UMBC

AN HONORS UNIVERSITY IN MARYLAND

IS 709/809: Computational Methods for IS Research

Algorithm Analysis

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What is an Algorithm?

- An algorithm is a clearly specified set of instructions to be followed to solve a problem
  - Solves a problem but requires a year is hardly of any use
  - Requires several gigabytes of main memory is not useful on most machines
- Problem
  - Specifies the desired input-output relationship
- Correct algorithm
  - Produces the correct output for every possible input in finite time
  - Solves the problem

Purpose

- Why bother analyzing algorithm or code; isn't getting it to work enough?
  - Estimate time and memory in the average case and worst case
  - Identify bottlenecks, i.e., where to reduce time and space
  - Speed up critical algorithms or make them more efficient

Algorithm Analysis

- Predict resource utilization of an algorithm
  - Running time
  - Memory usage
- Dependent on architecture
  - Serial
  - Parallel
  - Quantum

What to Analyze

- Our main focus is on running time
  - Memory/time tradeoff
  - Memory is cheap
- Our assumption: simple serial computing model
  - Single processor, infinite memory

What to Analyze (cont'd)

- Let T(N) be the running time
  - N (sometimes n) is typically the size of the input
    - Linear or binary search?
    - Sorting?
    - Multiplying two integers?
    - Multiplying two matrices?
    - Traversing a graph?
- T(N) measures number of primitive operations performed
  - E.g., addition, multiplication, comparison, assignment

An Example

```
                                          # of operations
int sum(int n) {                          ?

  int partialSum;                         ?

1. partialSum = 0;                         ?

2. for (int i = 1; i <= n; i++)            ?
3.   partialSum += i * i * i;             ?

4. return partialSum;                      ?

}
                                          T(n) = ?
```

Running Time Calculations

- The declarations count for no time
- Simple operations (e.g. +, *, <=, =) count for one unit each
- Return statement counts for one unit

Revisit the Example

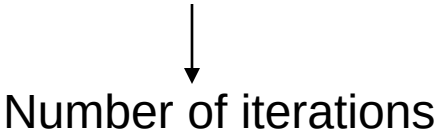
```
                                                    Cost
int sum (int n)                                     0
{
     int partialSum;                                0

1.    partialSum = 0;                               1
2.   for (int i = 1; i <= n; i++)                   1+(N+1)+N
3.         partialSum += i * i * i;                 N*(1+1+2)
4.    return partialSum;                            1
}
```

T(N) = 6N+4

- T(N) = 6N+4

Running Time Calculations (cont'd)

- General rules
  - Rule 1 – Loops
    - The running time of a loop is at most the running time of the statements inside the loop (including tests) **times** the number of iterations of the loop
  - Rule 2 – Nested loops
    - Analyze these inside out
    - The total running time of a statement inside a group of nested loops is the running time of the statement **multiplied** by the product of the sizes of all the loops



Running Time Calculations (cont'd)

- Examples
  - Rule 1 – Loops

```
for (int i = 0; i < N; i++)
  sum += i * i;
```

```
# of operations
? (1 + N + N)
?  3*N
T(N) = ?
```

  - Rule 2 – Nested loops

```
for (int i = 0; i < n; i++)
  for (int j = 0; j < n; j++)
    sum++;
```

```
# of operations
? (1 + N + N)
? N*(1 + N + N)
? N * N
T(N) = ?
```

Running Time Calculations (cont'd)

- General rules
  - Rule 3 – Consecutive statements
    - These just add
    - Only the maximum is the one that counts
  - Rule 4 – Conditional statements (e.g. if/else)
    - The running time of a conditional statement is never more than the running time of the test plus the largest of the running times of the various blocks of conditionally executed statements
  - Rule 5 – Function calls
    - These must be analyzed first

Running Time Calculations (cont'd)

- Examples
  - Rule 3 – Consecutive statements

```
                                          # of operations
for (int i = 0; i < n; i++)               ?
  a[i] = 0;                               ?
for (int i = 0; i < n; i++)               ?
  for (int j = 0; j < n; j++)             ?
    a[i] += a[j] + i * j;                 ?
                                          T(n) = ?
```

Running Time Calculations (cont'd)

- Examples
  - Rule 4 – Conditional statements

```
                                          # of operations
if (a > b && c < d) {                     ?
  for (int j = 0; j < n; j++)             ?
    a[i] += j;                            ?
}
else {
  for (int j = 0; j < n; j++)             ?
    for (int k = 1; k <= n; k++)          ?
      a[i] += j * k;                      ?
}
                                          T(n) = ?
```

Average and Worst-Case Running Times

- Estimating the resource use of an algorithm is generally a theoretical framework and therefore a formal framework is required
- Define some mathematical definitions
- Average-case running time $T_{avg}(N)$
- Worst-case running time $T_{worst}(N)$
- $T_{avg}(N) \leq T_{worst}(N)$
- Average-case performance often reflects typical behavior of an algorithm
- Worst-case performance represents a guarantee for performance on any possible input

Average and Worst-Case Running Times (cont'd)

- Typically, we analyze worst-case performance
  - Worst-case provides a guaranteed upper bound for all input
  - Average-case is usually much more difficult to compute

Asymptotic Analysis of Algorithms

- We are mostly interested in the performance or behavior of algorithms for very large input (i.e., as $N \rightarrow \infty$)
  - For example, let T(N) = 10,000 + 10N be the running time of an algorithm that processes N transactions
  - As N grows large ($N \rightarrow \infty$), the term 10N will dominate
  - Therefore, the smaller looking term 10N is more important if N is large
- Asymptotic efficiency of the algorithms
  - How the running time of an algorithm increases with the size of the input ***in the limit***, as the size of the input increases without bound

Asymptotic Analysis of Algorithms (cont'd)

- Asymptotic behavior of T(N) as N gets big
- Exact expression for T(N) is meaningless and hard to compare
- Usually expressed as fastest growing term in T(N), dropping constant coefficients
  - For example, $T(N) = 3N^2 + 5N + 1$

    Fastest growing term
  - Therefore, the term N^2 describes the behavior of T(N) as N gets big

