}_3, \tag{11.4b}$$
$$\dot{\boldsymbol{e}}_3 = \dot{\varphi}\sin\theta\,\boldsymbol{e}_1 - \dot{\theta}\,\boldsymbol{e}_2. \tag{11.4c}$$

Now let us examine the body frame. Referring to Figure 11.1 we have:

$$\boldsymbol{b}_1 = \cos\psi\,\boldsymbol{e}_1 + \sin\psi\,\boldsymbol{e}_2, \tag{11.5a}$$
$$\boldsymbol{b}_2 = -\sin\psi\,\boldsymbol{e}_1 + \cos\psi\,\boldsymbol{e}_2, \tag{11.5b}$$
$$\boldsymbol{b}_3 = \boldsymbol{e}_3. \tag{11.5c}$$

We differentiate these with respect to t, and substitute from (11.4) for the derivatives of $\boldsymbol{e}_i$, and obtain

$$\dot{\boldsymbol{b}}_1 = -(\dot{\varphi}\cos\theta + \dot{\psi})\sin\psi\,\boldsymbol{e}_1 + (\dot{\varphi}\cos\theta + \dot{\psi})\cos\psi\,\boldsymbol{e}_2 + (-\dot{\varphi}\sin\theta\cos\psi + \dot{\theta}\sin\psi)\,\boldsymbol{e}_3, \tag{11.6a}$$
$$\dot{\boldsymbol{b}}_2 = -(\dot{\varphi}\cos\theta + \dot{\psi})\cos\psi\,\boldsymbol{e}_1 - (\dot{\varphi}\cos\theta + \dot{\psi})\sin\psi\,\boldsymbol{e}_2 + (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)\,\boldsymbol{e}_3, \tag{11.6b}$$
$$\dot{\boldsymbol{b}}_3 = \dot{\varphi}\sin\theta\,\boldsymbol{e}_1 - \dot{\theta}\,\boldsymbol{e}_2. \tag{11.6c}$$

Remark 11.2. The expressions for $\dot{\boldsymbol{b}}_3$ in (11.6c) and $\dot{\boldsymbol{e}}_3$ in (11.4c) agree since $\boldsymbol{b}_3 = \boldsymbol{e}_3$.

Equations (11.6) may be used in conjunction with (7.5) to calculate the body's angular velocity vector. Toward that end we compute

$$\omega_1^b = \dot{\boldsymbol{b}}_2 \cdot \boldsymbol{b}_3 = -(\dot{\varphi}\cos\theta + \dot{\psi})\cos\psi\,(\boldsymbol{e}_1 \cdot \boldsymbol{b}_3) - (\dot{\varphi}\cos\theta + \dot{\psi})\sin\psi\,(\boldsymbol{e}_2 \cdot \boldsymbol{b}_3) \\ + (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)(\boldsymbol{e}_3 \cdot \boldsymbol{b}_3),$$

where the superscript b is to remind us that ω_1^b is the component of the vector $\boldsymbol{\omega}$ in the the *body* frame. On the right-hand side we substitute for $\boldsymbol{b}_3$ from (11.5), simplify the result, and arrive at $\omega_1^b = \dot\varphi \sin\theta \sin\psi + \dot\theta \cos\psi$. Computing the ω_2^b and ω_3^b components in the same way, we arrive at:

$$\begin{aligned}
\omega_1^b &= \dot{\boldsymbol{b}}_2 \cdot \boldsymbol{b}_3 = \dot\varphi \sin\theta \sin\psi + \dot\theta \cos\psi, \\
\omega_2^b &= \dot{\boldsymbol{b}}_3 \cdot \boldsymbol{b}_1 = \dot\varphi \sin\theta \cos\psi - \dot\theta \sin\psi, \\
\omega_3^b &= \dot{\boldsymbol{b}}_1 \cdot \boldsymbol{b}_2 = \dot\varphi \cos\theta + \dot\psi.
\end{aligned}$$

Then from the definition (7.3) of the angular velocity vector we conclude that

$$\boldsymbol{\omega} = (\dot\varphi \sin\theta \sin\psi + \dot\theta \cos\psi)\,\boldsymbol{b}_1 + (\dot\varphi \sin\theta \cos\psi - \dot\theta \sin\psi)\,\boldsymbol{b}_2 + (\dot\varphi \cos\theta + \dot\psi)\,\boldsymbol{b}_3. \tag{11.7}$$

This is the coin's angular velocity vector expressed in the $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ basis. By substituting for $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ from (11.5), we obtain and expression for $\boldsymbol{\omega}$ in the $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ basis:

$$\boldsymbol{\omega} = \dot\theta\,\boldsymbol{e}_1 + \dot\varphi \sin\theta\,\boldsymbol{e}_2 + (\dot\varphi \cos\theta + \dot\psi)\,\boldsymbol{e}_3. \tag{11.8}$$

We see that the components of $\boldsymbol{\omega}$ along the intermediate basis are particularly simple. Therein lies the significance of the choice of our intermediate basis. Those components play a central role in what comes next, therefore we name them simply ω_i,[13]

$$\omega_1 = \dot\theta, \quad \omega_2 = \dot\varphi \sin\theta, \quad \omega_3 = \dot\varphi \cos\theta + \dot\psi, \tag{11.9}$$

Note that each ω_i is a linear combination of the generalized velocities $\dot\theta$, $\dot\varphi$, and $\dot\psi$, and therefore each ω_i is a quasi-velocity as defined in Section 10.1.9. Well, actually ω_1 is not a true quasi-velocity since it is the derivative of the generalized velocity $\dot\theta$, but the other two are honest quasi-velocities since they are not derivatives of anything.

Solving (11.9) as a linear system of three equations for the three unknowns $\dot\theta$, $\dot\varphi$, and $\dot\psi$, we get:

$$\dot\theta = \omega_1, \quad \dot\varphi = \frac{1}{\sin\theta}\omega_2, \quad \dot\psi = \omega_3 - \omega_2 \cot\theta. \tag{11.10}$$

Equations (11.9) and (11.10) establish a one-to-one correspondence between the generalized velocities $\dot\theta$, $\dot\varphi$, $\dot\psi$, and the quasi-velocities ω_1, ω_2, ω_3. We may formulate the rest of the analysis in terms of one or the other set of variables. We will do it in terms of the ω's since the derivations are easier that way.

For future reference, let us make a note of the following formula whose derivation is left as an exercise:

$$\dot{\boldsymbol{\omega}} = [\dot\omega_1 + \omega_2\omega_3 - \omega_2^2 \cot\theta]\,\boldsymbol{e}_1 + [\dot\omega_2 - \omega_3\omega_1 + \omega_1\omega_2 \cot\theta]\,\boldsymbol{e}_2 + \dot\omega_3\,\boldsymbol{e}_3. \tag{11.11}$$

11.6 ▪ The no-slip constraint

The position vector of the coin's center in the stationary frame is

$$\boldsymbol{r}_c = x\,\boldsymbol{i} + y\,\boldsymbol{j} + a \sin\theta\,\boldsymbol{k}. \tag{11.12}$$

The vector $\boldsymbol{\rho} = -a\boldsymbol{e}_2$ extends from the coin's center to its contact point with the floor. Therefore, according to (7.7), the velocity, relative to the coin's center, of a point on the

[13]To be consistent with the b superscript introduced earlier for the components of $\boldsymbol{\omega}$ in the $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ basis, we should have named its $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ components with superscripts e. We refrain from doing that, however, to reduce clutter.

coin's rim at the contact point is given by $\boldsymbol{\omega}\times\boldsymbol{\rho}$. Consequently, the velocity of that point relative to the stationary frame is $\dot{\boldsymbol{r}}_c+\boldsymbol{\omega}\times\boldsymbol{\rho}$. Since the velocity of the ground at the contact point is zero, then to prevent slippage, we need

$$\dot{\boldsymbol{r}}_c+\boldsymbol{\omega}\times\boldsymbol{\rho}=\boldsymbol{0}. \tag{11.13}$$

Upon substituting for $\boldsymbol{r}_c$ from (11.12), for $\boldsymbol{\omega}$ from (11.8), for the $\boldsymbol{e}_i$ vectors from (11.3), and simplifying the result, and we get

$$\big[\dot{x}-a\dot{\theta}\sin\varphi\sin\theta+(\dot{\varphi}\cos\theta+\dot{\psi})\cos\varphi\big]\boldsymbol{i}+\big[\dot{y}+a\dot{\theta}\cos\varphi\sin\theta+(\dot{\varphi}\cos\theta+\dot{\psi})\cos\varphi\big]\boldsymbol{j}=\boldsymbol{0}.$$

Then we apply (11.10) to change over to quasi-velocities and arrive at

$$\big[\dot{x}+a\omega_3\cos\varphi-a\omega_1\sin\varphi\sin\theta\big]\boldsymbol{i}+\big[\dot{y}+a\omega_3\sin\varphi+a\omega_1\cos\varphi\sin\theta\big]\boldsymbol{j}=\boldsymbol{0}.$$

This leads us to a pair of scalar equations

$$\dot{x}=-a(\omega_3\cos\varphi-\omega_1\sin\varphi\sin\theta),\quad \dot{y}=-a(\omega_3\sin\varphi+\omega_1\cos\varphi\sin\theta) \tag{11.14}$$

which express the rolling coin's nonholonomic constraints. We use these to eliminate $\dot{x}$ and $\dot{y}$ from the rest of the computations.

