THE SPATIAL MODEL OF SOCIAL CHOICE AND VOTING

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Abstract

This chapter presents the basic elements of the standard spatial model commonly that is used as a framework for developing theories of legislative, electoral, and other forms of social choice and voting and is increasingly used in empirical analysis as well. It introduces the concepts of single-peaked and Euclidean preferences, win sets, the Condorcet winner, the core, median lines, the yolk, and the uncovered set, and presents such foundational results as Black's Median Voter Theorem, Plott's Majority Rule Equilibrium Theorem, McKelvey's Global Cycling Theorem, and Greenberg's Core Existence Theorem.

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4.1 *THE SPATIAL MODEL OF SOCIAL CHOICE AND VOTING*

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1. Overview

This chapter presents the basic elements of the standard spatial model commonly used as a framework for developing theories of legislative, electoral, and other forms of social choice and voting and increasingly used in empirical analyses as well. It builds on Chapters 3.1 and 3.2 by placing concepts introduced there in a spatial context, and it lays out groundwork for the remaining chapters in Section 4 and the first two chapters in Section 6.

The 'spatial model' is often associated with the work of Anthony Downs (1957, Chapter 8), who drew on some remarks by Harold Hotelling (1929) to propose that two candidates or parties competing for the support of an electoral majority 'converge to the center' (as discussed in Chapters 4.3 and 4.4). However, the origins of the formal spatial model of social choice lie in Duncan Black's (1948, 1958) attempt to build a 'pure science of politics' founded on 'a point-set representing motions' to be voted on by a 'committee' (i.e., a small set of voters). In effect, Black and Downs formalized the notion of the left-right political spectrum that originated with the seating arrangements in the National Assembly at the time of the French Revolution. Formal political theorists have taken this classic notion and run (very far) with it. Driven by desire for generality, explicitness of assumptions, and theoretical coherence, they have produced an elaborate theory with possibly intimidating terminology, notation, concepts and derivations. Here we attempt to simplify or sidestep (or relegate to footnotes) most of these complexities and still convey the main ideas.

The essential idea underlying Black's 'point-set representing motions' is that a geometrical space — a one-dimensional line, a two-dimensional plane, a three-dimensional solid, etc. — can represent the 'space' of (potential) 'policy alternatives', 'party platforms', etc., available for social choice, whether on some narrowly defined issue or in a more global sense. Given voters with plausible preferences and a voting rule, we can derive logical results pertaining to relationships among preferences, institutions and outcomes.

The spatial model captures our sense that there may be many — perhaps an infinite number of — policy alternatives potentially available for choice and that these alternatives may be related in a 'spatial' sense. For example, some alternatives may be 'close together' while others are 'far apart', and compromises may be available 'between' alternatives. More specifically, some alternatives may be 'left-wing' (or extreme in some other sense), others 'right-wing' (or extreme in an opposite sense), and still others 'centrist',with essentially infinite gradations in between. The spatial model further captures our sense that voter preferences with respect to such alternatives are likely to be 'spatially' structured as well. For example, some voters are 'close together' with mostly similar preferences but others are 'far apart', e.g., 'left-wingers' and 'right-wingers', with largely opposed preferences, while others are 'centrists', again with gradations in between.[1]

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[1] In discussing the one-dimensional spatial model, left-right terminology will be used, but other (non-ideological) language may be appropriate in particular situations (e.g., low- vs. high-spending proposals). Not being limited to one dimension, the spatial model allows for multiple ideological dimensions, e.g., left-right with respect to economic issues, left-right with respect to social issues, etc.

A common theme concerning the spatial model is that in a one-dimensional setting social choice — and majority rule in particular — is well behaved but in a higher-dimensional setting its operation becomes 'chaotic'. This chapter will both explain in what sense this is true and suggest why the 'chaotic' characterization may be overdrawn.

2. Voter Preferences

As discussed in Chapters 3.1 and 5.1, social choice theory assumes that individuals have *preference orderings* over all *alternatives* available for choice; this means that individual preference is *complete*, i.e., given any two alternatives a voter prefers one to the other or is indifferent between them, and *transitive*, i.e., a voter who prefers x to y and y to z also prefers x to z. Social choice theory typically puts no additional restrictions on preferences (hence the 'Unrestricted Domain' condition discussed in Chapters 5.1 and 5.2) but in a spatial context it is natural to make particular assumptions concerning voter preferences.