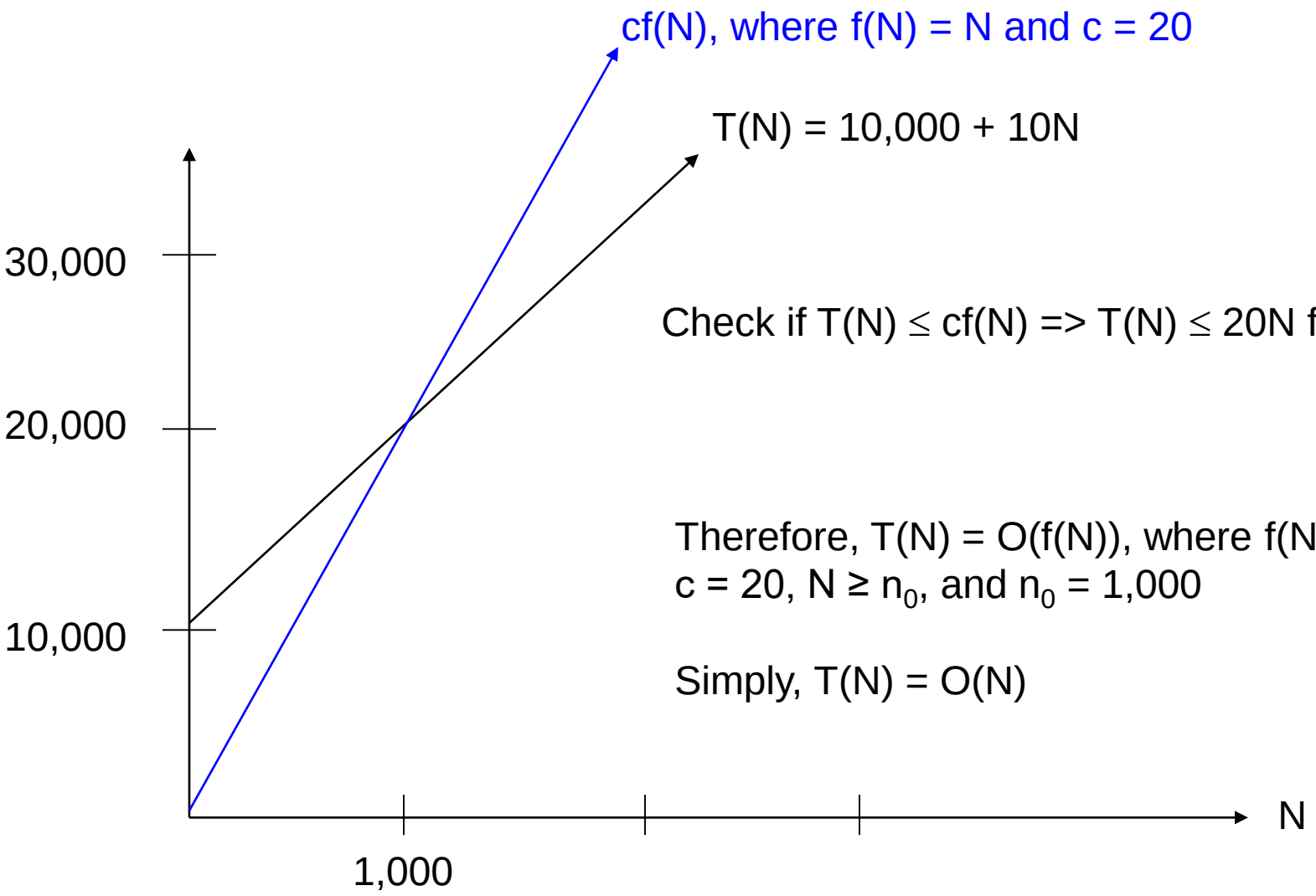
Mathematical Background

- Let T(N) be the running time of an algorithm
- Let f(N) be another function (preferably simple) that we will use as a bound for T(N)
- Asymptotic notations
  - "Big-Oh" notation O()
  - "Big-Omega" notation $\Omega()$
  - "Big-Theta" notation $\Theta()$
  - "Little-oh" notation o()

Mathematical Background (cont'd)

- "Big-Oh" notation
  - Definition: T(N) = O(f(N)) if there are positive constants c and n_0 such that $T(N) \leq cf(N)$ when $N \geq n_0$
  - Asymptotic **upper** bound on a function T(N)
  - "The **growth rate** of T(N) is $\leq$ that of f(N)"
    - Compare the ***relative rates of growth***
  - For example: T(N) = 10,000 + 10N
  - Is T(N) bounded by Big-Oh notation by some simple function f(N)? Try f(N) = N and c = 20
  - See graphs on the next slide

Mathematical Background (cont'd)



Check if $T(N) \leq cf(N) \Rightarrow T(N) \leq 20N$ for large N

Therefore, $T(N) = O(f(N))$, where $f(N) = N$, $c = 20$, $N \geq n_0$, and $n_0 = 1{,}000$

Simply, $T(N) = O(N)$

Mathematical Background (cont'd)

- "Big-Oh" notation
  - O(f(N)) is the SET of ALL functions T(N) that satisfy:
    - There exist positive constants c and n_0 such that, for all $N \geq n_0$, $T(N) \leq cf(N)$
  - O(f(N)) is an uncountably infinite set of functions

Mathematical Background (cont'd)

- "Big-Oh" notation
  - Examples
    - $1{,}000{,}000N \in O(N)$
      - Proof: Choose $c = 1{,}000{,}000$ and $n_0 = 1$

Thus, big-oh notation doesn't care about (most) constant factors

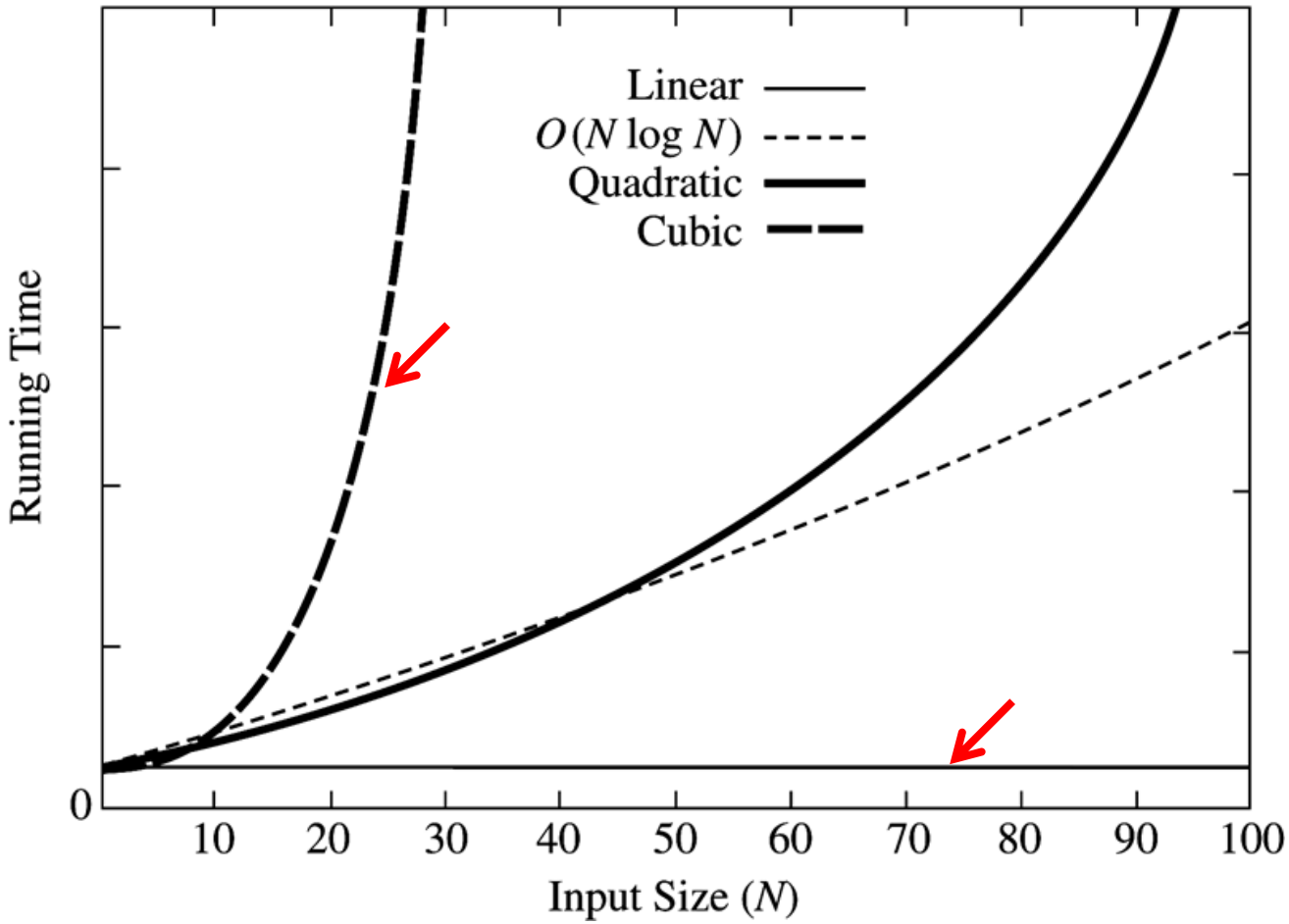
It is unnecessary to write O(2N). We can just simply write O(N)