11.7 ▪ The acceleration of the coin's center

From (11.12) we have $\dot{\boldsymbol{r}}_c=\dot{x}\,\boldsymbol{i}+\dot{y}\,\boldsymbol{j}+a\dot{\theta}\cos\theta\,\boldsymbol{k}$.. We substitute for $\dot{x}$ and $\dot{y}$ from (11.14), then we apply (11.10) and arrive at

$$\dot{\boldsymbol{r}}_c=-a(\omega_3\cos\varphi-\omega_1\sin\varphi\sin\theta)\boldsymbol{i}-a(\omega_3\sin\varphi+\omega_1\cos\varphi\sin\theta)\boldsymbol{i}+a\omega_1\cos\theta\,\boldsymbol{k}.$$

Then we differentiate this once again to find the acceleration:

$$\begin{aligned}\ddot{\boldsymbol{r}}_c&=a\big[\dot{\omega}_1\sin\varphi\sin\theta-\dot{\omega}_3\cos\varphi+\omega_1^2\sin\varphi\cos\theta+\omega_1\omega_2\cos\varphi+\omega_2\omega_3\sin\varphi/\sin\theta\big]\boldsymbol{i}\\&\quad+a\big[-\dot{\omega}_1\cos\varphi\sin\theta-\dot{\omega}_3\sin\varphi-\omega_1^2\cos\varphi\cos\theta+\omega_1\omega_2\sin\varphi-\omega_2\omega_3\cos\varphi/\sin\theta\big]\boldsymbol{j}\\&\quad+a\big[\dot{\omega}_1\cos\theta-\omega_1^2\sin\theta\big]\boldsymbol{k}.\end{aligned} \tag{11.15}$$

Here we have applied (11.10) to eliminate the generalized velocities $\dot{\theta}$, $\dot{\varphi}$, $\dot{\psi}$ in favor of the quasi-velocities ω_1, ω_2, ω_3. Finally, we compute $\|\ddot{\boldsymbol{r}}_c\|^2$ which forms a part of the problem's Gibbs function:

$$\|\ddot{\boldsymbol{r}}_c\|^2=a^2[\dot{\omega}_1^2+\dot{\omega}_3^2+2\dot{\omega}_1\omega_2\omega_3-2\omega_1\omega_2\dot{\omega}_3]+\cdots, \tag{11.16}$$

where the ellipsis indicate terms that involve no acceleration terms $\dot{\omega}_1$, $\dot{\omega}_2$, $\dot{\omega}_3$.

11.8 ▪ The rotational acceleration

The body frame $\{\boldsymbol{b}_1,\boldsymbol{b}_2,\boldsymbol{b}_3\}$ is lined up with the coin's principal moment of inertia axes, therefore the moment of inertia tensor relative to the coin's center is

$$\mathscr{I}=\alpha\boldsymbol{b}_1\otimes\boldsymbol{b}_1+\alpha\boldsymbol{b}_2\otimes\boldsymbol{b}_2+\beta\boldsymbol{b}_3\otimes\boldsymbol{b}_3.$$

where

$$\alpha=\frac{1}{4}ma^2,\quad \beta=\frac{1}{2}ma^2. \tag{11.17}$$

Therefore we have:

$$\mathscr{I}\boldsymbol{\omega} = \alpha(\boldsymbol{b}_1\cdot\boldsymbol{\omega})\boldsymbol{b}_1 + \alpha(\boldsymbol{b}_2\cdot\boldsymbol{\omega})\boldsymbol{b}_2 + \beta(\boldsymbol{b}_3\cdot\boldsymbol{\omega})\boldsymbol{b}_3,$$
$$\mathscr{I}\dot{\boldsymbol{\omega}} = \alpha(\boldsymbol{b}_1\cdot\dot{\boldsymbol{\omega}})\boldsymbol{b}_1 + \alpha(\boldsymbol{b}_2\cdot\dot{\boldsymbol{\omega}})\boldsymbol{b}_2 + \beta(\boldsymbol{b}_3\cdot\dot{\boldsymbol{\omega}})\boldsymbol{b}_3.$$

Inserting for $\dot{\boldsymbol{\omega}}$ from (11.11) we get:

$$\begin{aligned}\dot{\boldsymbol{\omega}}\cdot\mathscr{I}\dot{\boldsymbol{\omega}} &= \alpha(\dot\omega_1 + \omega_2\omega_3 - \omega_2^2\cot\theta)^2 + \alpha(\dot\omega_2 - \omega_3\omega_1 + \omega_1\omega_2\cot\theta)^2 + \dot\omega_3^2\\ &= \alpha\dot\omega_1^2 + \alpha\dot\omega_2^2 + \beta\dot\omega_3^2 - 2\alpha(\dot\omega_1\omega_2 - \omega_1\dot\omega_2)(\omega_2\cot\theta - \omega_3)\\ &\quad + \alpha(\omega_1^2+\omega_2^2)(\omega_2\cot\theta-\omega_3)^2. \end{aligned} \tag{11.18}$$

The final term is free of accelerations, therefore it may be dropped when forming the Gibbs function.

In a similar manner we also calculate

$$\begin{aligned}(\dot{\boldsymbol{\omega}}\times\boldsymbol{\omega})\cdot\mathscr{I}\boldsymbol{\omega} &= (\beta-\alpha)(\dot\omega_1\omega_2 - \omega_1\dot\omega_2)\omega_3\\ &\quad -(\beta-\alpha)(\omega_1^2+\omega_2^2)(\omega_2\cot\theta-\omega_3)\omega_3. \end{aligned}\tag{11.19}$$

Again, the final term is free of accelerations, therefore it may be dropped when forming the Gibbs function.

According to (11.2) and (10.7), The Gibbs function is

$$\begin{aligned}\mathfrak{G} &= \int_{\mathscr{B}} \frac{1}{2}\|\ddot{\boldsymbol{r}}\|^2\,dm - \int_{\mathscr{B}} \boldsymbol{F}\cdot\ddot{\boldsymbol{r}}\,dm\\ &= \frac{1}{2}m\|\ddot{\boldsymbol{r}}_c\|^2 + \frac{1}{2}\dot{\boldsymbol{\omega}}\cdot\mathscr{I}\dot{\boldsymbol{\omega}} + (\dot{\boldsymbol{\omega}}\times\boldsymbol{\omega})\cdot\mathscr{I}\boldsymbol{\omega} - (-mg\boldsymbol{k})\cdot\ddot{\boldsymbol{r}}_c,\end{aligned}$$

which may be evaluated by putting together (11.15), (11.16), (11.18), and (11.19):

$$\begin{aligned}\mathfrak{G} &= \frac{1}{2}ma^2[\dot\omega_1^2 + \dot\omega_3^2 + 2\dot\omega_1\omega_2\omega_3 - 2\omega_1\omega_2\dot\omega_3] + \frac{1}{2}\alpha\dot\omega_1^2 + \frac{1}{2}\alpha\dot\omega_2^2 + \frac{1}{2}\beta\dot\omega_3^2\\ &\quad - (\dot\omega_1\omega_2 - \omega_1\dot\omega_2)(\alpha\omega_2\cot\theta - \beta\omega_3) + mga\dot\omega_1\cos\theta + \cdots,\end{aligned}$$

where, as always, we have dropped terms which do not depend on the accelerations. Finally, substituting for α and β from (11.17), we arrive at

$$\begin{aligned}\mathfrak{G} = ma^2\Big[&\frac{5}{8}\dot\omega_1^2 + \frac{1}{8}\dot\omega_2^2 + \frac{3}{4}\dot\omega_3^2 - \frac{1}{4}\dot\omega_1\omega_2(\omega_2\cot\theta - 6\omega_3)\\ &+ \frac{1}{4}\omega_1\dot\omega_2(\omega_2\cot\theta - 2\omega_3) - \omega_1\omega_2\dot\omega_3\Big] + mga\dot\omega_1\cos\theta.\end{aligned}$$

Then from the equations $\partial\mathfrak{G}/\partial\dot\omega_1 = 0$, $\partial\mathfrak{G}/\partial\dot\omega_2 = 0$, $\partial\mathfrak{G}/\partial\dot\omega_3 = 0$ we get

$$\begin{aligned}&\frac{5}{4}\dot\omega_1 - \frac{1}{4}\omega_2(\omega_2\cot\theta - 6\omega_3) + \frac{g}{a}\cos\theta = 0,\\ &\frac{1}{4}\dot\omega_2 + \frac{1}{4}\omega_1(\omega_2\cot\theta - 2\omega_3) = 0,\\ &\frac{3}{2}\dot\omega_3 - \omega_1\omega_2 = 0.\end{aligned}$$

Clearly this set of differential equations is under-determined, since it depend on θ which is also an unknown. However, from (11.10) we see that $\dot\theta = \omega_1$. Adjoining this to the above gives us a system of four first order differential equations in the four unknowns ω_1, ω_2, ω_3, and θ.

