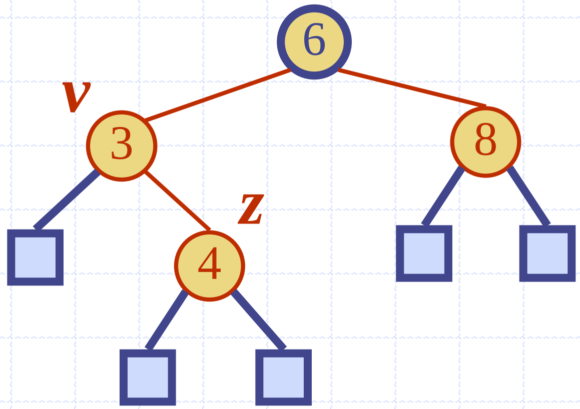
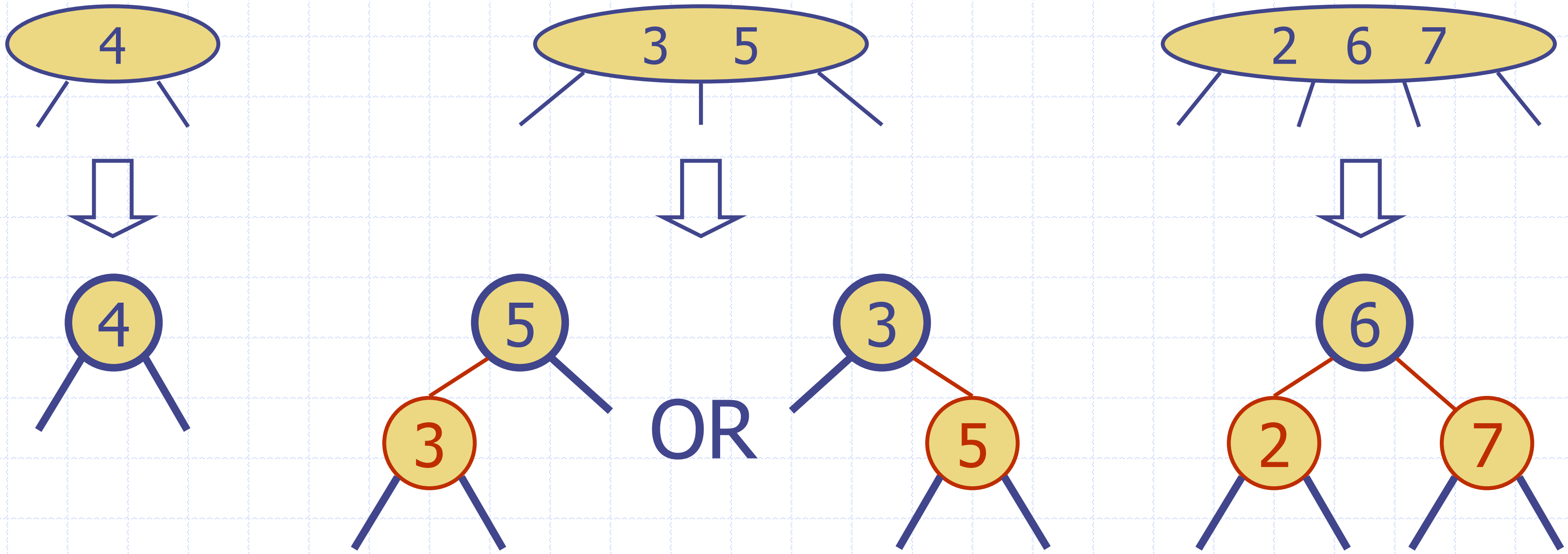
Red-Black Trees