    - $N \in O(N^3)$
      - Proof: Set $c = 1$, $n_0 = 1$
      - See graphs on the next slide

Big-Oh is an upper bound

Mathematical Background (cont'd)

- Graph of N vs. N^3



Mathematical Background (cont'd)

- "Big-Oh" notation
  - Example
    - $N^3 + N^2 + N \in O(N^3)$
      - Proof: Set $c = 3$, and $n_0 = 1$

Big-Oh notation is usually used to indicate dominating (fastest-growing) term

Mathematical Background (cont'd)

- "Big-Oh" notation
  - Another example: $1{,}000N \in O(N^2)$
    - Proof: Set $n_0 = 1{,}000$ and $c = 1$
    - We could also use $n_0 = 10$ and $c = 100$

    There are many possible pairs c and n_0

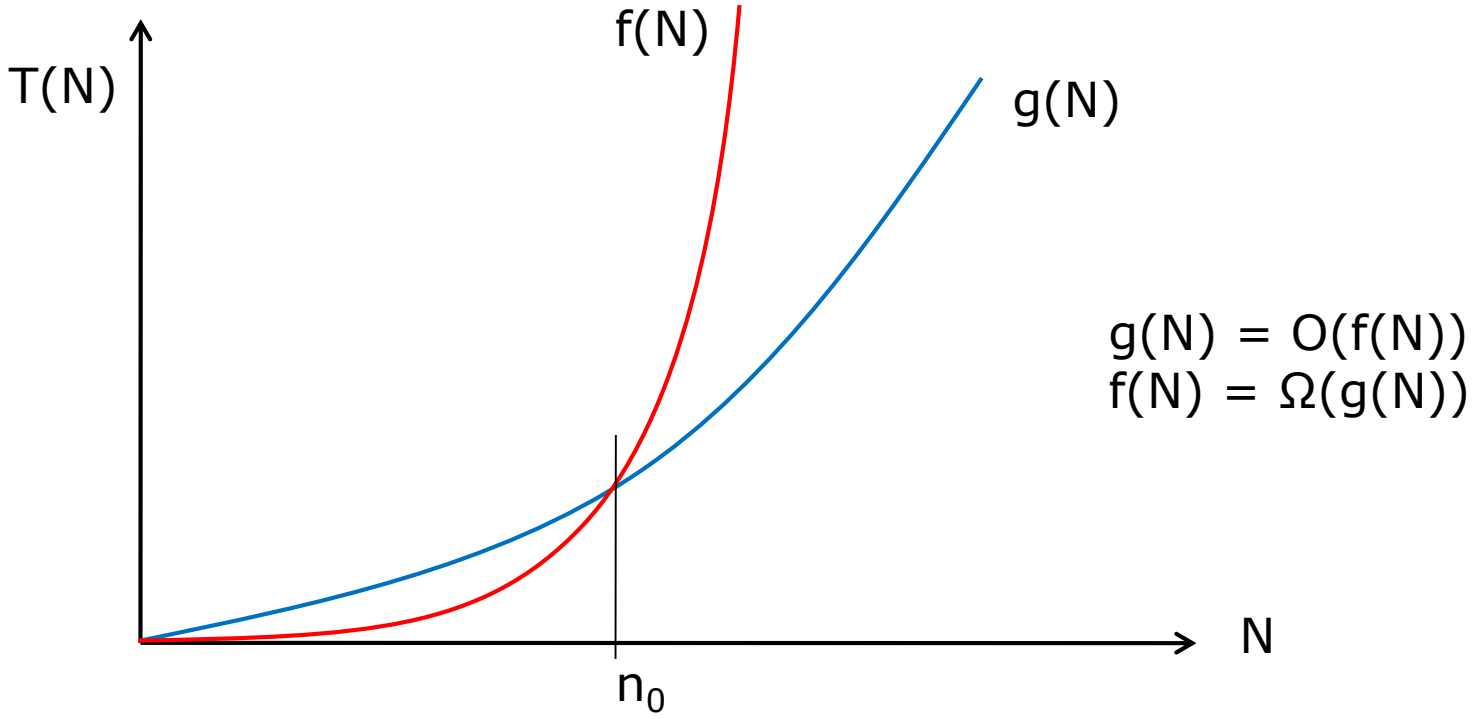
  - Another example: If $T(N) = 2N^2$
    - $T(N) = O(N^4)$
    - $T(N) = O(N^3)$
    - $T(N) = O(N^2)$

    All are technically correct, but the last one is the best answer

Mathematical Background (cont'd)

- "Big-Omega" notation
  - Definition: $T(N) = \Omega(g(N))$ if there are positive constants c and n_0 such that $T(N) \geq cg(N)$ when $N \geq n_0$
  - Asymptotic **lower** bound
  - "The growth rate of T(N) is ≥ that of g(N)"
  - Examples
    - $N^3 = \Omega(N^2)$ (Proof: c = ?, n_0 = ?)
    - $N^3 = \Omega(N)$ (Proof: c = 1, n_0 = 1)

Mathematical Background (cont'd)



- g(N) is asymptotically upper bounded by f(N)
- f(N) is asymptotically lower bounded by g(N)

Mathematical Background (cont'd)