In practice, we extend the system by adjoining all three of the equations from (11.10), and the two equations from (11.14). Thus, we obtain a system of eight first order differential equations in the eight unknowns ω_1, ω_2, ω_3, θ, φ, ψ, x, and y, whose solution completely determines the coin's motion. Here is the system in its full glory:

$$\begin{aligned}
&\dot\omega_1 - \frac{1}{5}\omega_2(\omega_2\cot\theta - 6\omega_3) + \frac{4g}{5a}\cos\theta = 0,\\
&\dot\omega_2 + \omega_1(\omega_2\cot\theta - 2\omega_3) = 0,\\
&\dot\omega_3 - \frac{2}{3}\omega_1\omega_2 = 0,\\
&\dot\theta = \omega_1,\\
&\dot\varphi = \frac{1}{\sin\theta}\omega_2,\\
&\dot\psi = \omega_3 - \omega_2\cot\theta.\\
&\dot x = -a(\omega_3\cos\varphi - \omega_1\sin\varphi\sin\theta),\\
&\dot y = -a(\omega_3\sin\varphi + \omega_1\cos\varphi\sin\theta).
\end{aligned} \tag{11.20}$$

Remark 11.3. We need eight initial conditions to go with these equations. The initial conditions on x, y, θ, φ, and ψ determine the coin's initial position relative to the stationary axes, so they may be specified in the obvious way. The initial conditions on ω_1, ω_2, and ω_3 require a more careful consideration. You may recall from (11.9) that the ω's were defined as the components of the coin's angular velocity along the intermediate frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. Since the intermediate frame has no immediate physical manifestation, it is not easy to make up meaningful initial values for the ω's. We do note, however, that (11.9) defines the ω's in terms of the generalized coordinates and their velocities, which are easier to grasp. Therefore in practice you will make up initial values for $\dot\theta$, $\dot\varphi$, and $\dot\psi$ as desired, then use (11.9) to determine the initial values for ω_1, ω_2, and ω_3.

Exercises

11.1. Derive (11.11).

11.2. Give the details of the computation which leads from (11.13) to (11.14).

11.3. Derive (11.19).

11.4. Equip a rigid body $\mathscr{B}$ with a body reference frame $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ whose origin coincides with the body's center of mass. Additionaly, suppose that the $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ is aligned with the body's moment of inertia axes, so that the moment of inertia tensor takes the form

$$\mathscr{I} = I_1\boldsymbol{b}_1\otimes\boldsymbol{b}_1 + I_2\boldsymbol{b}_1\otimes\boldsymbol{b}_2 + I_3\boldsymbol{b}_1\otimes\boldsymbol{b}_3.$$

Let ω_1, ω_2, ω_3 be the components of the body's angular velocity along the $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ vectors, that is

$$\boldsymbol{\omega} = \omega_1\boldsymbol{b}_1 + \omega_2\boldsymbol{b}_2 + \omega_3\boldsymbol{b}_3.$$

Show that:

1. $\dot{\boldsymbol{\omega}} \cdot \mathscr{I} \dot{\boldsymbol{\omega}} = I_1 \dot{\omega}_1{}^2 + I_2 \dot{\omega}_2{}^2 + I_3 \dot{\omega}_3{}^2$.
2. $(\dot{\boldsymbol{\omega}} \times \boldsymbol{\omega}) \cdot \mathscr{I} \boldsymbol{\omega} = I_1(\dot{\omega}_2 \omega_3 - \dot{\omega}_3 \omega_2)\omega_1$
$$+ I_2(\dot{\omega}_3 \omega_1 - \dot{\omega}_1 \omega_3)\omega_2 + I_3(\dot{\omega}_1 \omega_2 - \dot{\omega}_2 \omega_1)\omega_3.$$

Conclude that the equations of motion of a freely spinning rigid body are

$$\begin{aligned}
I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2 \omega_3 &= 0, \\
I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_3 \omega_1 &= 0, \\
I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1 \omega_2 &= 0.
\end{aligned}$$

11.5. Solve the system of differential equations (11.20) and produce an animation of the coin's motion.

Chapter 12
Quaternions

12.1 ▪ The quaternion algebra

A *quaternion*, as a mathematical object, is a pair $[\![a, \boldsymbol{u}]\!]$, where $a \in \mathbf{R}$ and $\boldsymbol{u} \in E_3$. Multiplication by a scalar $c\mathbb{p}$, the sum $\mathbb{p}+\mathbb{q}$, and the product $\mathbb{p} \circ \mathbb{q}$ of the quaternions $\mathbb{p} = [\![a, \boldsymbol{u}]\!]$ and $\mathbb{q} = [\![b, \boldsymbol{v}]\!]$ are defined according to

$$c\mathbb{p} = c[\![a, \boldsymbol{u}]\!] = [\![ca, c\boldsymbol{u}]\!], \qquad (c \in \mathbf{R}) \tag{12.1}$$

$$\mathbb{p}+\mathbb{q} = [\![a, \boldsymbol{u}]\!] + [\![b, \boldsymbol{v}]\!] = [\![a+b, \boldsymbol{u}+\boldsymbol{v}]\!], \tag{12.2}$$

$$\mathbb{p} \circ \mathbb{q} = [\![a, \boldsymbol{u}]\!] \circ [\![b, \boldsymbol{v}]\!] = [\![ab - \boldsymbol{u}\cdot\boldsymbol{v}, a\boldsymbol{v} + b\boldsymbol{u} + \boldsymbol{u}\times\boldsymbol{v}]\!]. \tag{12.3}$$

Note that addition is commutative but multiplication, due to the presence of the $\boldsymbol{u}\times\boldsymbol{v}$ term, is not.

As a special case of (12.3), we have

$$[\![c, \mathbf{0}]\!] \circ [\![a, \boldsymbol{u}]\!] = c[\![a, \boldsymbol{u}]\!] \qquad \text{and} \qquad [\![a, \boldsymbol{u}]\!] \circ [\![c, \mathbf{0}]\!] = c[\![a, \boldsymbol{u}]\!],$$

therefore the quaternion $[\![c, \mathbf{0}]\!]$ acts exactly like the scalar c under quaternion multiplication. In particular, the quaternion $[\![1, \mathbf{0}]\!]$ is the multiplicative identity in the quaternion algebra.

In view of the observation above, we adopt the convention of writing c instead of of $[\![c, \mathbf{0}]\!]$ when there is no chance of confusion. This is analogous to writing α, instead of $\alpha + 0i$ when dealing with complex numbers. By the same token, we write $\boldsymbol{u}$ instead of $[\![0, \boldsymbol{u}]\!]$. The quaternion $c = [\![c, \mathbf{0}]\!]$ is called a *scalar quaternion* or just a *scalar* if the meaning is clear from the context. Similarly, the quaternion $\boldsymbol{u} = [\![0, \boldsymbol{u}]\!]$ is called a *vectorial quaternion*, or just a *vector*, for short.[14]

As a consequence of the convention adopted above, the notation $\boldsymbol{u} \circ \boldsymbol{v}$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are vectors, makes sense, and evaluates to

$$\boldsymbol{u} \circ \boldsymbol{v} = [\![0, \boldsymbol{u}]\!] \circ [\![0, \boldsymbol{v}]\!] = [\![-\boldsymbol{u}\cdot\boldsymbol{v}, \boldsymbol{u}\times\boldsymbol{v}]\!]. \tag{12.4}$$

You may want to think of $\boldsymbol{u} \circ \boldsymbol{v}$ as the "quaternion product" of the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, but note that the product is a quaternion, not a vector, unless $\boldsymbol{u}\cdot\boldsymbol{v} = 0$, in which case $\boldsymbol{u} \circ \boldsymbol{v} = \boldsymbol{u}\times\boldsymbol{v}$.

[14]The terminology is not quite standardized. What we have called a scalar quaternion is also called a *real* quaternion, and what we have called a vectorial quaternion is also called a *pure* quaternion or *imaginary* quaternion.