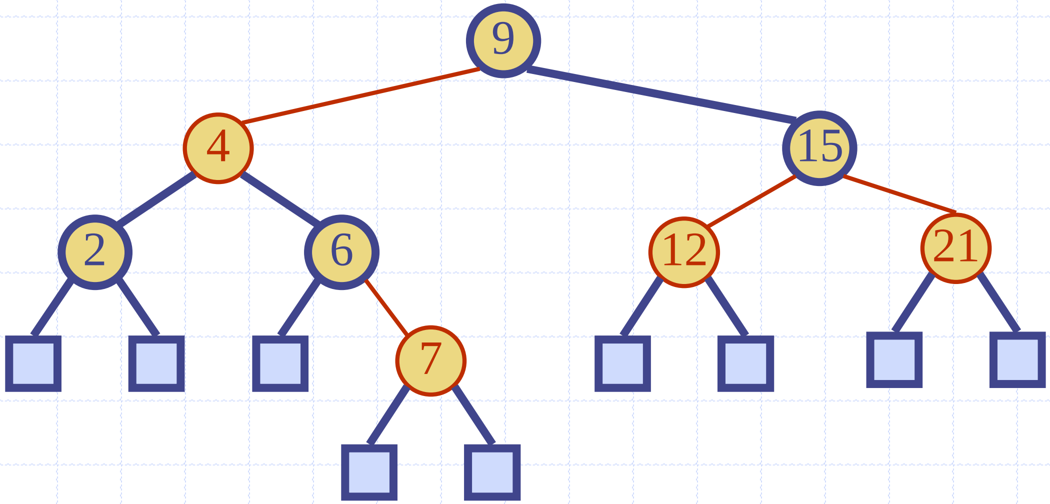
From (2,4) to Red-Black Trees

- A red-black tree is a representation of a (2,4) tree by means of a binary tree whose nodes are colored **red** or **black**
- In comparison with its associated (2,4) tree, a red-black tree has
  - same logarithmic time performance
  - simpler implementation with a single node type



Red-Black Trees

- A red-black tree can also be defined as a binary search tree that satisfies the following properties:
  - Root Property: the root is black
  - External Property: every leaf is black
  - Internal Property: the children of a red node are black
  - Depth Property: all the leaves have the same black depth



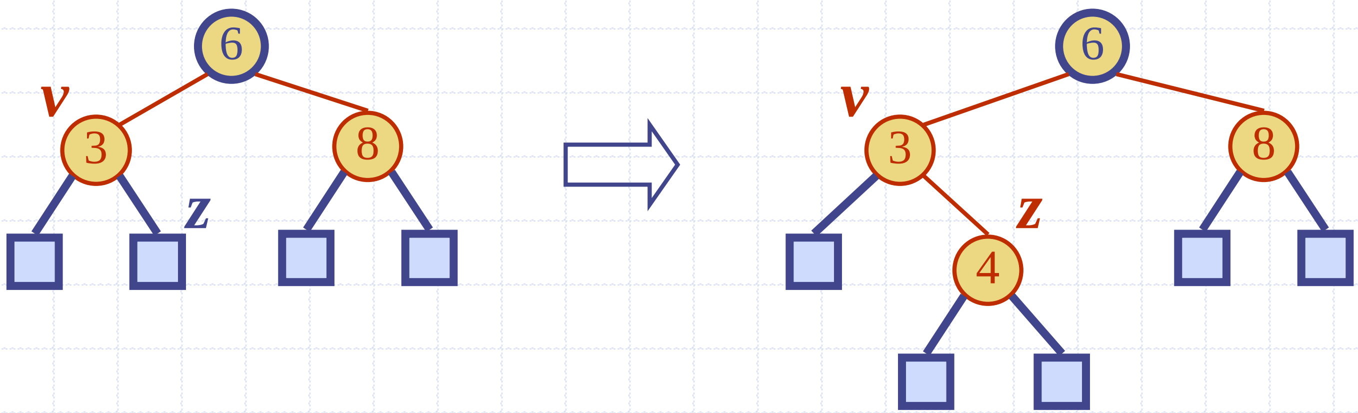
Height of a Red-Black Tree

- Theorem: A red-black tree storing n entries has height $O(\log n)$

  Proof:
  - The height of a red-black tree is at most twice the height of its associated (2,4) tree, which is $O(\log n)$
- The search algorithm for a binary search tree is the same as that for a binary search tree
- By the above theorem, searching in a red-black tree takes $O(\log n)$ time

Insertion

- To perform operation put(k, o), we execute the insertion algorithm for binary search trees and color red the newly inserted node z unless it is the root
  - We preserve the root, external, and depth properties
  - If the parent v of z is black, we also preserve the internal property and we are done
  - Else (v is red) we have a double red (i.e., a violation of the internal property), which requires a reorganization of the tree
- Example where the insertion of 4 causes a double red:

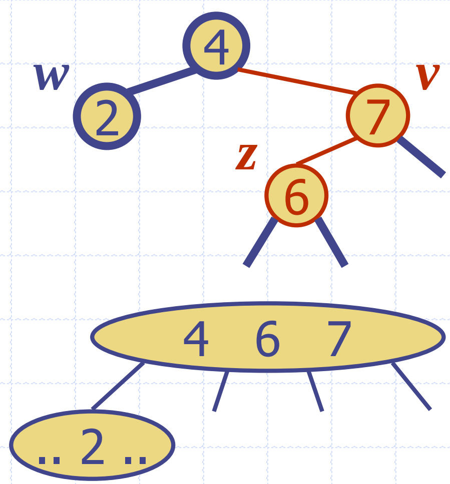


Remedying a Double Red

- Consider a double red caused by adding new child z, parent v, and let w be the sibling of v (i.e. z's "aunt")

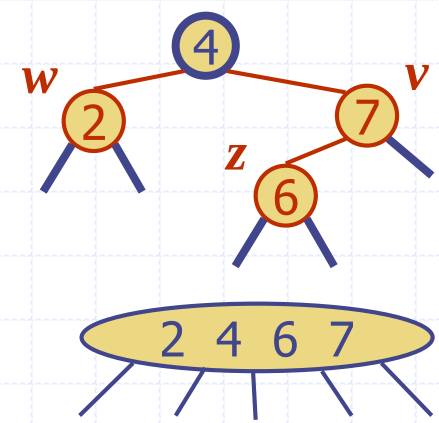
Case 1: w is black

- The double red is an incorrect replacement of a 4-node
- Restructuring: we change the 4-node replacement