2.1 Single-Peaked Preferences

Consider a one-dimensional alternative space, i.e., a single ideological dimension or a single issue concerning which alternatives differ in only one respect. Each point along the line represents an alternative potentially available for social choice — indeed, we will often use the term 'point' in place of 'alternative' — and voters have preferences over these alternatives. It is natural to assume that each voter i has a point of maximum preference called his *ideal point* x_i (and, in that sense, each voter has a spatial location), and that the voter's preferences among alternatives relate to their distance from this ideal point. In the case of a one-dimensional alternative space, a standard assumption is that voters have what Black (1948, 1958) called *single-peaked* preferences: given two alternatives x and y that lie on the same side of his ideal point, a voter prefers the closer one, but if x and y lie on opposite sides of his ideal point, the voter may have either preference (or be indifferent) between x and y.

Figure 1 illustrates why such preferences are called 'single-peaked'. The horizontal axis represents the alternative space and the vertical axis represents a voter's degree of preference (or 'utility') for alternatives. The preference graph of each voter 1 through 5 is literally single-peaked, rising without interruption until it reaches a (single) peak at his ideal point and declining without interruption thereafter, but it may be asymmetric about the peak (as in the case of voter 1).[2] Accordingly, two voters with identical ideal points may have different preferences with respect to alternatives on opposite sides of that ideal point.

The set of points $P_i(x)$ that voter i prefers to x is called his *preferred-to* set with respect to x. Inspection of Figure 1 supports the following conclusions. First, each $P_i(x)$ is a line segment that extends from x through x_i to a point x' such that i is indifferent between x and x'.[3] Second, if two

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[2] Figure 1 depicts single-peaked preferences that decline linearly with distance from the ideal point, but any uninterrupted (or 'monotonic') decline will suffice. (See Figure 3 in Chapter 6.1 for examples of symmetric but non-linear single-peaked preferences.) Likewise, the facts that the preferences of voters 2 through 5 are shown with the same slopes, and also that all peaks are shown with the same height, have no special significance.

[3] $P_i(x)$ is the 'open' interval from x_i to x', i.e., excluding the end points (since i does not prefer x to x and is by definition indifferent between x and x').

voters i and j have ideal points on opposite sides of x, their preference sets are disjoint, in the manner of $P_1(x)$ and $P_2(x)$ in Figure 1; if they have ideal points on the same side of x, one is a subset of the other, so the preferred-to sets with respect to x of all voters with ideal points on the same side of x are nested inside one another, as for voters 2 through 5 in Figure 1.

Single-peakedness can be generalized to two or more dimensions by assuming that voter preferences over the points on every straight line through the space are single-peaked. In two dimensions, this implies that a voter's *indifference curves*, i.e., sets of points among which the voter is indifferent, are closed curves concentric about the voter's ideal point that everywhere bend inward (for example, ellipses) and that $P_i(x)$ is the set of points enclosed by i's indifference curve that passes through x.

2.2 Euclidean Preferences

A more demanding assumption is that voters have *Euclidean* preferences. This means that between *any* two points, a voter prefers the one that is closer to his ideal point (and is indifferent between equidistant points), so the single-peaked graph is *symmetric* about a voter's ideal point, as shown for voters 2 through 5 (but not 1) in Figure 1. Given that preferences are Euclidean, each voter's preferences are fully specified by the location of the voter's ideal point and, if a voter prefers x to y, we can infer that his ideal point is closer to x than y.

Inspection of Figure 1 shows that we can strengthen two conclusions about preference sets when preferences are Euclidean. First, each $P_i(x)$ is a line segment extending from x through x_i to point x' at an *equal distance* beyond x_i. Second, if two voters i and j have ideal points on the same side of x, $P_j(x)$ is a subset of $P_i(x)$ if and only if x_j is closer to x than is x_i, so the nesting of preferred-to sets of voters with ideal points on the same side of x follows the order in which their ideal points are distant from x (as in Figure 1).

The definition of Euclidean preferences applies directly to any number of dimensions. In two (or more) dimensions, it implies that a voter's indifference curves are concentric circles (or spheres, etc.) about his ideal point.

2.3 Social Preference

The following analysis focuses on the *social preference relation* over alternatives, given some number n of voters with single-peaked or Euclidean preferences and some *proper k-majority rule*, where k is an integer such that $n/2 < k \leq n$ — that is, *simple majority rule* (where k is the smallest integer greater than $n/2$) or some more 'demanding' *supermajority rule* (i.e., with some larger k, as discussed in Chapter 3.2) all the way up to *unanimity rule* (with $k = n$). Alternative x is *socially preferred* to y if at least k voters out of n prefer x to y.[4] Since this is an awkward phrase, we shall usually say that 'x beats y'. Social preference relations, especially simple majority rule, govern many

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[4] This defines the *absolute* variant of k-majority rule, which also has a *relative* variant in which n is replaced by the (perhaps smaller) number of voters who are not indifferent between x and y. While indifference does occur in a spatial context (and is critical in defining indifference curves and the boundaries of many sets, e.g., $P_i(x)$), it is sufficiently rare that definitions and results are essentially the same under both variants.

social choice processes (e.g., committee decisions, legislative voting, and mass elections) analyzed using the spatial model.