The *conjugate* $\mathbb{p}^*$ of the quaternion $\mathbb{p} = [\![a, \boldsymbol{u}]\!]$ is defined as $\mathbb{p}^* = [\![a, -\boldsymbol{u}]\!]$. Let us note that

$$\mathbb{p}^* \circ \mathbb{p} = [\![a^2 + ||\boldsymbol{u}||^2, 0]\!] = (a^2 + ||\boldsymbol{u}||^2)[\![1, \boldsymbol{0}]\!] = a^2 + ||\boldsymbol{u}||^2 = |\mathbb{p}|^2, \qquad (12.5)$$

where we have defined

$$|\mathbb{p}| = (a^2 + ||\boldsymbol{u}||^2)^{1/2}.$$

The quantity $|\mathbb{p}|$ is called the *norm* (or *modulus*) of the quaternion $\mathbb{p}$. A quaternion $\mathbb{p}$ such that $|\mathbb{p}| = 1$ is called a *unit quaternion*.

If $|\mathbb{p}|$ is nonzero, then we may rearrange (12.5) into

$$\left(\frac{1}{|\mathbb{p}|^2} \mathbb{p}^* \right) \circ \mathbb{p} = 1,$$

which shows that the parenthesized expression is the left multiplicative inverse of $\mathbb{p}$. Repeating the calculation, beginning with $\mathbb{p} \circ \mathbb{p}^*$, we see that the parenthesized expression is also the right multiplicative inverse of $\mathbb{p}$. We conclude that if $|\mathbb{p}|$ is nonzero, then $\mathbb{p}$ has a multiplicative inverse, $\mathbb{p}^{-1}$, given by

$$\mathbb{p}^{-1} = \frac{1}{|\mathbb{p}|^2} \mathbb{p}^*, \qquad (|\mathbb{p}| \neq 0).$$

The special case unit criterion arises quite frequently:

$$\mathbb{p}^{-1} = \mathbb{p}^*, \qquad (|\mathbb{p}| = 1).$$

Remark 12.1. This section has touched on just a few basic concepts of the quaternion algebra which will be needed in what follows. For an in-depth study of quaternions see Altmann [1]. For a leisurely introduction to quaternions, with applications to computer graphics, see Hanson [9].

12.2 ▪ The geometry of the quaternions

The goal of this section is to elucidate the geometric interpretations of a few features related to quaternion which play significant roles in the dynamics of rigid bodies.

12.2.1 ▪ The reflection operator

As we noted in connection with (12.4), the quaternion product of two vectors is not a vector in general. Thus, it may come as a surprise that the triple product $\boldsymbol{v} \circ \boldsymbol{u} \circ \boldsymbol{v}$ is always a vector:

$$\boldsymbol{v} \circ \boldsymbol{u} \circ \boldsymbol{v} = ||\boldsymbol{v}||^2 \boldsymbol{u} - 2(\boldsymbol{v} \cdot \boldsymbol{u})\boldsymbol{v}, \qquad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in E_3. \qquad (12.6)$$

The proof of this identity is left as an exercise.

In particular, if $\boldsymbol{v}$ is a unit vector, let's call it $\boldsymbol{n}$, then we have

$$\boldsymbol{n} \circ \boldsymbol{u} \circ \boldsymbol{n} = \boldsymbol{u} - 2(\boldsymbol{n} \cdot \boldsymbol{u})\boldsymbol{n}, \qquad \text{if } ||\boldsymbol{n}|| = 1. \qquad (12.7)$$

The right-hand side has a well-known geometric interpretation. Suppose the tails of the vectors $\boldsymbol{n}$ and $\boldsymbol{u}$ are attached to a common point o, and let P be a plane through o and perpendicular to $\boldsymbol{n}$, as illustrated in Figure 12.1, and let Q be the plane of the vectors $\boldsymbol{n}$

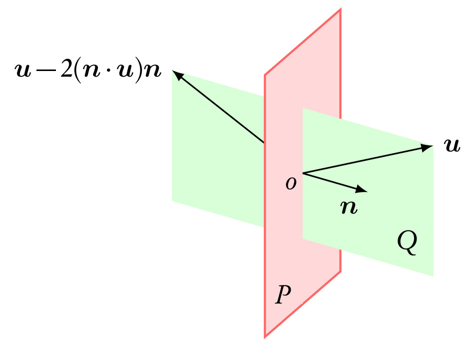


Figure 12.1: whatever

and $\boldsymbol{u}$. Then it is simple exercise to show that $\boldsymbol{u}-2(\boldsymbol{n}\cdot\boldsymbol{u})\boldsymbol{n}$ is the mirror reflection of the vector $\boldsymbol{u}$ through the plane P. Thus, we have established the following:

Theorem 12.1. *The reflection operator $R_{\boldsymbol{n}}$ into a plane with a unit normal $\boldsymbol{n}$ is given by*

$$R_{\boldsymbol{n}}\boldsymbol{u} = \boldsymbol{n}\circ\boldsymbol{u}\circ\boldsymbol{n}, \quad \boldsymbol{u}\in E_3. \tag{12.8}$$

12.2.2 ▪ The rotation operator

Consider two planes, P_1 and P_2, which have unit normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ and make a dihedral angle $\varphi/2$ with each other. In the sketch shown in Figure 12.2a, the planes are shown edgewise, therefore they appear as lines. The line of intersection of the two planes appears as a point in that figure since we are looking at it head on. The unit vector $\boldsymbol{n} = (\boldsymbol{n}_1 \times \boldsymbol{n}_2)/||\boldsymbol{n}_1 \times \boldsymbol{n}_2||$ lies in the direction of that line of intersection.

Let $R_{\boldsymbol{n}_1}$ and $R_{\boldsymbol{n}_2}$ be the reflection operators, as in the previous subsection, into the planes P_1 and P_2. Pick an arbitrary vector $\boldsymbol{u}$, apply $R_{\boldsymbol{n}_1}$ to it, and then apply $R_{\boldsymbol{n}_2}$ the the result. Figure 12.2b shows the effect.

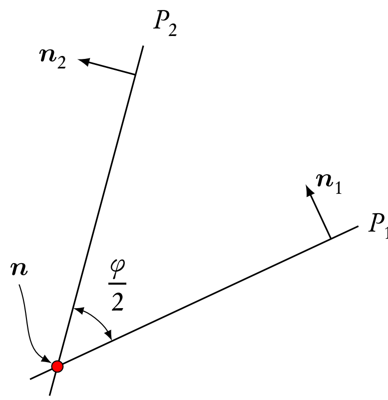
To gain insight into the structure of this geometric construction, it helps to introduce the plane P_1' which is the reflection of P_1 into P_2. The plane P_1', seen edgewise, is shown as a dashed line in Figure 12.2c.

In Figure 12.2d we have added the vector $R_{\boldsymbol{n}_2}\boldsymbol{u}$. It should be evident from the symmetries of the diagram that $R_{\boldsymbol{n}_2}\boldsymbol{u}$ coincides with the reflection of $R_{\boldsymbol{n}_2}R_{\boldsymbol{n}_1}\boldsymbol{u}$ into P_1'.

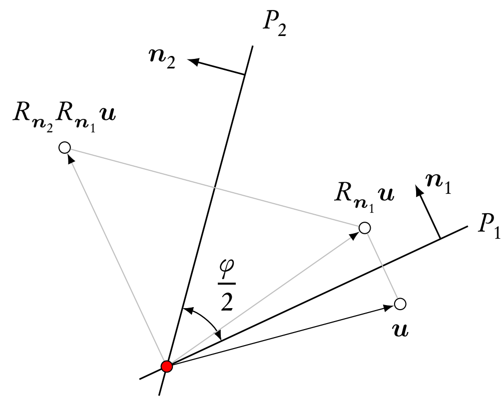
As a rotation by the angle φ about the vector $\boldsymbol{n}$ takes the plane P_1 to the plane P_1', it is evident that the same rotation takes the vector $\boldsymbol{u}$ to the vector $R_{\boldsymbol{n}_2}R_{\boldsymbol{n}_1}\boldsymbol{u}$. Thus, we have established the following:

Lemma 12.2. *Suppose the planes P_1 and P_2, with unit normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, make a dihedral angle of $\varphi/2$ with each other. Let $R_{\boldsymbol{n}_1}$ and $R_{\boldsymbol{n}_2}$ be the reflection operators into the planes P_1 and P_2, and let $R_{\boldsymbol{n},\varphi}$ be the rotation operator by angle φ about the vector $\boldsymbol{n} = (\boldsymbol{n}_1\times\boldsymbol{n}_2)/||\boldsymbol{n}_1\times\boldsymbol{n}_2||$. Then $R_{\boldsymbol{n},\varphi} = R_{\boldsymbol{n}_2}R_{\boldsymbol{n}_1}$.*

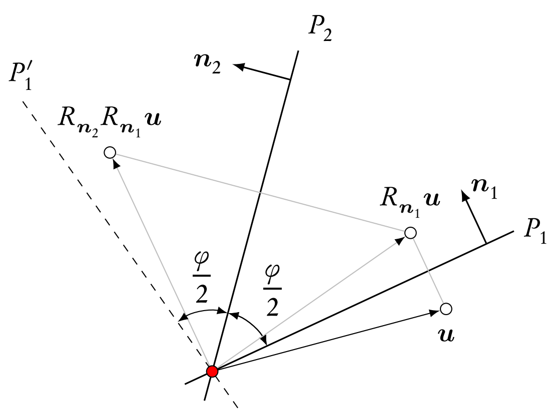
In the previous section we characterized a reflection operator as a triple quaternion product. That, along with the lemma above enables us to express a rotation operator in terms of quaternions. This is the subject of the following:



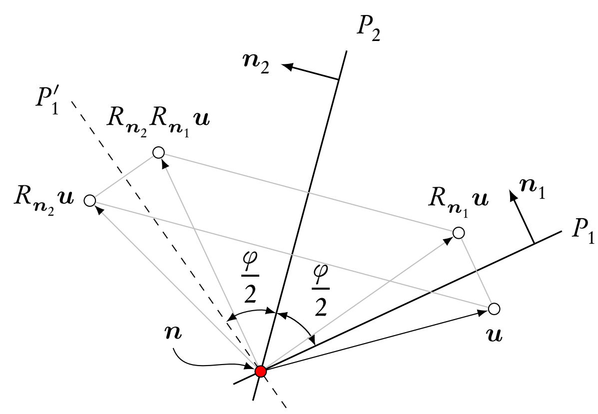
(a) Planes P_1 and P_2, shown edgewise, have unit normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, and make a dihedral angle of $\varphi/2$. The unit vector $\boldsymbol{n} = (\boldsymbol{n}_1 \times \boldsymbol{n}_2)/\|\boldsymbol{n}_1 \times \boldsymbol{n}_2\|$ lies along their intersection, and points out of the picture, toward you, therefore it is not visible.



(b) Reflecting the arbitrary vector $\boldsymbol{u}$ into the plane P_1 produces $R_{\boldsymbol{n}_1}\boldsymbol{u}$. Reflecting the latter into the plane P_2 results in $R_{\boldsymbol{n}_2}R_{\boldsymbol{n}_1}\boldsymbol{u}$.



(c) We introduce the auxiliary plane P_1' (shown as a dashed line) which is the reflection of the plane P_1 into the plane P_2, therefore it makes a dihedral angle of $\varphi/2$ with P_2.



(d) The vector $R_{\boldsymbol{n}_2}$ produced by reflecting $\boldsymbol{u}$ into P_2 coincides with the reflection of $R_{\boldsymbol{n}_2}R_{\boldsymbol{n}_1}\boldsymbol{u}$ into P_1'. From the symmetries of the diagram it is evident that a rotation by an angle φ about the vector $\boldsymbol{n}$ takes the plane plane P_1 to the plane P_1'. It follows that the same rotation takes the vector $\boldsymbol{u}$ to the vector $R_{\boldsymbol{n}_2}R_{\boldsymbol{n}_1}\boldsymbol{u}$, thus proving that $R_{\boldsymbol{n},\varphi} = R_{\boldsymbol{n}_2}R_{\boldsymbol{n}_1}$.

Figure 12.2: The sequence of diagrams is in effect a "proof without words" of Lemma 12.2. It shows clearly that the composition of the two reflections $R_{\boldsymbol{n}_1}$ and $R_{\boldsymbol{n}_2}$ into the planes P_1 and P_2 is equivalent to a rotation by an angle φ about the vector $\boldsymbol{n}$ which lies in the intersection of those planes.

Theorem 12.3. *The operator $R_{\boldsymbol{n},\varphi}$ of rotation by angle φ about a unit vector $\boldsymbol{n}$ has an associated unit quaternion*

$$\mathbb{q} = [\![\cos\frac{\varphi}{2}, \boldsymbol{n}\sin\frac{\varphi}{2}]\!] \tag{12.9}$$

so that

$$R_{\boldsymbol{n},\varphi}\boldsymbol{u} = \mathbb{q} \circ \boldsymbol{u} \circ \mathbb{q}^*, \quad \textit{for all } \boldsymbol{u} \in E_3. \tag{12.10}$$

Proof. Pick any two planes P_1 and P_2 which intersect along a line parallel to $\boldsymbol{n}$, and whose dihedral angle is $\varphi/2$. Then according to Theorem 12.1 we have

$$R_{\boldsymbol{n}_1}\boldsymbol{u} = \boldsymbol{n}_1 \circ \boldsymbol{u} \circ \boldsymbol{n}_1. \qquad R_{\boldsymbol{n}_2}\boldsymbol{u} = \boldsymbol{n}_2 \circ \boldsymbol{u} \circ \boldsymbol{n}_2,$$

and according to Lemma 12.2 we have $R_{\boldsymbol{n},\varphi} = R_{\boldsymbol{n}_2}R_{\boldsymbol{n}_1}$. Thus, we compute

$$\begin{aligned} R_{\boldsymbol{n},\varphi}\boldsymbol{u} &= R_{\boldsymbol{n}_2}R_{\boldsymbol{n}_1}\boldsymbol{u} \\ &= R_{\boldsymbol{n}_2}(\boldsymbol{n}_1 \circ \boldsymbol{u} \circ \boldsymbol{n}_1) \\ &= \boldsymbol{n}_2 \circ (\boldsymbol{n}_1 \circ \boldsymbol{u} \circ \boldsymbol{n}_1) \circ \boldsymbol{n}_2 \\ &= (\boldsymbol{n}_2 \circ \boldsymbol{n}_1) \circ \boldsymbol{u} \circ (\boldsymbol{n}_1 \circ \boldsymbol{n}_2). \end{aligned}$$

Let $\mathbb{q} = \boldsymbol{n}_2 \circ \boldsymbol{n}_1$. We leave it as an exercise for you to show that $\mathbb{q}^* = \boldsymbol{n}_1 \circ \boldsymbol{n}_2$. We conclude that $R_{\boldsymbol{n},\varphi} = \mathbb{q} \circ \boldsymbol{u} \circ \mathbb{q}^*$, which proves (12.10). To show that $\mathbb{q}$ has the form given in (12.9), observe that the dihedral angle between the planes P_1 and P_2 is $\varphi/2$, therefore the vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$, which are perpendicular to those planes, also form an angle of $\varphi/2$. Then applying (12.4) we see that

$$\mathbb{q} = \boldsymbol{n}_2 \circ \boldsymbol{n}_1 = [\![-\boldsymbol{n}_2 \cdot \boldsymbol{n}_1, \boldsymbol{n}_2 \times \boldsymbol{n}_1]\!] = [\![-\cos\frac{\varphi}{2}, -\boldsymbol{n}\sin\frac{\varphi}{2}]\!] = -[\![\cos\frac{\varphi}{2}, \boldsymbol{n}\sin\frac{\varphi}{2}]\!].$$

The negative sign in the final expression is immaterial since $\mathbb{q}$ appears twice in the rotation formula (12.10). Finally, the assertion that $\mathbb{q}$ is a *unit quaternion* follows immediately from the form of (12.9). □

Remark 12.2. The planes P_1 and P_2 and their associated unit normals $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ which enter the proof above are conveniences for the proof. They do not appear in the theorem's final result.

Remark 12.3. If we split the quaternion $\mathbb{q}$ in (12.10) in (12.10) into components, as in $\mathbb{q} = [\![q_0, \boldsymbol{q}]\!]$, and expand the triple quaternion product, we get

$$R_{\boldsymbol{n},\varphi}\boldsymbol{u} = \mathbb{q} \circ \boldsymbol{u} \circ \mathbb{q}^* = (q_0^2 - \|\boldsymbol{q}\|^2)\boldsymbol{u} + 2(\boldsymbol{q} \cdot \boldsymbol{u})\boldsymbol{q} + 2q_0(\boldsymbol{q} \times \boldsymbol{u}). \tag{12.11}$$

We may be more explicit by setting $q_0 = \cos\varphi/2$ and $\boldsymbol{q} = \boldsymbol{n}\sin\varphi/2$ and simplifying (12.10) to get

$$R_{\boldsymbol{n},\varphi}\boldsymbol{u} = \mathbb{q} \circ \boldsymbol{u} \circ \mathbb{q}^* = (\boldsymbol{n} \cdot \boldsymbol{u})\boldsymbol{n} + \big(\boldsymbol{u} - (\boldsymbol{n} \cdot \boldsymbol{u})\boldsymbol{n}\big)\cos\varphi + (\boldsymbol{n} \times \boldsymbol{u})\sin\varphi,$$

which has an obvious geometric interpretation if you look at it closely.