- "Big-Theta" notation
  - Definition: $T(N) = \Theta(h(N))$ if and only if $T(N) = O(h(N))$ and $T(N) = \Omega(h(N))$
  - Asymptotic **tight** bound
  - "The growth rate of T(N) equals the growth rate of h(N)"
  - Examples
    - $2N^2 = \Theta(N^2)$
    - Suppose $T(N) = 2N^2$ then $T(N) = O(N^4)$; $T(N) = O(N^3)$; $T(N) = O(N^2)$

    all are technically correct, but last one is the best answer. Now writing $T(N) = \Theta(N^2)$ says not only that $T(N) = O(N^2)$, but also the result is as ***good*** (***tight***) as possible

Mathematical Background (cont'd)

- "Little-oh" notation
  - Definition: T(N) = o(g(N)) if for all constants *c* there exists an n_0 such that T(N) < cg(N) when N > n_0
    - That is, T(N) = o(g(N)) if T(N) = O(g(N)) and T(N) $\neq \Theta$(g(N))
    - The growth rate of T(N) less than (<) the growth rate of g(N)
    - Denote an upper bound that is **not asymptotically tight**

- The definition of O-notation and o-notation are similar
  - The main difference is that in T(N)=O(g(N)), the bound $0 \leq T(N) \leq cg(N)$ holds for ***some*** constant c > 0, but in T(N)=o(g(N)), the bound $0 \leq T(N) < cg(N)$ holds for all constants c > 0
  - For example , $N = o(N^2)$, but $2N^2 \neq o(N^2)$

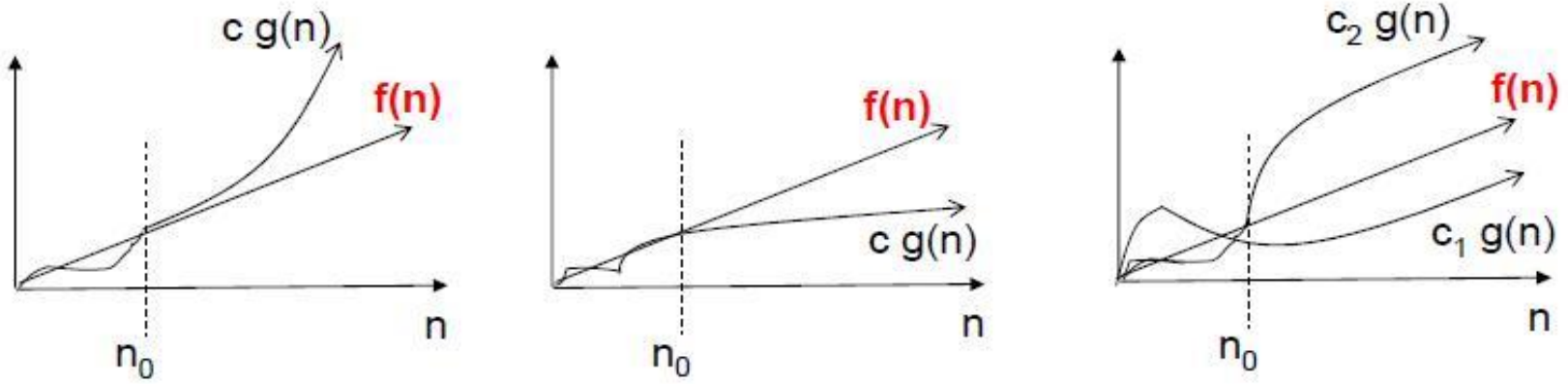
Mathematical Background (cont'd)

- Examples
  - $N^2 = O(N^2) = O(N^3) = O(2^N)$
  - $N^2 = \Omega(1) = \Omega(N) = \Omega(N^2)$
  - $N^2 = \Theta(N^2)$
  - $N^2 = o(N^3)$
  - $2N^2 + 1 = \Theta(?)$
  - $N^2 + N = \Theta(?)$

Mathematical Background (cont'd)

- O() – upper bound
- Ω() – lower bound
- Θ() – tight bound
- o() – strict upper bound

Mathematical Background (cont'd)



f(n) = O(g(n)) f(n) = Ω(g(n)) f(n) = Θ(g(n))

- O-notation gives an upper bound for a function to within a constant factor
- Ω-notation gives an lower bound for a function to within a constant factor
- Θ-notation bounds a function to within a constant factor
  - The value of f(n) always lies between c_1 g(n) and c_2 g(n) inclusive

Mathematical Background (cont'd)

- Rules of thumb when using asymptotic notations
  - When asked to analyze an algorithm's complexity
    - 1st preference: Use $\Theta()$ → Tight bound
    - 2nd preference: Use O() or o() → Upper bound
    - 3rd preference: Use $\Omega()$ → Lower bound

Mathematical Background (cont'd)

- Rules of thumb when using asymptotic notations
  - Always express an algorithm's complexity in terms of its worst-case, unless specified otherwise
    - Note: Worst-case can be expressed in any of the asymptotic notations: O(), $\Omega()$, $\Theta()$, or o()

Mathematical Background (cont'd)

- Rules of thumb when using asymptotic notations
  - Way's to answer a problem's complexity
    - Q1) This problem is at least as hard as … ?
      - Use lower bound here
    - Q2) This problem cannot be harder than … ?
      - Use upper bound here
    - Q3) This problem is as hard as … ?
      - Use tight bound here

Mathematical Background (cont'd)

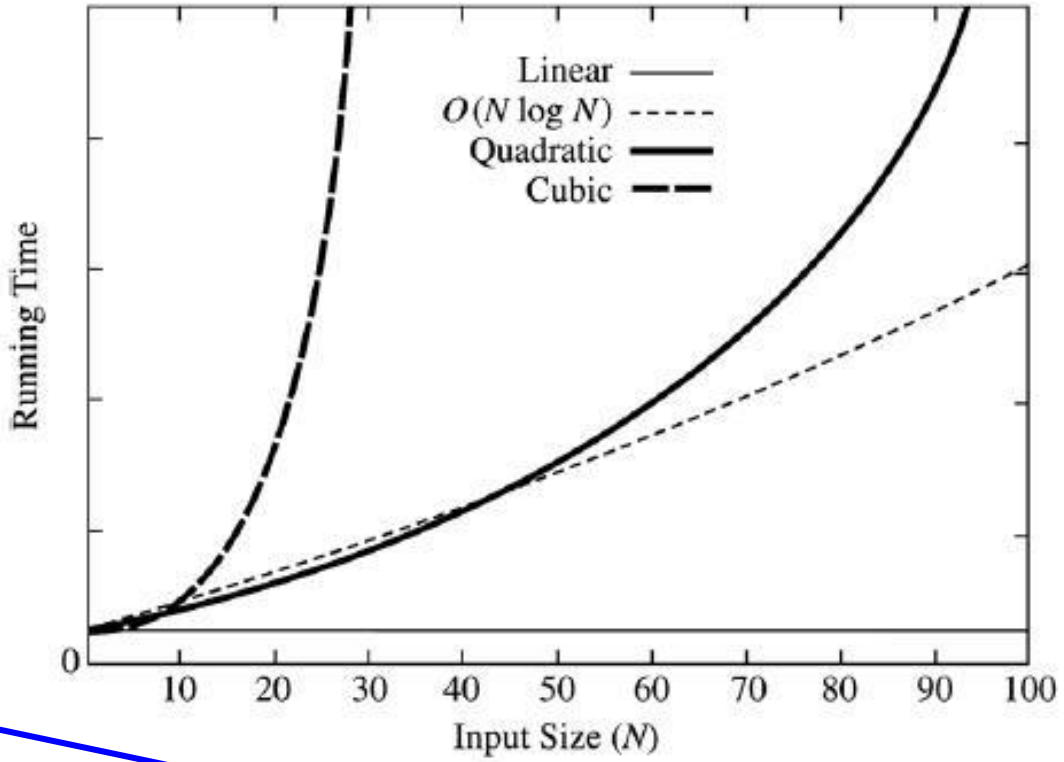
- Some rules
  - Rule 1: If $T_1(N) = O(f(N))$ and $T_2(N) = O(g(N))$, then
    - $T_1(N) + T_2(N) = O(f(N) + g(N))$ less formally it is max $(O(f(N)), O(g(N)))$
    - $T_1(N) * T_2(N) = O(f(N) * g(N))$
  - Rule 2: If $T(N)$ is a polynomial of degree k, then $T(N) = \Theta(N^k)$
  - Rule 3: $\log^k N = O(N)$ for any constant k
    - Logarithm grows very slowly as $\log N \leq N$ for $N \geq 1$
  - Rule 4: $\log_a N = \Theta(\log_b N)$ for any constants a and b