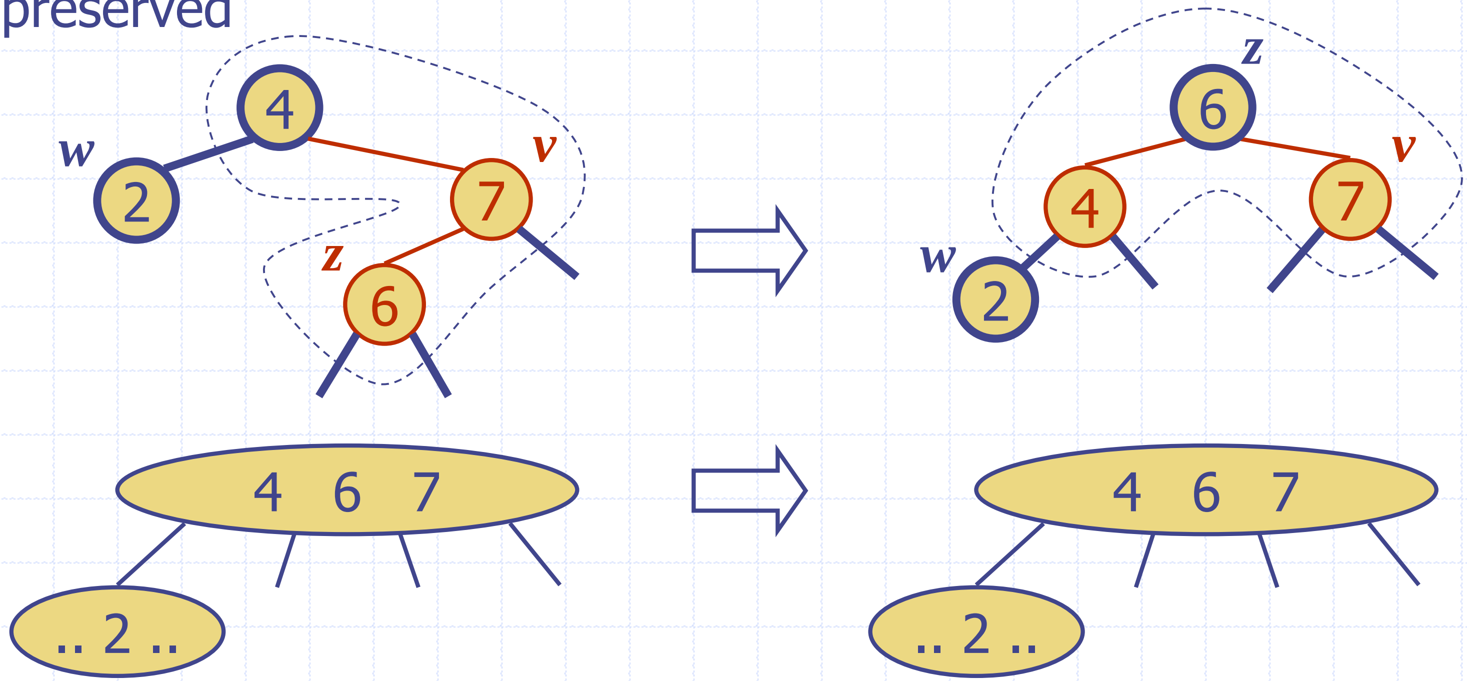
Case 2: w is red

- The double red corresponds to an overflow
- Recoloring: we perform the equivalent of a split



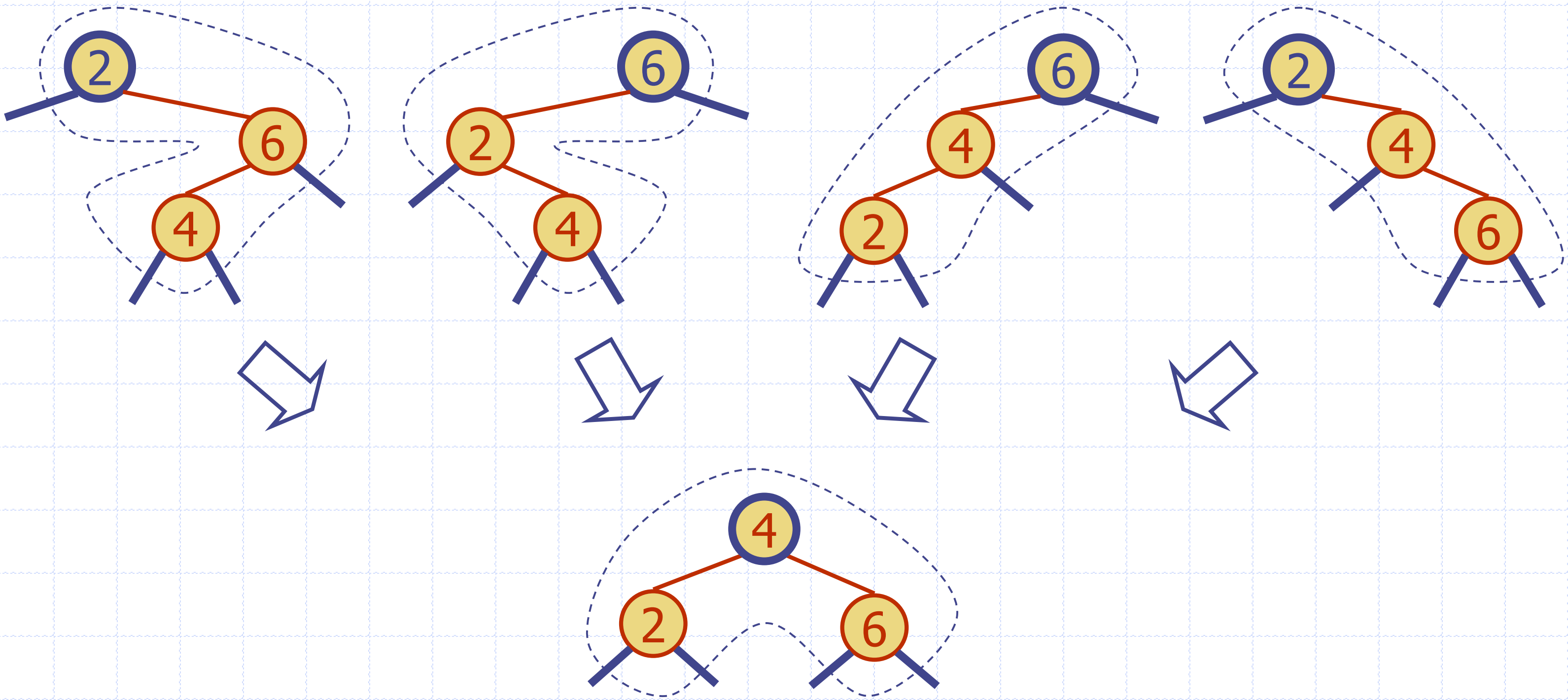
Restructuring

- A restructuring remedies a child-parent double red when the parent red node has a black sibling
- It is equivalent to restoring the correct replacement of a 4-node
- The internal property is restored and the other properties are preserved