12.3 ▪ Angular velocity

Theorem 12.3 shows that a rotation operator in E_3 may be expressed in terms of an associated unit quaternion $\mathbb{q}$ defined in (12.9). A rotation that varies with the time t then may be expressed in terms of a time-dependent unit quaternion $\mathbb{q}(t)$ of that form. An arbitrarily fixed vector $\boldsymbol{r}_0 \in E_3$ then rotates to the position $\boldsymbol{r}(t)$ according

$$\boldsymbol{r}(t) = \mathbb{q}(t) \circ \boldsymbol{r}_0 \circ \mathbb{q}^*(t). \tag{12.12}$$

Suppose that the instantaneous angular velocity vector under this rotation at time t is $\boldsymbol{\omega}(t)$. Then, according to (7.7), we have:

$$\dot{\boldsymbol{r}}(t) = \boldsymbol{\omega}(t) \times \boldsymbol{r}(t). \tag{12.13}$$

Comparing (12.12) and (12.13) we expect a relationship between $\mathbb{q}(t)$ and $\boldsymbol{\omega}(t)$. This is stated in the following:

Theorem 12.4. *Let* $\mathbb{q}(t)$ *be any time-varying unit quaternion, and let* $\boldsymbol{\omega}(t)$ *the angular velocity of the rotation* (12.12) *engendered by* $\mathbb{q}(t)$*. Then we have*

$$\boldsymbol{\omega} = 2\dot{\mathbb{q}} \circ \mathbb{q}^*. \tag{12.14}$$

Proof. Since $\mathbb{q}(t)$ is a unit quaternion, we have $|\mathbb{q}(t)|^2 = \mathbb{q}(t) \circ \mathbb{q}^*(t) = 1$, and therefore $\dot{\mathbb{q}} \circ \mathbb{q}^* + \mathbb{q} \circ \dot{\mathbb{q}}^* = 0$. But since $\mathbb{q} \circ \dot{\mathbb{q}}^* = (\dot{\mathbb{q}} \circ \mathbb{q}^*)^*$, it follows that $\dot{\mathbb{q}} \circ \mathbb{q}^* + (\dot{\mathbb{q}} \circ \mathbb{q}^*)^* = 0$, which indicates that the scalar part of the quaternion $\dot{\mathbb{q}} \circ \mathbb{q}^*$ is zero, that is, $\dot{\mathbb{q}} \circ \mathbb{q}^*$ is a vectorial quaternion. This justifies defining the vector $\boldsymbol{\omega}$ through the quaternion product (12.14). It remains to show that $\boldsymbol{\omega}$ thus defined is indeed the motion's angular velocity.

Toward that end, let us compute $\dot{\boldsymbol{r}}$ by differentiating (12.12):

$$\dot{\boldsymbol{r}} = \dot{\mathbb{q}} \circ \boldsymbol{r}_0 \circ \mathbb{q}^* + \mathbb{q} \circ \boldsymbol{r}_0 \circ \dot{\mathbb{q}}^*.$$

We note $\mathbb{q}^{-1} = \mathbb{q}^*$ by virtue of $\mathbb{q}$ being a unit quaternion, therefore $\boldsymbol{r} = \mathbb{q} \circ \boldsymbol{r}_0 \circ \mathbb{q}^*$ implies that

$$\boldsymbol{r}_0 \circ \mathbb{q}^* = \mathbb{q}^* \circ \boldsymbol{r} \quad \text{and} \quad \mathbb{q} \circ \boldsymbol{r}_0 = \boldsymbol{r} \circ \mathbb{q}.$$

Therefore, the expression for $\dot{\boldsymbol{r}}$ takes the form

$$\begin{aligned}
\dot{\boldsymbol{r}} &= \dot{\mathbb{q}} \circ \mathbb{q}^* \circ \boldsymbol{r} + \boldsymbol{r} \circ \mathbb{q} \circ \dot{\mathbb{q}}^* \\
&= \dot{\mathbb{q}} \circ \mathbb{q}^* \circ \boldsymbol{r} + \boldsymbol{r} \circ (\dot{\mathbb{q}} \circ \mathbb{q}^*)^* \\
&= \Big(\frac{1}{2}\boldsymbol{\omega}\Big) \circ \boldsymbol{r} + \boldsymbol{r} \circ \Big(\frac{1}{2}\boldsymbol{\omega}\Big)^* = \frac{1}{2}(\boldsymbol{\omega} \circ \boldsymbol{r} + \boldsymbol{r} \circ \boldsymbol{\omega}^*).
\end{aligned}$$

We leave it as an easy exercise to show that evaluating the right-hand side with the help of the identity (12.4) leads to

$$\dot{\boldsymbol{r}} = \boldsymbol{\omega} \times \boldsymbol{r} \tag{12.15}$$

which shows that $\boldsymbol{\omega}$ defined in (12.14) is indeed the sought-after angular velocity. ◻

12.4 ▪ A differential equation for the quaternion rotation

Recall the definitions of the three frames of reference in Section 11.1 in connection with the motion of a rigid body. The *stationary frame* is equipped with a non-moving orthonormal triad $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. The *body frame* is equipped with an orthonormal triad $\{\boldsymbol{b}_1, \boldsymbol{b}_3, \boldsymbol{b}_3\}$ which is fixed to the body and movies with it.

At any time t, the body triad $\{\boldsymbol{b}_1, \boldsymbol{b}_3, \boldsymbol{b}_3\}$ may be viewed as the rotated version of the stationary triad $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. Let $\mathbb{q}(t)$ be the unit quaternion associated with that rotation. Thus,

$$\boldsymbol{b}_1(t) = \mathbb{q}(t) \circ \boldsymbol{i} \circ \mathbb{q}^*(t), \quad \boldsymbol{b}_2(t) = \mathbb{q}(t) \circ \boldsymbol{j} \circ \mathbb{q}^*(t), \quad \boldsymbol{b}_3(t) = \mathbb{q}(t) \circ \boldsymbol{k} \circ \mathbb{q}^*(t), \quad (12.16)$$

which may be expanded with the help of (12.11), if desired.

Let $\boldsymbol{\omega}$ be the body's angular velocity vector, and ω_1, ω_2, and ω_3 be $\boldsymbol{\omega}$'s components in the body frame:

$$\boldsymbol{\omega} = \omega_1 \boldsymbol{b}_1 + \omega_1 \boldsymbol{b}_2 + \omega_1 \boldsymbol{b}_3. \quad (12.17)$$

Substituting from (12.16) we get

$$\begin{aligned} \boldsymbol{\omega} &= \omega_1 \mathbb{q} \circ \boldsymbol{i} \circ \mathbb{q}^* + \omega_2 \mathbb{q} \circ \boldsymbol{j} \circ \mathbb{q}^* + \omega_3 \mathbb{q} \circ \boldsymbol{k} \circ \mathbb{q}^* \\ &= \mathbb{q} \circ (\omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k}) \circ \mathbb{q}^*. \end{aligned}$$

Then from (12.14) it follows that

$$2\dot{\mathbb{q}} \circ \mathbb{q}^* = \mathbb{q} \circ (\omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k}) \circ \mathbb{q}^*,$$

whence

$$\dot{\mathbb{q}} = \frac{1}{2} \mathbb{q} \circ (\omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k}). \quad (12.18)$$

This system of first order differential equations expresses the evolution of $\mathbb{q}$ in time. Together with the equations in (12.16), these determine the body's orientation in space. We will adjoin them to the differential equations of dynamics to obtain a complete system of differential equations that describes the body's motion.

The differential equation's initial condition, $\mathbb{q}(0)$, which specifies the body's initial orientation, is a unit quaternion, as any quaternion associated with a rotation ought to be. A question arises whether the solution $\mathbb{q}(t)$ of the differential equation is a unit quaternion for all t. The answer is yes, as it follows from Theorem 12.5 below.

Remark 12.4. Equation (12.17) defines the components ω_1, ω_2, ω_3 of the angular velocity vector $\boldsymbol{\omega}$ along the body triad. Beware that the sum $\omega_1 \boldsymbol{i} + \omega_2 \boldsymbol{j} + \omega_3 \boldsymbol{k}$ that appears in (12.18) *does not* add up to $\boldsymbol{\omega}$.

Theorem 12.5. *The differential equation*

$$\dot{\mathbb{q}}(t) = \mathbb{q}(t) \circ \boldsymbol{a}(t), \quad \mathbb{q}(0) = \mathbb{q}_0$$

for the quaternion $\mathbb{q}(t)$, where $\boldsymbol{a}(t)$ is an arbitrary time-dependent vector, preserves the norm, that is $|\mathbb{q}(t)| = |\mathbb{q}_0|$ for all t. In particular, if $\mathbb{q}_0$ is a unit quaternion, then $\mathbb{q}(t)$ is a unit quaternion for all t.