Mathematical Background (cont'd)

- Rate of Growth

Considered "efficient"

Function	Name
c	Constant
$\log N$	Logarithmic
$\log^2 N$	Log-squared
N	Linear
$N \log N$	
N^2	Quadratic
N^3	Cubic
2^N	Exponential



Considered "useless"

Mathematical Background (cont'd)

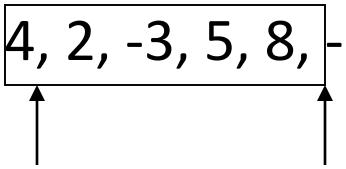
- Some examples
  - Prove that: $n \log n = O(n^2)$.
    - We know that $\log n \leq n$ for $n \geq 1$ (here, $n_0 = 1$).
    - Multiplying both sides by n: $n \log n \leq n^2$
  - Prove that: $6n^3 \neq O(n^2)$.
    - Proof by contradiction
    - If $6n^3 = O(n^2)$, then $6n^3 \leq cn^2$

Maximum subsequence sum problem

- Maximum subsequence sum problem
  - Given (possibly negative) integers A_1, A_2, …, A_N, find the maximum value (≥ 0) of:

$$\sum_{k=i}^{j} A_k$$

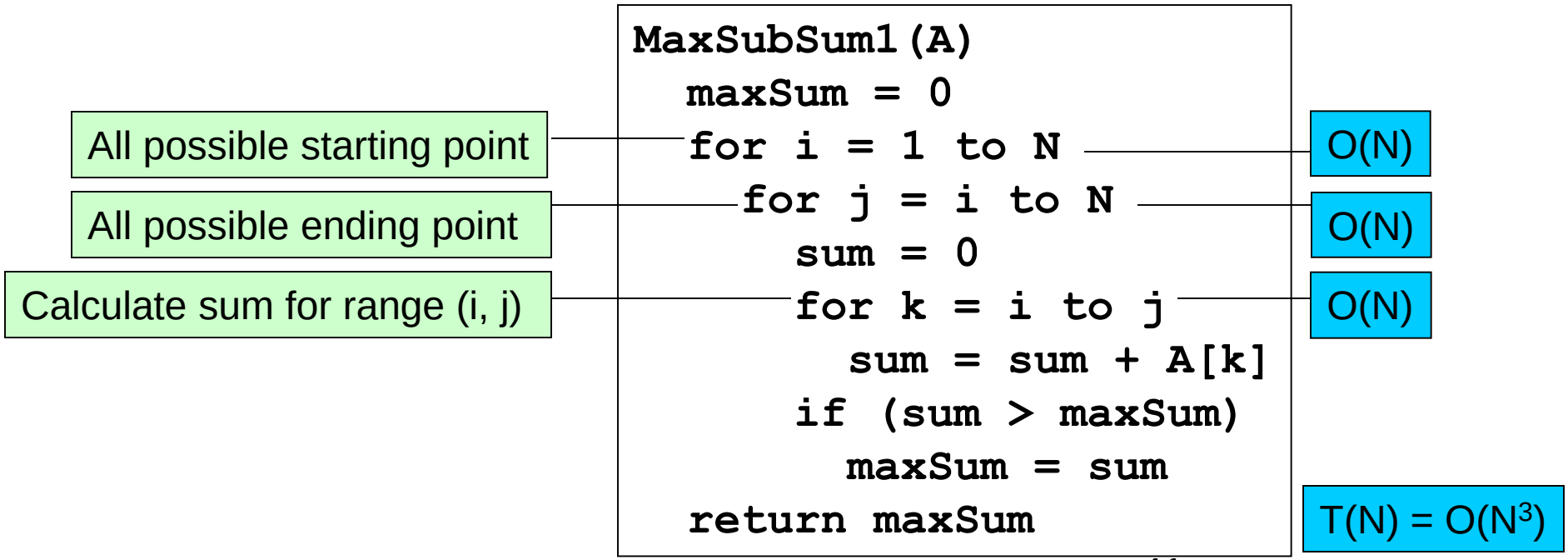
  - We don't need the actual sequence (i, j), just the sum
  - If the final sum is negative, the maximum sum is 0
  - E.g. <1, -4, 4, 2, -3, 5, 8, -2>



i j The maximum sum is 16

Solution 1

- MaxSubSum: Solution 1
  - Idea: Compute the sum for all possible subsequence ranges (i, j) and pick the maximum sum



Algorithm 1

```
1   /**
2    * Cubic maximum contiguous subsequence sum algorithm.
3    */
4   int maxSubSum1( const vector<int> & a )
5   {
6       int maxSum = 0;
7
8       for( int i = 0; i < a.size( ); i++ )
9           for( int j = i; j < a.size( ); j++ )
10          {
11              int thisSum = 0;
12
13              for( int k = i; k <= j; k++ )
14                  thisSum += a[ k ];
15
16              if( thisSum > maxSum )
17                  maxSum = thisSum;
18          }
19
20      return maxSum;
21  }
```

Solution 1 (cont'd)

- Analysis of Solution 1
  - Three nested *for* loops, each iterating at most N times
  - Operations inside *for* loops take constant time
  - But, *for* loops don't always iterate N times
  - More precisely;

$$T(N) = \sum_{i=0}^{N-1} \sum_{j=i}^{N-1} \sum_{k=i}^{j} 1$$

Solution 1 (cont'd)

- Analysis of Solution 1
  - Detailed calculation of T(N)

$$T(N) = \sum_{i=0}^{N-1} \sum_{j=i}^{N-1} \sum_{k=i}^{j} 1$$

  - Will be derived in the class;