The set of alternatives that beat x — the 'socially preferred-to' set — is commonly called the *win set* $W(x)$ of x. If $W(x)$ is empty, x is unbeaten; the set of unbeaten points is called the *core*. Because social preference may be *cyclical* (as discussed in Chapter 3.1 with respect to simple majority rule), the core may be empty. Keeping voter preferences constant, it is clear that win sets contract and the core expands as the voting rule becomes more demanding; conversely, win sets expand and the core contracts as the voting rule becomes less demanding. An empty core implies *instability*, in that for every alternative x there is a set of voters with the power (in the absence of agenda control) and desire to replace x with some other alternative; a large core implies *non-responsiveness* (or 'gridlock') in that a status quo point that belongs to the core under one set of preference is likely to remain in the new core as preferences change. (This theme is developed in the following chapter.)

3. The One-Dimensional Spatial Model

Let there be n voters with ideal points $x_1, x_2, \ldots, x_n$, numbered (as in Figure 1) according to their spatial locations along the single dimension with x_1 the leftmost and x_n the rightmost. (The numbering of ideal points that coincide is arbitrary.)

By definition, y beats x under k-majority rule if and only if y lies in (the intersection of) the $P_i(x)$ sets of at least k voters; given single-peaked preferences, the ideal points of these voters all lie on the y side of x and these preferred-to sets are nested inside one another. Thus, $W(x)$ is either empty (if fewer than k ideal points lie on either side of x) or $W(x)$ is equal to the preferred-to-x set of some voter, specifically the k^{th} smallest of the nested $P_i(x)$ sets (if at least k ideal points lie on the same side of x). Thus in Figure 1, $W(x) = P_3(x)$ if $k = 3$, $W(x) = P_2(x)$ if $k = 4$, and $W(x)$ is empty if $k = 5$. Since each win set is identical to some individual preferred-to set, it is a line segment extending from x in the direction of the k ideal points, which implies that, if x is beaten by y, x is also beaten by any point between x and y.

Any proper k-majority rule creates two *pivotal* voters — the *right pivot* whose ideal point is x_k and the *left pivot* whose ideal point is x_{k*} (where $k* = n - k + 1$). The points in the interval between the two pivotal ideal points are unbeaten, while any point outside this interval is beaten by every point between it and the nearest pivotal ideal point. It follows that the core is the interval from one pivotal ideal point to the other.[5] Moreover, if point x is beaten by any point to its left (right), x beats all points to its right (left); thus relative to any point in the space, social preference pulls in at most one direction.

Finally, if x beats y, $W(x)$ is a subset of $W(y)$. Otherwise there would be some point z that beats x but is beaten by y (producing cyclical social preference), but this leads to a contradiction. Points x, y, and z must all lie outside the core interval, but social preference among points that lie on the same side of the pivots depends on distance from the nearest pivot, so if x, y, and z all lie on the

[5] The core is the 'closed' interval from x_{k*} to x_k, i.e., including the end points.

same side, social preference among them is transitive. Thus two points must lie on one side and the third point on the other side, but if the third point is beaten by the more distant point on the other side, it must be beaten by the closer point as well. It then follows that the cyclical social preference phenomenon discussed in Chapter 3.1 cannot arise in the one-dimensional case with single-peaked preferences.

The core under unanimity rule, commonly called the *Pareto set*, is the interval from x_1 to x_n; the core contracts as the voting rule becomes less demanding. At the limit, if the number of voters n is odd and simple majority rule is used, so $k = k^* = (n+1)/2$, the *median voter* (with ideal point x_{med} where $med = (n+1)/2$), is both the left and right pivot, so the core is this single point. (Two or more voters may share the median ideal point.) This gives us what is probably the best known theorem pertaining to the spatial model of voting.

Black's (1948, 1958) Median Voter Theorem. Given an odd number of voters with single-peaked preferences and simple majority rule, x_{med} is the unique unbeaten point; moreover, x_{med} is the Condorcet winner.