Proof. We calculate the derivative of $|\mathbb{q}(t)|^2$ and substitute for $\dot{\mathbb{q}}$ from the differential equation:

$$\frac{d}{dt}|\mathbb{q}|^2 = \frac{d}{dt}(\mathbb{q} \circ \mathbb{q}^*) = \dot{\mathbb{q}} \circ \mathbb{q}^* + \mathbb{q} \circ \dot{\mathbb{q}}^*$$
$$= (\mathbb{q} \circ \boldsymbol{a}) \circ \mathbb{q}^* + \mathbb{q} \circ (\mathbb{q} \circ \boldsymbol{a})^* = (\mathbb{q} \circ \boldsymbol{a}) \circ \mathbb{q}^* + \mathbb{q} \circ (\boldsymbol{a}^* \circ \mathbb{q}^*).$$

In the last step we have used the quaternion conjugation property $(\mathbb{p} \circ \mathbb{q})^* = \mathbb{q}^* \circ \mathbb{p}^*$; see Exercise 12.5. Now note that for any vector $\boldsymbol{a}$ we have $\boldsymbol{a}^* = [\![0, \boldsymbol{a}]\!]^* = [\![0, -\boldsymbol{a}]\!] = -\boldsymbol{a}$. Then it follows that $\frac{d}{dt}|\mathbb{q}|^2 = 0$, therefore the norm of $\mathbb{q}$ remains constant. □

For computing purposes, we express $\mathbb{q}$ in components, as in

$$\mathbb{q} = [\![q_0,\, q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}]\!],$$

whereby (12.18) expands to the system of differential equations

$$\frac{d}{dt}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{pmatrix}\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}. \tag{12.19}$$

12.5 ▪ Unbalanced ball rolling on a horizontal plane

Consider a solid ball of radius R and uniformly distributed total mass of M. We know that the ball's moment of inertia tensor relative to its center is $2/5MR^2$ times the identity tensor.

We embed a point of mass m, let's call it a "slug", within that ball at a distance $a \le R$ away from its center. We wish to study the motion of the composite object when it rolls without slipping on a horizontal plane.

We set up the usual $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ orthonormal triad as a stationary frame, with the origin on the horizontal plane and $\boldsymbol{k}$ pointing upward. We set up the body frame with its origin at the ball's center, and the body triad $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ oriented so that the slug is at $a\boldsymbol{b}_3$ relative to the ball's center, and we let $\mathbb{q}(t)$ be the unit quaternion associated with the rotation that takes $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ to $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$.

Let $\boldsymbol{r}_c = x\,\boldsymbol{i} + y\,\boldsymbol{j} + R\,\boldsymbol{k}$ be the position vector of the ball's center relative to the stationary frame. The body's position at any time is given by the vector $\boldsymbol{r}_c$ and the quaternion $\mathbb{q}$.

We express the ball's angular velocity $\boldsymbol{\omega}$ in terms of its components along the body triad as in (12.17), and note that the derivative $\dot{\boldsymbol{\omega}}$ is given by (7.4) on page 44.

12.5.1 ▪ The no-slip condition

The velocity of the point on the ball which is at contact with the horizontal plane is $\dot{\boldsymbol{r}}_c + \boldsymbol{\omega} \times (-R\,\boldsymbol{k})$, which should be zero if there is to be no slippage at the contact point. Therefore we have $\dot{\boldsymbol{r}}_c = R\boldsymbol{\omega} \times \boldsymbol{k}$, that is, $\dot{x}\,\boldsymbol{i} + \dot{y}\,\boldsymbol{j} = R\boldsymbol{\omega} \times \boldsymbol{k}$. It follows that $\dot{x} = R\boldsymbol{\omega} \times \boldsymbol{k} \cdot \boldsymbol{i} = R\boldsymbol{k} \times \boldsymbol{i} \cdot \boldsymbol{\omega} = R\,\boldsymbol{j} \cdot \boldsymbol{\omega}$. Similarly, $\dot{y} = -R\,\boldsymbol{i} \cdot \boldsymbol{\omega}$. The conditions

$$\dot{x} = R\,\boldsymbol{j} \cdot \boldsymbol{\omega} \qquad \text{and} \qquad \dot{y} = -R\,\boldsymbol{i} \cdot \boldsymbol{\omega} \tag{12.20}$$

are this problem's nonholonomic constraints.

The from $\dot{\boldsymbol{r}}_c = \dot{x}\,\boldsymbol{i} + \dot{y}\,\boldsymbol{j}$ and $\ddot{\boldsymbol{r}}_c = \ddot{x}\,\boldsymbol{i} + \ddot{y}\,\boldsymbol{j}$ it follows that

$$\dot{\boldsymbol{r}}_c = R\big[(\boldsymbol{j}\cdot\boldsymbol{\omega})\boldsymbol{i} - (\boldsymbol{i}\cdot\boldsymbol{\omega})\boldsymbol{j}\big], \tag{12.21a}$$

$$\ddot{\boldsymbol{r}}_c = R\big[(\boldsymbol{j}\cdot\dot{\boldsymbol{\omega}})\boldsymbol{i} - (\boldsymbol{i}\cdot\dot{\boldsymbol{\omega}})\boldsymbol{j}\big]. \tag{12.21b}$$

We will use (12.21b) to eliminate $\ddot{x}$ and $\ddot{y}$ from the Gibbs function.

12.5.2 ▪ The Gibbs function and the equations of motion

The Gibbs function of the system is the sum of the Gibbs functions corresponding to the homogeneous ball and to the slug. We calculate them separately, and then add them up.

The Gibbs function of the homogeneous ball is

$$\mathfrak{G}_1 = \frac{1}{2}\Big(M\|\ddot{\boldsymbol{r}}_c\|^2 + \frac{2}{5}MR^2(\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2)\Big).$$

Let us note that due to (12.21b) we have

$$\begin{aligned}\|\ddot{\boldsymbol{r}}_c\|^2 = R^2\big[(\boldsymbol{j}\cdot\dot{\boldsymbol{\omega}})^2 + (\boldsymbol{i}\cdot\dot{\boldsymbol{\omega}})^2\big] &= R^2\big[\|\dot{\boldsymbol{\omega}}\|^2 - (\boldsymbol{k}\cdot\dot{\boldsymbol{\omega}})^2\big]\\ &= R^2\big(\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2 - (\boldsymbol{k}\cdot\dot{\boldsymbol{\omega}})^2\big),\end{aligned} \tag{12.22}$$

therefore

$$\mathfrak{G}_1 = \frac{1}{2}MR^2\Big(\frac{7}{5}(\dot{\omega}_1^2 + \dot{\omega}_2^2 + \dot{\omega}_3^2) - (\boldsymbol{k}\cdot\dot{\boldsymbol{\omega}})^2\Big).$$

To compute the slug's Gibbs function, let us write $\boldsymbol{r}$ for its position vector. We have:

$$\boldsymbol{r} = \boldsymbol{r}_c + a\boldsymbol{b}_3.$$

Then we calculate the slug's velocity

$$\dot{\boldsymbol{r}} = \dot{\boldsymbol{r}}_c + a\dot{\boldsymbol{b}}_3 = \dot{\boldsymbol{r}}_c + a\boldsymbol{\omega}\times\boldsymbol{b}_3,$$

and its acceleration

$$\begin{aligned}\ddot{\boldsymbol{r}} &= \ddot{\boldsymbol{r}}_c + a\big(\dot{\boldsymbol{\omega}}\times\boldsymbol{b}_3 + \boldsymbol{\omega}\times\dot{\boldsymbol{b}}_3\big)\\ &= \ddot{\boldsymbol{r}}_c + a\big(\dot{\boldsymbol{\omega}}\times\boldsymbol{b}_3 + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{b}_3)\big)\\ &= \ddot{\boldsymbol{r}}_c + a\big(\dot{\boldsymbol{\omega}}\times\boldsymbol{b}_3 + (\boldsymbol{\omega}\cdot\boldsymbol{b}_3)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\omega})\boldsymbol{b}_3\big).\end{aligned}$$

Noting that $\boldsymbol{\omega}\cdot\boldsymbol{b}_3 = \omega_3$ and

$$\dot{\boldsymbol{\omega}}\times\boldsymbol{b}_3 = (\dot{\omega}_1\boldsymbol{b}_1 + \dot{\omega}_2\boldsymbol{b}_2 + \dot{\omega}_3\boldsymbol{b}_3)\times\boldsymbol{b}_3 = -\dot{\omega}_1\boldsymbol{b}_2 + \dot{\omega}_2\boldsymbol{b}_1,$$

the acceleration simplifies to

$$\begin{aligned}\ddot{\boldsymbol{r}} &= \ddot{\boldsymbol{r}}_c + a\big(-\dot{\omega}_1\boldsymbol{b}_2 + \dot{\omega}_2\boldsymbol{b}_1 + \omega_3\boldsymbol{\omega} - \|\boldsymbol{\omega}\|^2\boldsymbol{b}_3\big)\\ &= \ddot{\boldsymbol{r}}_c + a\big(-\dot{\omega}_1\boldsymbol{b}_2 + \dot{\omega}_2\boldsymbol{b}_1 + \omega_3(\omega_1\boldsymbol{b}_1 + \omega_2\boldsymbol{b}_2 + \omega_3\boldsymbol{b}_3) - (\omega_1^2 + \omega_2^2 + \omega_3^2)\boldsymbol{b}_3\big),\end{aligned}$$

which we group as

$$\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}_c + a\big((\dot{\omega}_2 + \omega_3\omega_1)\boldsymbol{b}_1 - (\dot{\omega}_1 - \omega_2\omega_3)\boldsymbol{b}_2 - (\omega_1^2 + \omega_2^2)\boldsymbol{b}_3\big),$$

and then substitute from (12.21b), for $\ddot{\boldsymbol{r}}_c$ and arrive at

$$\ddot{\boldsymbol{r}} = R\big[(\boldsymbol{j}\cdot\dot{\boldsymbol{\omega}})\boldsymbol{i} - (\boldsymbol{i}\cdot\dot{\boldsymbol{\omega}})\boldsymbol{j}\big] + a\big((\dot{\omega}_2 + \omega_3\omega_1)\boldsymbol{b}_1 - (\dot{\omega}_1 - \omega_2\omega_3)\boldsymbol{b}_2 - (\omega_1^2 + \omega_2^2)\boldsymbol{b}_3\big).$$