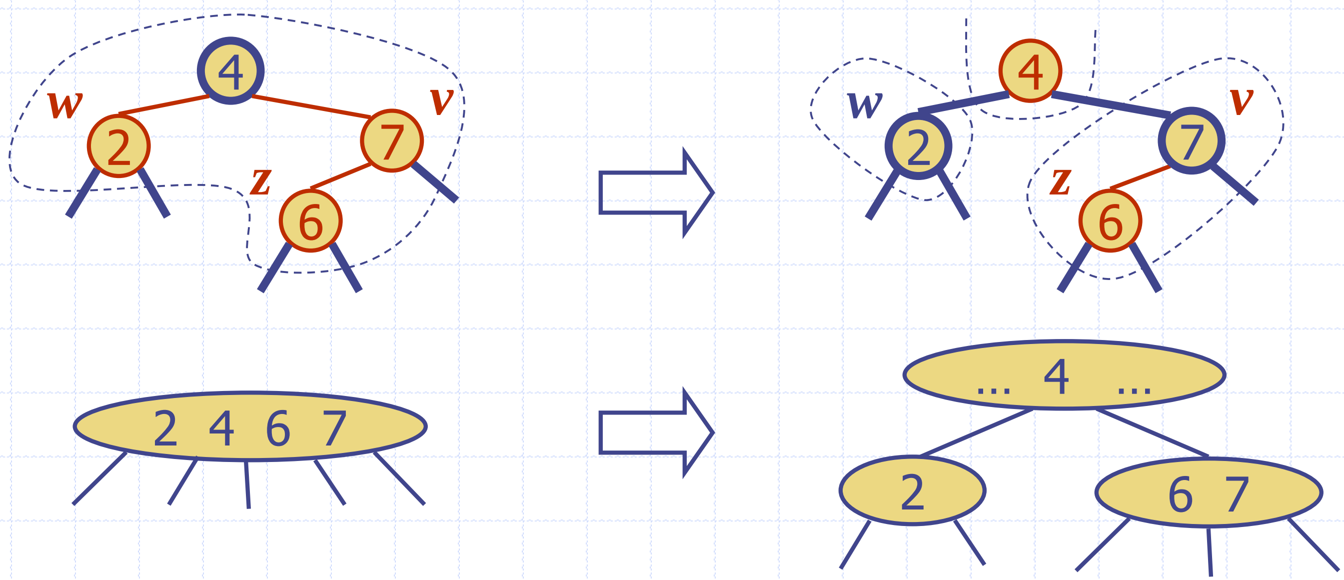
Restructuring (cont.)

- There are four restructuring configurations depending on whether the double red nodes are left or right children



Recoloring

- A recoloring remedies a child-parent double red when the parent red node has a red sibling
- The parent v and its sibling w become black and the grandparent u becomes red, unless it is the root
- It is equivalent to performing a split on a 5-node
- The double red violation may propagate to the grandparent u



Analysis of Insertion

Algorithm *put*(k, o)