As discussed in Chapter 3.1, a *Condorcet winner* is an alternative that beats every other alternative under simple majority rule. Thus, in the absence of some kind of agenda control, x_{med} is the expected final voting outcome. Whatever the status quo, some voter (indeed, a majority of them) has an incentive to propose x_{med}, which can then beat the status quo and any rival proposals. This property of x_{med} drives the 'convergence to the center' (i.e., to the median ideal point) associated with Downs (1957). It also means that in legislative or electoral models of majority rule voting with single-peaked preferences, the median voter can, with respect to determining the winning outcome, 'stand in' for the set of all voters.[6]

We now consider how these conclusions can be strengthened when we assume that preferences are not merely single-peaked but also Euclidean. With Euclidean preferences, every voter whose ideal point lies of the x side of the midpoint between x and y prefers x to y and every voter whose ideal point lies on the y side prefers y to x. It follows that x beats y if and only if x is closer to both pivots than y is, so both pivots prefer x to y. Given an odd number of voters and simple majority rule, x beats y if and only x is closer to x_{med} than y is, so the median voter prefers x to y, and $W(x) = P_{med}(x)$ for all points x. The other $n-1$ voters become irrelevant, and the median voter can in every respect 'stand in' for the set of all voters, in the sense that changes in location, or the addition or deletion, of other voters that do not change the location of the median voter have no effect on social preference. For example, if voter 1 also had Euclidean preferences, Figure 1 could show only the ideal point of voter 3 without any loss of information regarding majority preference.

In sum, given single-peaked preferences over a one-dimensional space, social choice operates in an orderly fashion and unbeaten alternatives always exist. If an alternative x is beaten, it is beaten

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[6] If simple majority rule is used but the number of voters n is even, so that $k = n/2 + 1$ and $k^* = n/2$, there is an interval of unbeaten points between $x^{n/2}$ and $x^{n/2+1}$ and no Condorcet winner (unless these two ideal points happen to coincide).

only by alternatives on one side of x, reflecting the preferences of a single set of voters of sufficient size given the voting rule. Given simple majority rule with an odd number of voters, social preference is an ordering like individual preference; indeed, if preferences are Euclidean, social preference is identical to the preference ordering of the median voter.

4. The Two-Dimensional Spatial Model

We now consider a two-dimensional alternative space — that is, any point on a plane represents a possible alternative and voter ideal points are distributed over the plane. We now restrict our attention to Euclidean preferences and, in contrast to the one-dimensional case, first consider the case of simple majority rule with an odd number of voters. In the following section, we briefly consider what happens as k-majority rule becomes more demanding and the number of dimensions increases.

4.1. Win Sets in Two Dimensions

In the one-dimensional case, the preferred-to sets of any two voters with respect to x are line segments that can be related in only one of two ways: they are disjoint or one is a subset of the other. This implies that x can be beaten through the preferences of only one majority (or larger set) of voters and that every win set is identical to the preferred-to set of some voter. But in two dimensions, $P_i(x)$ is the area enclosed by a circle (i's indifference curve) centered on x_i and passing through x. Thus, $P_i(x)$ and $P_j(x)$ are disjoint or one is a subset of the other if and only if x, x_i, and x_j lie on a common straight line; otherwise they intersect without one being a subset of the other. This implies that a win set is almost never identical to the preferred-to set of any individual voter; rather it is an amalgamation of the preferred-to sets of many (typically all) voters, reflecting the fact that different majorities prefer different sets of alternatives to x. This phenomenon has important consequences for social choice.

Both panels of Figure 2 show the same configuration of five voter ideal points. Panel (a) shows the Pareto set, which is the set of points enclosed by the polygon (a triangle in this case) with vertices at the 'non-interior' ideal points. It also shows the circular preferred-to sets of each voter with respect to point x as shaded areas (the boundary of $P_1(x)$ is explicitly labelled) that become darker they intersect. Given simple majority rule, the win set of x includes all points that lie in the intersection of preferred-to sets of at least three voters. The boundary of $W(x)$ is shown by a dark line and has four distinct 'petals' (two of which overlap) pointing in different directions, reflecting the preferences of four distinct three-voter majorities: voters 1, 2, and 4 with respect to the left-pointing petal; voters 1, 3, and 5 with respect to the right-pointing petal; voters 2, 3, and 4 with respect to the left downward-pointing petal; and voters 2, 3, and 5 with respect to the right downward-pointing petal.[7] Thus in this case, and in contrast to any one-dimensional case, x is beaten by points that lie on multiple sides of x.

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[7] The intersection of the latter two petals constitutes the single petal of the win set of x under supermajority rule with $k = 4$. The win set of x under unanimity rule (with $k = 5$) is empty, because x lies in the Pareto set.