Finally, we compute

$$\mathfrak{G}_2 = \frac{1}{2} m \|\ddot{\boldsymbol{r}}\|^2 - (-m g\, \boldsymbol{k}) \cdot \ddot{\boldsymbol{r}},$$

and then substitute for $\boldsymbol{b}_1$, $\boldsymbol{b}_2$, and $\boldsymbol{b}_3$ from (12.16).

The fully expanded expression of $\mathfrak{G}_2$ is horrendously large. For all practical purposes it is impossible to compute it with bare hands. Doing it with a symbolic computational software, such as MAPLE or MATHEMATICA, however, is not a problem.

The composite ball's Gibbs function os $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$. The equations of motion are

$$\frac{\partial \mathfrak{G}}{\partial \dot{\omega}_1} = 0, \quad \frac{\partial \mathfrak{G}}{\partial \dot{\omega}_2} = 0, \quad \frac{\partial \mathfrak{G}}{\partial \dot{\omega}_3} = 0,$$

These along with the two equations (12.20) and the four equations (12.19) form a system of nine first order differential equations in the nine unknowns x, y, ω_1, ω_2, ω_3, q_0, q_1, q_2, q_3.

The initial conditions for the first five have immediate physical meanings and their specifications are intuitive and easy. The x and y place the ball's center at a desired location, and ω_1, ω_2, ω_3 impart it an initial velocity. (Remember that are the components of the angular velocity in the *body triad*. The ball's initial orientation is specified through q_0, q_1, q_2, q_3 as follows.

Begin with the ball oriented so that the body triad $\{\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3\}$ is lined up with the stationary $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ triad. Then apply a rotation $R_{\boldsymbol{n},\varphi}$ about a unit vector $\boldsymbol{n}$ and rotation angle φ to orient the ball as desired. Any orientation in space may be achieved through the appropriate choices of $\boldsymbol{n}$ and φ. The quaternion associated with that rotation is given in (12.9). Therefore

$$q_0(0) = \cos\frac{\varphi}{2}, \quad q_1(0) = n_1 \sin\frac{\varphi}{2}, \quad q_2(0) = n_2 \sin\frac{\varphi}{2}, \quad q_3(0) = n_3 \sin\frac{\varphi}{2}. \tag{12.23}$$

Don't forget that $\boldsymbol{n} = n_1\, \boldsymbol{i} + n_2\, \boldsymbol{j} + n_3\, \boldsymbol{k}$ must be a unit vector.

Remark 12.5. To make an animation of the rolling ball, it is suffices to design the ball graphics in a reference position, say where the body axes is aligned with the stationary axes. Then, after having solved the differential equations and computed $x(t)$, $y(t)$, $q_0(t)$, $q_1(t)$, $q_2(t)$, $q_3(t)$, rotate the reference ball about the vector $q_1(t)\,\boldsymbol{i} + q_2(t)\,\boldsymbol{j} + q_3(t)\,\boldsymbol{k}$ by the angle $\varphi(t)$ obtained from $q_0(t) = \cos\varphi(t)/2$, that is, $\varphi(t) = 2\cos^{-1} q_0(t)$. Then translate the rotated ball's center to the location $x(t)\,\boldsymbol{i} + y(t)\,\boldsymbol{j} + R\,\boldsymbol{k}$. MAPLE's `plottools` package provides the commands `rotate()` and `translate()` for rotating and translating graphics objects. Figure 12.3 shows the ball in its reference position (on the left), and a frame from an animation sequence (on the right).

Exercises

12.1. Prove the identity (12.6).

12.2. In Figure 12.1, $\boldsymbol{n}$ is a unit vector to the plane P, and $\boldsymbol{u}$ is an arbitrary vector. Why is $\boldsymbol{u} - 2(\boldsymbol{n} \cdot \boldsymbol{u})\boldsymbol{n}$ the mirror reflection of $\boldsymbol{u}$ through the plane P?

12.3. Let $\mathbb{q} = \boldsymbol{n}_2 \circ \boldsymbol{n}_1$, where $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$ are any two vectors. Show that $\mathbb{q}^* = \boldsymbol{n}_1 \circ \boldsymbol{n}_2$.

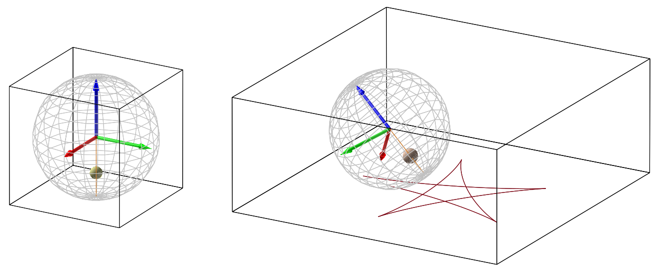


Figure 12.3: On the left is a sample graphics of the unbalanced ball in the reference position. On the right is a frame from an animation sequence which includes the trace of the ball's contact point with the floor. This particular animation was produced with the parameters $R = 1$, $a = -3/5$. $M = 100$, $m = 25$, $g = 1$. The initial orientation was set through $\boldsymbol{n} = \boldsymbol{i}$, $\varphi = \pi/2$; see (12.23). The remaining initial conditions were $x(0)$, $y(0)$, $\omega_1(0)$, $\omega_3(0)$ all zeros, and $\omega_2(0) = 0.1$.

12.4. Verify that the quaternion multiplication is associative, that is

$$\mathbb{p} \circ (\mathbb{q} \circ \mathbb{r}) = (\mathbb{p} \circ \mathbb{q}) \circ \mathbb{r}.$$

12.5. Show that $(\mathbb{p} \circ \mathbb{q})^* = \mathbb{q}^* \circ \mathbb{p}^*$ for all quaternion $\mathbb{p}$ and $\mathbb{q}$.

12.6. Show that $|\mathbb{p} \circ \mathbb{q}| = |\mathbb{p}|\,|\mathbb{q}|$ for all quaternions $\mathbb{p}$ and $\mathbb{q}$.

12.7. Show that $\boldsymbol{n} \circ \boldsymbol{n} = -1$ for all unit vectors $\boldsymbol{n}$. Note the parallel with the imaginary numbers: $i^2 = -1$.

12.8. Supply the details that lead to (12.15).

12.9. Produce an animation of the unbalanced ball of Section 12.5.

Bibliography

[1] Simon L. Altmann, *Rotations, Quaternions, and Double Groups*, Clarendon Press, Oxford, 1986.

[2] Stuart S. Antman, *The simple pendulum is not so simple*, SIAM Rev. **40** (1998), no. 4, 927–930.

[3] Roy Bowen M. and C.-C. Wang, *Introduction to vectors and tensors*, Vol. 1, Plenum Press, New York, 1976.

[4] Edward A. Desloge, *Classical Mechanics*, Vol. 1, John Wiley, 1982.

[5] Edward A. Desloge, *Classical Mechanics*, Vol. 2, John Wiley, 1982.

[6] Edward A. Desloge, *The Gibbs-Appell equations of motion*, American Journal of Physics **56** (1988), no. 9, 841–846.

[7] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, 1963.

[8] F. Gantmacher, *Lectures on Analytical Mechanics*, Foundations of Engineering Mechanics, Mir Publishers, Moscow, 1975. Translated from the Russian by George Yankovsky.

[9] Andrew J. Hanson, *Visualizing Quaternions*, Morgan Kaufmann Publishers, San Franciso, 2006.

[10] Heinrich Hertz, *The Principles of Mechanics Presented in a New Form (English Translation)*, translated by D. E. Jones and J. T. Walley, Macmillan and Co., New York, 1899.

[11] A. I. Lur'e, *Analiticheskaya mekhanika*, Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1961 (Russian).

[12] A. I. Lurie, *Analytical mechanics*, Foundations of Engineering Mechanics, Springer-Verlag, Berlin, 2002. Translated from the Russian by A. Belyaev.

[13] James Stewart, *Multivariable Calculus*, Brooks/Cole, 2008.