1. We search for key k to locate the insertion node z

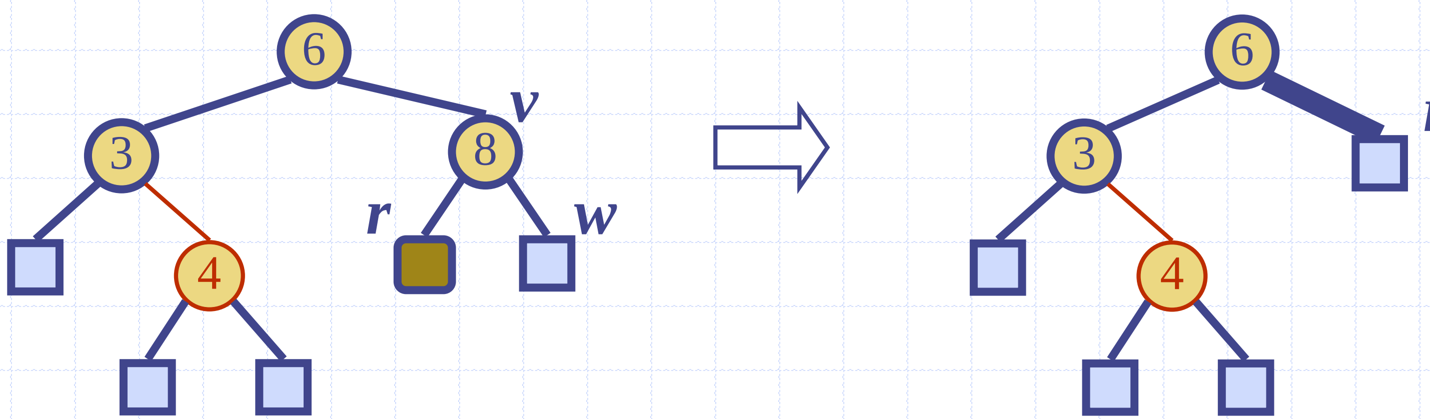
2. We add the new entry (k, o) at node z and color z red

3. **while** *doubleRed*(z)
    if *isBlack*(*sibling*(*parent*(z)))
        $z \leftarrow$ *restructure*(z)
        return
    else { *sibling*(*parent*(z) is red }
        $z \leftarrow$ *recolor*(z)

- Recall that a red-black tree has $O(\log n)$ height
- Step 1 takes $O(\log n)$ time because we visit $O(\log n)$ nodes
- Step 2 takes $O(1)$ time
- Step 3 takes $O(\log n)$ time because we perform
    - $O(\log n)$ recolorings, each taking $O(1)$ time, and
    - at most one restructuring taking $O(1)$ time
- Thus, an insertion in a red-black tree takes $O(\log n)$ time

Deletion

- To perform operation erase(k), we first execute the deletion algorithm for binary search trees
- Let v be the internal node removed, w the external node removed (there must be at least one), and r the sibling of w
  - If either v or r was red, we color r black and we are done
  - Else (v and r were both black) we color r ***double black***, which is a violation of the depth, property requiring a reorganization of the tree
- Example where the deletion of 8 causes a double black:



Remedying a Double Black

- The algorithm for remedying a double black node r with sibling y considers three cases