4.2 Median Lines and Condorcet Winners

Any straight line L partitions the set of voter ideal points into three subsets: those that lie on one side of L, those that lie on the other side of L, and those that lie on L itself. A *median line* M partitions the set of ideal points so that fewer than half of the ideal points lie on either side of M. Given an odd number of ideal points, a median line must pass through at least one ideal point, no two median lines are parallel, and there is exactly one median line perpendicular to any line L. While almost all median lines pass through exactly one ideal point, a finite number of *limiting* median lines pass through two (or possibly more) ideal points. Typically pairs of limiting median lines pass through a given ideal point, with non-limiting median lines sandwiched between them. Figure 2(b) shows all limiting median lines (passing through pairs of ideal points) in the five-voter configuration, and the shaded cone indicates where all the (non-limiting) median lines that pass through x_1 lie.

Given any two points x and y, we can draw the line L through them and erect the line L^* that is perpendicular to L and intersects L at the midpoint between x and y. Given Euclidean preferences and in the manner of a midpoint between x and y in one dimension, all voters with ideal points on the x side of L^* prefer x to y while all voters with ideal points on the y side prefer y to x. Some median line M is perpendicular to L (and parallel to L^*). If M lies on the x side of L^*, x beats y; if M lies on the y side of L^*, y beats x; and the point at the intersection of M and L beats every other point on L (just as x_{med} beats every other point in a one-dimensional space).

Generalizing, a point that lies off *any* median line M is beaten by some point on M. It follows that a point c in a two-dimensional space is a Condorcet winner if and only if *every* median line passes through c. Provided the number of voters is odd (as we are assuming), at least one ideal point must be located at c. Provided further that several ideal points do not coincide at c, this condition requires that the configuration of ideal points exhibit *Plott symmetry*, as specified in the following famous theorem.

Plott's (1967) Majority Rule Equilibrium Theorem. Given a two-dimensional space, an odd number of voters with Euclidean preferences, and simple majority rule, a point c is a Condorcet winner if one ideal point is located at c and the other $n-1$ ideal points can be paired off in such a way that the two points in each pair lie on a straight line with, and on opposite sides of, c.[8]

Enelow and Hinich (1983) subsequently observed that, if several ideal points coincide at c, the symmetry requirement can be weakened, so the condition that all median lines have a common intersection may be referred to as *generalized Plott symmetry*. If this condition holds, majority preference is identical to the preference of the voter with ideal point at c and is therefore transitive (Davis et al. 1972); the win set of any point x is the preferred-to set of this voter, i.e., the set of points contained in a circle centered on c and passing through x, and y beats x if and only if y is closer to c than is x.

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[8] Plott (1967) stated his theorem for any number of dimensions (it is always satisfied in one dimension) and generalized single-peaked preferences, in which case the condition requires that voters can be paired off in such a way that their 'contract curves' all pass through c.

Thus a voter with ideal point at c is the two-dimensional counterpart of the one-dimensional median voter, and Plott's Theorem is in a sense the two-dimensional generalization of Black's Median Voter Theorem, but with this fundamental difference: while a median voter always exists, (generalized) Plott symmetry is an extremely stringent condition, making a Condorcet winner in two-dimensions highly unlikely. Thus we need to consider what happens in the absence of Plott symmetry.

4.3 The Yolk and Global Cycling

The discussion in Chapter 3.1 indicates that, when there no Condorcet winner, we may expect to find the final voting outcome in the *top cycle set*, i.e., the minimal set of alternatives such that every alternative in the set beats every alternative outside the set.[9] However, in the absence of generalized Plott symmetry, McKelvey (1976 and 1979) demonstrated that, for Euclidean and much more general preferences respectively, the top cycle set is not confined even to the Pareto set but rather encompasses the entire alternative space. From this result, McKelvey (1976) drew some implications for voting that have to some extent bedeviled social choice theorists ever since. He observed that an agenda setter could design a sequence of pairwise majority votes leading from *any* status quo point to *any other* point, most plausibly his own ideal point but also a point well outside the Pareto set, as the final voting outcome and thereby have total control over social choice. More generally, some have concluded that such global cycling means that, in two or more dimensions, majority rule is 'chaotic' and that 'anything can happen'.

However, this conclusion presents a puzzling discontinuity: given generalized Plott symmetry, majority rule is wholly coherent but, given the slightest perturbation of ideal points, the character of majority rule changes entirely and becomes totally chaotic. We now introduce a concept that both smooths out this discontinuity and provides the basis for an intuitive understanding of McKelvey's theorem and its implications.