T(N)= $(N^3 + 3N^2 + 2N)/6 = O(N^3)$

Solution 2

- MaxSubSum: Solution 2
  - Observation: $\sum_{k=i}^{j} A_k = A_j + \sum_{k=i}^{j-1} A_k$
  - So, we can re-use the sum from previous range

```
MaxSubSum2(A)
  maxSum = 0
  for i = 1 to N
    sum = 0
    for j = i to N
      sum = sum + A[j]
      if (sum > maxSum)
        maxSum = sum
  return maxSum
```

O(N)

O(N)

$T(N) = O(N^2)$

Algorithm 2

```
1   /**
2    * Quadratic maximum contiguous subsequence sum algorithm.
3    */
4   int maxSubSum2( const vector<int> & a )
5   {
6       int maxSum = 0;
7
8       for( int i = 0; i < a.size( ); i++ )
9       {
10          int thisSum = 0;
11          for( int j = i; j < a.size( ); j++ )
12          {
13              thisSum += a[ j ];
14
15              if( thisSum > maxSum )
16                  maxSum = thisSum;
17          }
18      }
19
20      return maxSum;
21  }
```

Solution 2 (cont'd)

- Analysis of Solution 2
  - Two nested for loops, each iterating at most N times
  - Operations inside for loops take constant time
  - More precisely;

$$T(N) = \sum_{i=0}^{N-1} \sum_{j=i}^{N-1} 1$$

Solution 2 (cont'd)

- Analysis of Solution 2
  - Detailed calculation of T(N)

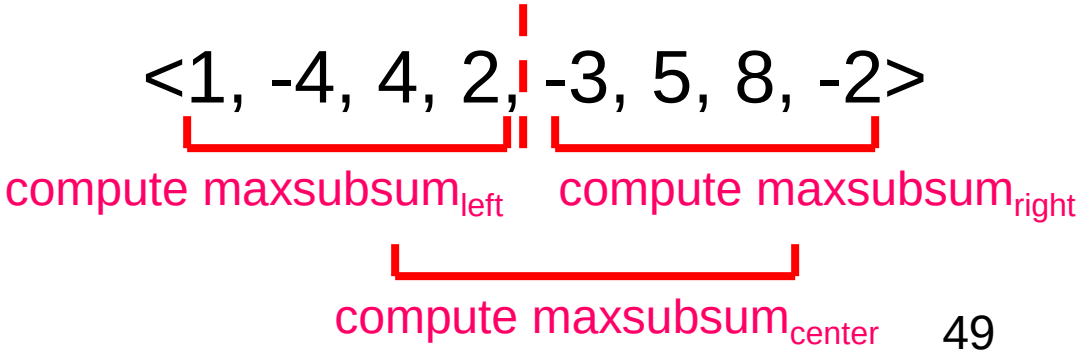
$$T(N) = \sum_{i=0}^{N-1} \sum_{j=i}^{N-1} 1$$

  - Will be derived in the class;

T(N)= N(N+1)/2 = $O(N^2)$

Solution 3

- MaxSubSum: Solution 3
  - Idea: Recursive, "divide and conquer"
    - Divide sequence in half: $A_{1..center}$ and $A_{(center + 1)..N}$
    - Recursively compute MaxSubSum of left half
    - Recursively compute MaxSubSum of right half
    - Compute MaxSubSum of sequence constrained to use A_{center} and $A_{(center + 1)}$
    - For example



Solution 3 (cont'd)

- MaxSubSum: Solution 3
  - Idea: Recursive, "divide and conquer"
  - Divide: split the problem into two roughly equal subproblems, which are then solved recursively
  - Conquer: patching together the two solutions of the subproblems, and possibly doing a small amount of additional work to arrive at a solution for the whole problem
  - The maximum subsequence sum can be in one of three places
    - Entirely in the left half of the input
    - Entirely in the right half
    - Or it crosses the middle and is in both halves
    - First two cases can be solved recursively
    - Last case: find the largest sum in the first half that includes the last element in the first half and the largest sum in the second half that includes the first element in the second half. These two sums then can be added together.

Example

- For example, consider the sequence

4, -3, 5, -2 || -1, 2, 6,-2, where || marks the half-way point

  - The maximum subsequence sum of the left half is 6: 4 + -3 + 5.
  - The maximum subsequence sum of the right half is 8: 2 + 6.
  - The maximum subsequence sum of sequences having -2 as the right edge is 4: 4 + -3 + 5 + -2; and the maximum subsequence sum of sequences having -1 as the left edge is 7: -1 + 2 + 6.
  - Comparing 6, 8 and 11 (4 + 7), the maximum subsequence sum is 11 where the subsequence spans both halves: 4 + -3 + 5 + -2 + -1 + 2 + 6.

Solution 3 (cont'd)

- MaxSubSum: Solution 3

```
MaxSubSum3(A, i, j)
  maxSum = 0
  if (i == j)
    if (A[i] > 0)
      maxSum = A[i]
  else
    k = floor((i + j) / 2)
    maxSumLeft = MaxSubSum3(A, i, k)
    maxSumRight = MaxSubSum4(A, k + 1, j)
    compute maxSumThruCenter
    maxSum = Maximum(maxSumLeft, maxSumRight, maxSumThruCenter)
  return maxSum
```

Solution 3 (cont'd)

```
1   /**
2    * Recursive maximum contiguous subsequence sum algorithm.
3    * Finds maximum sum in subarray spanning a[left..right].
4    * Does not attempt to maintain actual best sequence.
5    */
6   int maxSumRec( const vector<int> & a, int left, int right )
7   {
8       if( left == right )  // Base case
9           if( a[ left ] > 0 )
10              return a[ left ];
11          else
12              return 0;
13
14      int center = ( left + right ) / 2;
15      int maxLeftSum  = maxSumRec( a, left, center );
16      int maxRightSum = maxSumRec( a, center + 1, right );
```

// How to find the maximum subsequence sum that passes through the center

```
18      int maxLeftBorderSum = 0, leftBorderSum = 0;
19      for( int i = center; i >= left; i-- )
20      {
21          leftBorderSum += a[ i ];
22          if( leftBorderSum > maxLeftBorderSum )
23              maxLeftBorderSum = leftBorderSum;
24      }
25
26      int maxRightBorderSum = 0, rightBorderSum = 0;
27      for( int j = center + 1; j <= right; j++ )
28      {
29          rightBorderSum += a[ j ];
30          if( rightBorderSum > maxRightBorderSum )
31              maxRightBorderSum = rightBorderSum;
32      }
33
34      return max3( maxLeftSum, maxRightSum,
35                   maxLeftBorderSum + maxRightBorderSum );
36  }
```

Keep right end fixed at center and vary left end

Keep left end fixed at center + 1 and vary right end

Add the two to determine maximum subsequence sum through center

Solution 3 (cont'd)

```
38  /**
39   * Driver for divide-and-conquer maximum contiguous
40   * subsequence sum algorithm.
41   */
42  int maxSubSum3( const vector<int> & a )
43  {
44      return maxSumRec( a, 0, a.size( ) - 1 );
45  }
```