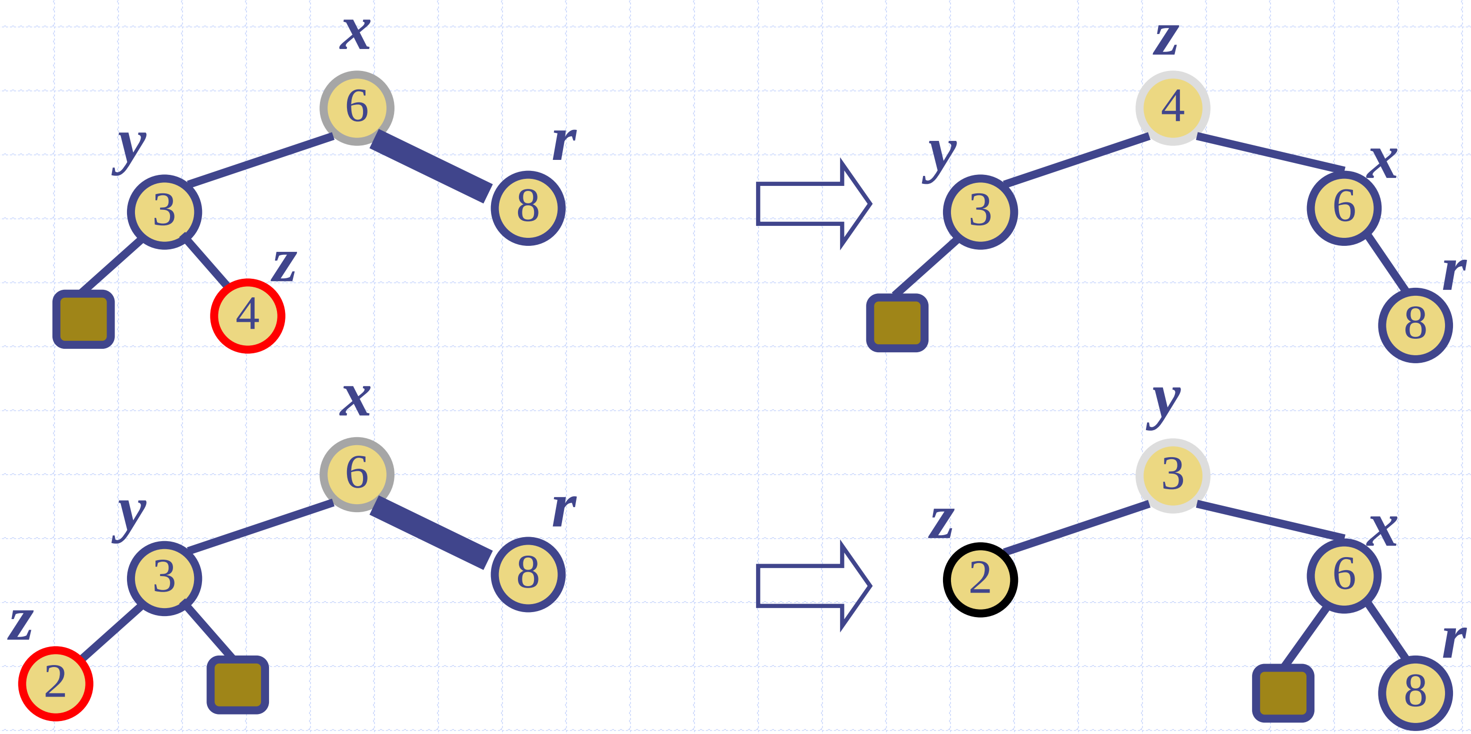
  Case 1: y is black and has a red child
  - We perform a restructuring, equivalent to a transfer , recolor, and we are done

  Case 2: y is black and its children are both black
  - We perform a recoloring, equivalent to a fusion, which may propagate up the double black violation

  Case 3: y is red
  - We perform an adjustment, equivalent to choosing a different representation of a 3-node, after which either Case 1 or Case 2 applies
- Deletion in a red-black tree takes $O(\log n)$ time

Deletion: Case 1

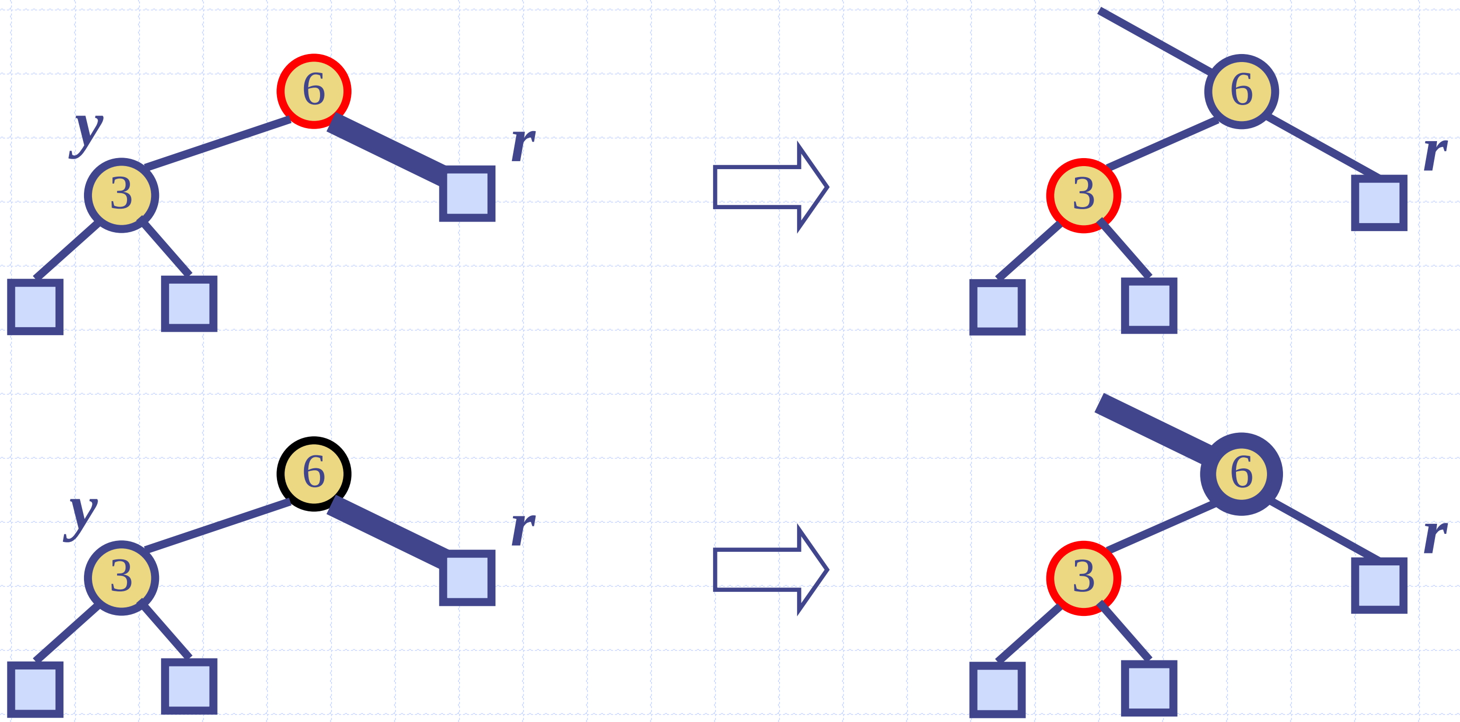
- Case 1: sibling y of r is black and has a red child
  - We perform a restructuring, equivalent to a transfer , and we are done