While it is very unlikely that ideal points are distributed in such a way that all median lines have a common intersection, it is likely that they all pass through a fairly small central region of the space. Following McKelvey (1986), we define the *yolk* as the area enclosed by the circle of minimum radius that intersects every median line. The yolk has a *center* c — which indicates the generalized center, in the sense of the median, of the configuration of ideal points — and a *radius* r — which indicates the extent to which the configuration departs from generalized Plott symmetry. Figure 2(b) shows the yolk for the five-voter configuration.

The symbol c can appropriately designate both the Condorcet winner (when one exists) and the center of the yolk, because a Condorcet winner exists only when all median lines intersect at a common point, in which case this point is both the Condorcet winner and the center of a yolk with zero radius. In this event, the win set of any point x that lies at a distance d from c is the area enclosed by a circle centered on c with a radius of d. But as the size of the yolk increases, the boundary of the win set becomes more irregular, in some place extending beyond this circle and in other places falling short, so that x is beaten by some points at a distance of more than d from c and x beats some points at a distance of less than d from c.

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[9] This set must contain a complete cycle, hence its name.

In Figure 2(b), the three most widely separated median lines are tangent to the yolk and form the 'yolk triangle'. Any point x, at a distance d from c, must lie together with c on the same side of at least one of these three median lines M. Consider the line L that passes through x and is perpendicular to M, intersecting L at point z. Point x is beaten by every point on L between x and the point x' equidistant from z on the opposite side of M. Since c lies on the same side of M as x, x is beaten by points in the vicinity of x'that lie at a distance greater than d from c. Moreover, we can determine that the maximum amount by which the distance between x' and c can exceed d; this occurs when c lies on L between x and x', necessarily (since M is tangent to the yolk) at a distance of r from M, so x' is $d + 2r$ from c. Thus we can state the *2r rule for win sets*: point x is beats all points more than $d + 2r$ from the center of the yolk; conversely (provided $d > 2r$), x is beaten by all points less than $d - 2r$ from c (Miller et al. 1989).

The fact that, in the absence of generalized Plott symmetry, every point x is beaten by some points more distant from c than is x drives McKelvey's theorem, for we can always construct a trajectory of this form: x is beaten by some point y further from the center of the yolk than x is; in turn, y is beaten by some point w further from the center of the yolk than y is; and so forth, forming a trajectory from x to some point u at least $2r$ more distant from c than is x, so x beats u. Thus, for any pair of points x and u, we can construct a majority preference cycle including both x and u. Furthermore, if two cycles have a point in common, a single cycle encompasses all the points in both cycles (Black 1958: 48). This yields the following result.

McKelvey's (1979) Global Cycling Theorem. In the absence of generalized Plott symmetry, a global majority rule cycle encompasses the entire alternative space.

While this argument supports McKelvey's theorem, it also indicates that the trajectory required by McKelvey's devilish agenda setter to move from a relatively centrist point to a relatively extreme one requires many intermediate steps if the yolk is small, since each step in the trajectory can lead *at most* $2r$ further from the center of the yolk. Moreover, if the maximum outward movement of $2r$ per step is to be approached, the trajectory must jump wildly back and forth across the space, since the points most distant from the center of the yolk that beat a given point x are located on the far side of the yolk from x. Such considerations suggest significant limitations on the kind of 'agenda control' implied by McKelvey's theorem or on the claim that 'anything can happen' in the absence of Plott symmetry.[10]

4.4 The Uncovered Set

While the absence of a Condorcet winner suggests looking for the final voting outcome in the top cycle set, this is unhelpful in the spatial context, because McKelvey's theorem tells us that the top cycle includes everything. As discussed in Chapter 3.1, the 'uncovered set' is a subset of the top cycle set that may be considerably smaller. Moreover, theoretical analyses (e.g., Miller 1980, McKelvey 1986) indicate that many social choice processes (for example, electoral competition, strategic voting under amendment agendas, voting with coalition formation, open agenda formation) lead to outcomes

[10] This line of argument is pursued in Feld et al. (1989). The same considerations suggest that other 'chaotic' implications sometimes drawn from McKelvey's theorem may be overstated (e.g., Feld and Grofman, 1991 and 1996; Tovey, 2010a).

in the uncovered set, as do various experimental and empirical results (see Chapter 6.2). Thus, we now consider the uncovered set in a spatial context.