Solution 3 (cont'd)

- Analysis of Solution 3
  - T(1) = O(1)
  - T(N) = 2T(N / 2) + O(N)
  - T(N) = O(?)
    - Will be derived in the class

Solution 4

- MaxSubSum: Solution 4
  - Observations
    - Any negative subsequence cannot be a prefix to the maximum subsequence
    - Or, only a positive, contiguous subsequence is worth adding
    - Example: <1, -4, 4, 2, -3, 5, 8, -2>

```
1  /**
2   * Quadratic maximum contiguous subsequence sum algorithm.
3   */
4  int maxSubSum2( const vector<int> & a )
5  {
6      int maxSum = 0;
7
8      for( int i = 0; i < a.size( ); i++ )
9      {
10         int thisSum = 0;
11         for( int j = i; j < a.size( ); j++ )
12         {
13             thisSum += a[ j ];
14
15             if( thisSum > maxSum )
16                 maxSum = thisSum;
17         }
18     }
19
20     return maxSum;
21 }
```

T(N) = O(N)

```
MaxSubSum4(A)
   maxSum = 0
   sum = 0
   for j = 1 to N
      sum = sum + A[j]
      if (sum > maxSum)
         maxSum = sum
      else if (sum < 0)
         sum = 0
   return maxSum
```

Solution 4 (cont'd)

```
1   /**
2    * Linear-time maximum contiguous subsequence sum algorithm.
3    */
4   int maxSubSum4( const vector<int> & a )
5   {
6       int maxSum = 0, thisSum = 0;
7
8       for( int j = 0; j < a.size( ); j++ )
9       {
10          thisSum += a[ j ];
11
12          if( thisSum > maxSum )
13              maxSum = thisSum;
14          else if( thisSum < 0 )
15              thisSum = 0;
16      }
17
18      return maxSum;
19  }
```

Solution 4 (cont'd)

- Online Algorithm
  - constant space and runs in linear time
  - just about as good as possible

MaxSubSum Running Times

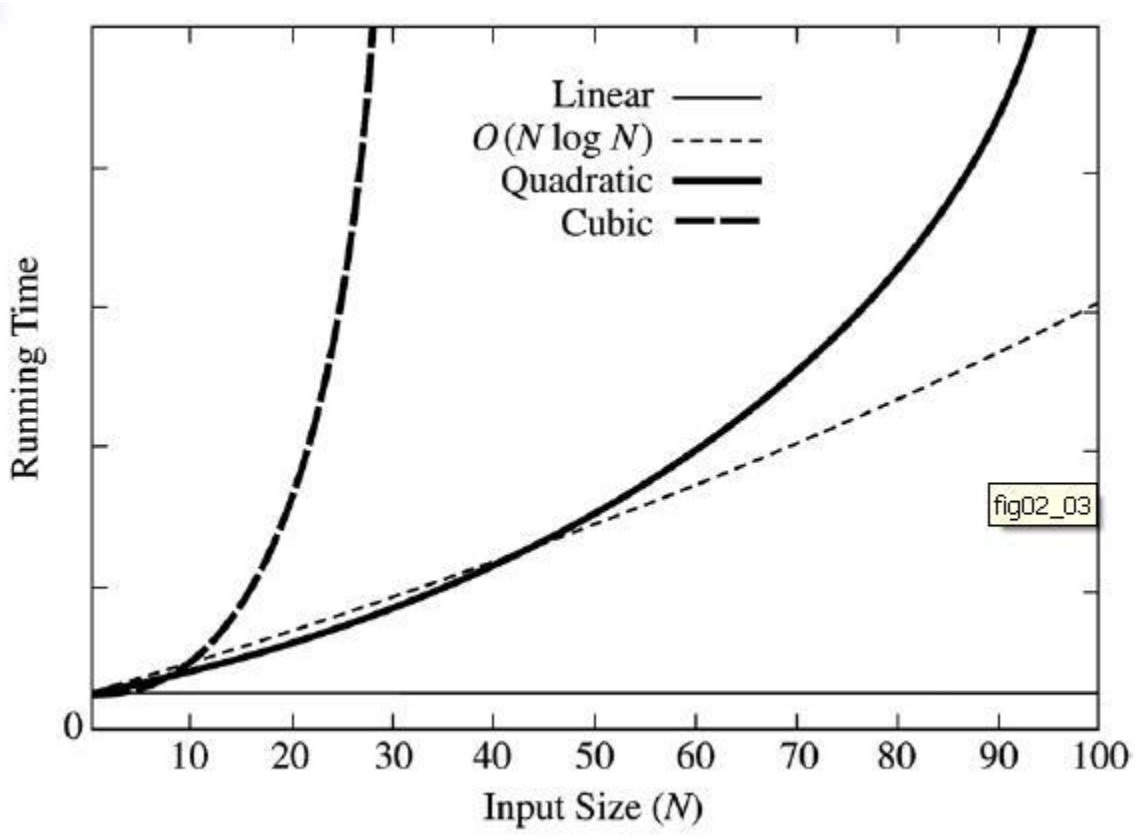
Input Size	Algorithm Time 1 $O(N^3)$	2 $O(N^2)$	3 $O(N \log N)$	4 $O(N)$
$N = 10$	0.000009	0.000004	0.000006	0.000003
$N = 100$	0.002580	0.000109	0.000045	0.000006
$N = 1{,}000$	2.281013	0.010203	0.000485	0.000031
$N = 10{,}000$	NA	1.2329	0.005712	0.000317
$N = 100{,}000$	NA	135	0.064618	0.003206