Deletion: Case 2

Case 2: sibling y is black and its children are both black

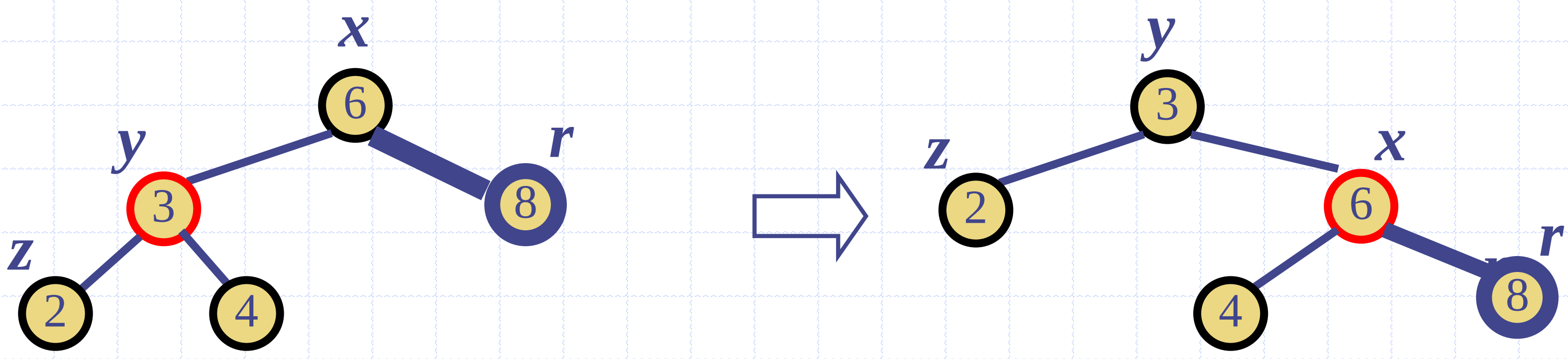
- We perform a recoloring, equivalent to a fusion, which may propagate up the double black violation



Deletion: Case 3

Case 3: sibling y of r is red

- We perform an adjustment, equivalent to a restructuring , which converts the structure to a form of Case 1 or Case 2
- Take child z of y on same side as y is of x: do trinode restructuring, then recolor x red, y black. Sibling of r is now black: have Case 1 or 2



Red-Black Tree Reorganization

Insertion	remedy double red	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	change of 4-node representation	double red removed
recoloring	split	double red removed or propagated up

Deletion	remedy double black	
Red-black tree action	**(2,4) tree action**	**result**
restructuring	transfer	double black removed
recoloring	fusion	double black removed or propagated up
adjustment	change of 3-node representation	restructuring or recoloring follows