Alternative *x covers y* if x beats y and also beats every alternative that y beats.[11] In this event, $W(x)$ is a proper subset of $W(y)$ and, being based on set inclusion, the covering relation is transitive (but incomplete) and there is an *uncovered set*, comprising all alternatives not covered by other alternatives. If a Condorcet winner exists, it constitutes the uncovered set. Otherwise, an uncovered alternative x beats every other alternative in no more than two steps — that is, if x fails to beat y, x beats some alternative z that beats y (for otherwise y would cover x). Moreover, if x beats y under unanimity rule and y beats z under majority rule, the transitivity of individual preference implies that x beats z through the same majority. Hence the uncovered set (unlike the top cycle set) is contained in the Pareto set.

If x covers y, $W(x)$ is a subset of $W(y)$ so, in a spatial context, the boundary of $W(y)$ encloses $W(x)$. Sometimes x may cover a neighboring point y, in which case $W(x)$ is simply a slightly shrunken replica of $W(y)$. This is true if x is closer to every ideal point than is y, which implies that y lies outside the Pareto set. Such covering can also operate within the Pareto set but only if unlikely symmetries in the distribution of ideal points imply that some voters' indifference curves never form part of the boundary of win sets, most prominently in the case of generalized Plott symmetry (Miller 2007).

But x covers y more typically when they are not neighboring points and x is substantially closer to the center of the yolk than is y. In this event, $W(x)$ is not a slightly shrunken replica of $W(y)$; rather the two win sets may be quite differently shaped but $W(x)$ is sufficiently smaller than $W(y)$ that its different shape is enclosed within the boundary of $W(y)$. The $2r$ rule together with the two-step property imply that the set of points covered by x is contained in the circle centered on c with a radius $d + 4r$ (where d is the distance from x to c). Thus the set of points not covered by c is contained in the circle centered on c with a radius of $4r$. Since the uncovered set is the set of points not covered by *any* other point, it is a subset of the set of points not covered by c and lies within the same $4r$ bound, as was first demonstrated by McKelvey (1986).

Little more was known about the size and location of the uncovered set in a two-dimensional space until the development about ten years ago of computer programs that can demarcate the uncovered set for any configuration of ideal points. These include *CyberSenate*, developed by Joseph Godfrey, and a similar program developed by Bianco et al. (2004; also see Chapter 6.2). On the basis of an examination of many ideal point configurations using *CyberSenate*, Miller (2007) concluded that the uncovered set is relatively compact and typically lies within a circle centered on c with a radius of about $2r$ to $2.5r$.

4.5 The Size and Location of the Yolk

Since the radius of the yolk determines the irregularity of win sets and r and c largely determine the size and location of the uncovered set, it is useful to have some sense of the size and location of the yolk relative to the Pareto set.

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[11] When ties are possible, complexities arise in the definition of covering that need not concern us here; see Penn (2006) and Duggan (2013).

With a small number of ideal points, the yolk is typically quite large relative to the Pareto set (as in Figure 2(b)), and the (understandably) common use of such examples has reinforced the impression that this is more generally true. However, from the time the concept was first propounded, there has been an intuition that the yolk tends to shrink as the number and diversity of ideal points increase. However, it was difficult to confirm this intuition or even to state it in a theoretically precise fashion. Feld et al. (1988) took a few very modest first steps. Shortly thereafter Tovey (2010b) took a much larger step by showing that, if ideal point configurations are random samples drawn from a 'centered' continuous distribution, the expected yolk radius decreases as the number of ideal points increases and approaches zero in the limit.[12]

Two important further questions arise. First, at what rate does the yolk shrinks as the number of 'centered' ideal points increases. Second, what effect does the kind of 'non-centered' clustering of ideal points such as we often see in empirical data (for example, in Figures 6 and 7 in Chapter 6.1) have on the size and location of the yolk.

Simulations by Koehler (1990), Hug (1999), and Bräuninger (2007) indicate that the expected size of the yolk declines quite rapidly as larger samples of ideal points are drawn out of a bivariate uniform distribution. Miller (2007) replicated these findings for bivariate normal distributions. Once a low threshold of fewer than a dozen ideal points is crossed, the expected yolk radius shrinks as the number of voters increases: given configurations of about a hundred voters, the expected yolk radius is about one-quarter (and yolk area about 6%) of that for most small configurations with the same dispersion. For larger configurations, the expected yolk radius appears to follow an inverse square root law with respect to sample size (in the manner of sampling error more generally). With just a few hundred ideal points, the yolk is extremely small relative to the Pareto set, and win sets of points at any substantial distance from the yolk come very close to forming perfect circles (as if Plott symmetry existed). Figure 3(a) provides an example for the politically relevant case of 435 voters[13]. In such a configuration, McKelvey's theorem, though technically correct, has little practical relevance, and the uncovered set is confined to a tiny area centrally located in the Pareto set.