38 min

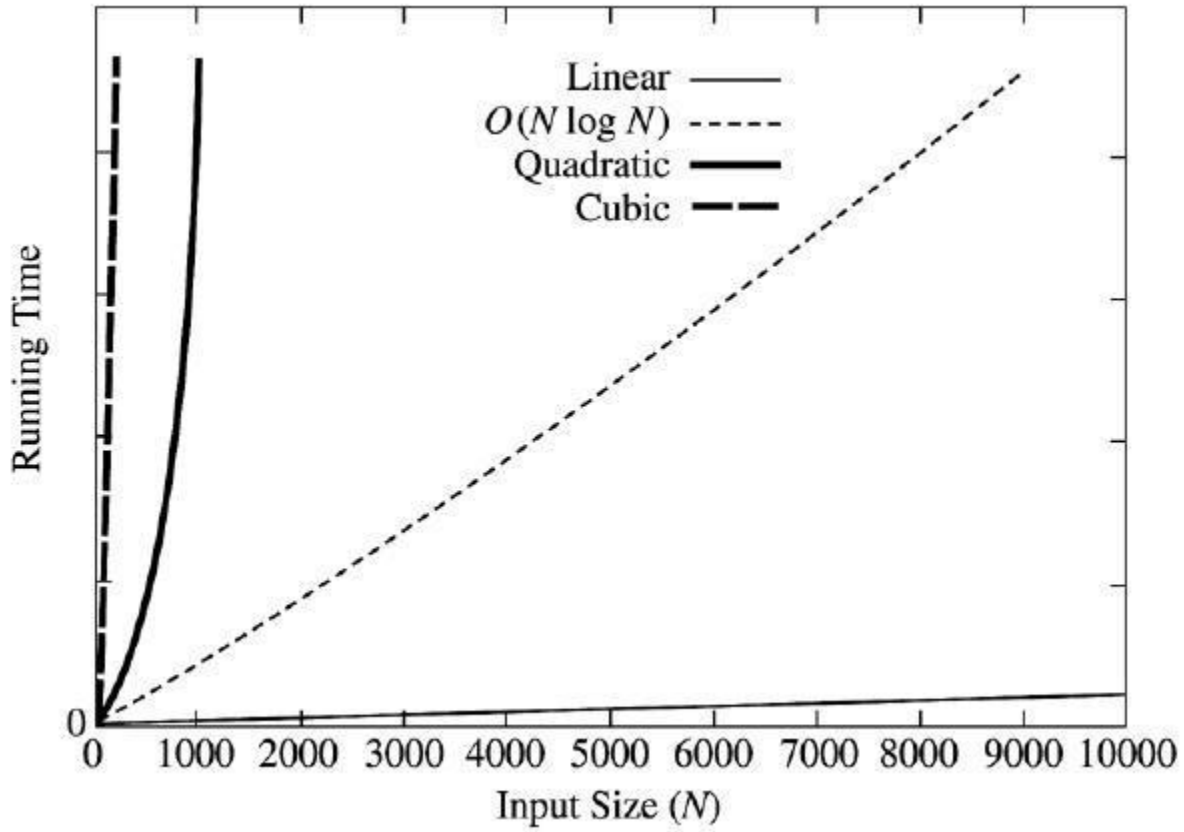
26 days

Time in seconds.
Does not include time to read array.

MaxSubSum Running Times (cont'd)



MaxSubSum Running Times (cont'd)



Logarithmic Behavior

- $T(N) = O(\log_2 N)$
- An algorithm is $O(\log_2 N)$ if it takes constant $O(1)$ time to cut the problem size by a fraction (which is usually ½)
- Usually occurs when
  - Problem can be halved in constant time
  - Solutions to sub-problems combined in constant time
- Examples
  - Binary search
  - Euclid's algorithm
  - Exponentiation

Binary Search

- Given an integer X and integers A_0, A_1, …, A_{N-1}, which are **presorted** and already in memory, find i such that A_i = X, or return i = -1 if X is not in the input
- Obvious Solution: scanning through the list from left to right and runs in linear time
  - Does not take advantage of the fact that list is sorted
  - Not likely to be best
- Better Strategy: Check if X is the middle element
  - If so, the answer is at hand
  - If X is smaller than middle element, apply the same strategy to the sorted subarray to the left of the middle element
  - Likewise, if X is larger than middle element, we look at the right half
- $T(N) = O(\log_2 N)$

```
1   /**
2    * Performs the standard binary search using two comparisons per level.
3    * Returns index where item is found or -1 if not found.
4    */
5   template <typename Comparable>
6   int binarySearch( const vector<Comparable> & a, const Comparable & x )
7   {
8       int low = 0, high = a.size( ) - 1;
9
10      while( low <= high )
11      {
12          int mid = ( low + high ) / 2;
13
14          if( a[ mid ] < x )
15              low = mid + 1;
16          else if( a[ mid ] > x )
17              high = mid - 1;
18          else
19              return mid;    // Found
20      }
21      return NOT_FOUND;      // NOT_FOUND is defined as -1
22  }
```

Euclid's Algorithm

- Compute the greatest common divisor gcd(M, N) between the integers M and N
  - That is, the largest integer that divides both
  - Example: *gcd* (50,15) = 5
  - Used in encryption

Euclid's Algorithm (cont'd)

```
1   long gcd( long m, long n )
2   {
3       while( n != 0 )
4       {
5           long rem = m % n;
6           m = n;
7           n = rem;
8       }
9       return m;
10  }
```

Example: gcd(3360,225)
- m = 3360, n = 225
- m = 225, n = 210
- m = 210, n = 15
- m = 15, n = 0

Euclid's Algorithm (cont'd)

- Estimating the running time: how long the sequence of remainders is?
  - log N is a good answer, but value of the remainder does not decrease by a constant factor
  - Indeed the remainder does not decrease by a constant factor in one iteration, however we can prove that after two iterations the remainder is at most half of its original value
  - Number of iterations is at most 2 log N = O(log N)
- $T(N) = O(\log_2 N)$

Euclid's Algorithm (cont'd)

- Analysis
  - Note: After two iterations, remainder is at most half its original value
    - Theorem 2.1: If M > N, then M mod N < M / 2
  - $T(N) = 2 \log_2 N = O(\log_2 N)$
    - $\log_2 225 = 7.8$, $T(225) = 16$ (overestimate)
  - Better worst-case: $T(N) = 1.44 \log_2 N$
    - $T(225) = 11$
  - Average-case: $T(N) = (12 \ln 2 \ln N) / \pi^2 + 1.47$
    - $T(225) = 6$

Exponentiation

- Compute $X^N = X * X * \ldots * X$ (N times), integer N
- Obvious algorithm:
  - To compute X^N uses (N-1) multiplications
- Observations
  - A recursive algorithm can do better
  - N <= 1 is the base case
  - $X^N = X^{N/2} * X^{N/2}$ (for even N)
  - $X^N = X^{(N-1)/2} * X^{(N-1)/2} * X$ (for odd N)
- Minimize number of multiplications
- $T(N) = 2 \log_2 N = O(\log_2 N)$

```
pow(x, n)
  result = 1
  for i = 1 to n
    result = result * x
  return result
```

T(N) = O(N)

Exponentiation (cont'd)

```
1   long pow( long x, int n )
2   {
3       if( n == 0 )
4           return 1;
5       if( n == 1 )
6           return x;
7       if( isEven( n ) )
8           return pow( x * x, n / 2 );
9       else
10          return pow( x * x, n / 2 ) * x;
11  }
```

$T(N) = \Theta(1), N \leq 1$
$T(N) = T(N/2) + \Theta(1), N > 1$

$T(N) = O(\log_2 N)$
$T(N) = \Theta(\log_2 N)$?

Exponentiation (cont'd)

- To compute X^{62} , the algorithm does the following calculations, which involve only 9 multiplications

  - $X^3 = (X^2) \cdot X$ then $X^7 = (X^3)^2 \cdot X$ then $X^{15} = (X^7)^2 \cdot X$ then $X^{31} = (X^{15})^2 \cdot X$ then $X^{62} = (X^{31})^2$

  - The number of multiplications required is at most 2 logN, because at most 2 multiplications (if N is odd) are required to halve the problem

Summary

- Algorithm analysis
- Bound running time as input gets big
- Rate of growth: O() and Θ()
- Compare algorithms
- Recursion and logarithmic behavior