However, clustering of ideal points can greatly increase the size of the yolk and also push it off-center. Figure 3(b) shows 435 ideal points distributed in two distinct clusters of virtually equal size, resembling many empirically estimated configurations of ideal points in the contemporary 'polarized' U.S. House of Representatives. The figure shows limiting median lines, almost all of which form a 'bow tie' pattern and pass through a small area about midway between the two clusters. This might suggest that the yolk lies in this small central region, but there must be at least one additional median line that lies across the centrist face of the larger cluster, with the entire minority cluster and the empty space between them on the other side. Since it intersects all median lines, the yolk must lie within the majority side of the bow tie; as such, it is much larger than in Figure 3(a) and

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[12] Tovey's paper was originally written and widely circulated 20 years prior to publication.

[13] The U.S. House of Representatives has 435 members. Figures 4(a) and 4(b) were generated by *CyberSenate*, as were Figures 2(a) and 2(b).

is non-centrally located, being nestled against the centrist face of the larger cluster. Win sets are therefore quite irregular, and the uncovered set is quite large and substantially penetrates the majority cluster. Moreover, it is evident that the yolk does not shrink as the number of clustered ideal points increases.

5. Multiple Dimensions and *k*-Majority Rule

In this brief concluding section, we consider voting rules more demanding than simple majority rule with an odd number of voters, which leads to some consideration of dimensions beyond two.

In the two-dimensional case, once we consider k-majority rule more demanding than simple majority rule, a median line becomes a 'pivotal strip' lying between two parallel lines such that no more than $n - k$ ideal points lie on either side of the strip. No point x on the strip can be beaten by points on the line that passes through x and is perpendicular to the strip. But any point y lying off such a strip is beaten by the point on the strip closest to y. Thus a point is unbeaten if and only if it lies in the intersection of all such strips. Strips with positive width are more likely to have a common intersection than median lines of zero width. Moreover, the more demanding the voting rule, the thicker the strips are likely to be, and the thicker they are the more likely to intersect. This supports the intuition that, in two dimensions, more demanding rules are more likely to entail unbeaten points and that a sufficiently demanding rule will guarantee them.

Let us consider how demanding k-majority rule can be and still fail to guarantee unbeaten points in the two-dimensional case. This requires that (at least) three pivotal strips fail to have a common intersection. Consider the triangle formed by the three lines that form the 'inside' boundaries of three non-intersecting strips. At least k ideal points must lie on or 'outside' each of these lines, and simple calculation shows that this cannot hold if $k > 2n/3$. Yet it can hold if $k = 2n/3$, provided that precisely one-third of the ideal points are clustered at or 'outside' each vertex of the triangle so that they are simultaneously on or 'outside' two of the three sides of the triangle.

This argument can be generalized to higher dimensions. Given three dimensions, the three lines enclosing the (plane) triangle with three vertices become four planes enclosing a (solid) tetrahedron with four vertices, so a core is not assured until $k > 3n/4$. Further generalization yields the following result.

Greenberg's (1979) Core Existence Theorem. Given an m-dimensional space, there is a non-empty core under any k-majority rule with $k > n\,(m\,/m{+}1)$, but given $k \leq n\,(m\,/m{+}1)$, the core may be empty for some configurations of ideal points.[14]

If k is close to $2n/3$ — or, more generally, to $n\,(m\,/m{+}1)$ — unbeaten points are likely in the absence of the kind of clustering of ideal points described above, but as k approaches simple majority rule unbeaten points become less and less likely. Given the distribution shown in Figures 3(a), a core

[14] Greenberg stated his theorem for generalized single-peaked preferences.

exists with k as small as 230 (or about 53%-rule); perhaps more surprisingly, the same is true with the distribution shown in Figure 3(b).[15]

In conclusion, we note that adding complexity to voting institutions — for example, turning a unicameral voting body into a bicameral one, or 'checking' a voting body with an executive armed with a veto — has an effect similar to making the voting rule more demanding. This kind of analysis of the spatial model is pursued in the following chapter.

[15] More generally, Caplin and Nalebuff (1988) have shown that, given a 'concave density' (a condition that rules out clustering) of a very large number of ideal points of voters with Euclidean preferences over an m-dimensional space, unbeaten points exist provided $k \geq n - [(m/(m+1)]^m$. This is approximately 56%-rule in two-dimensions and approaches a limit of about 64%-rule as the number of dimensions increases without limit.

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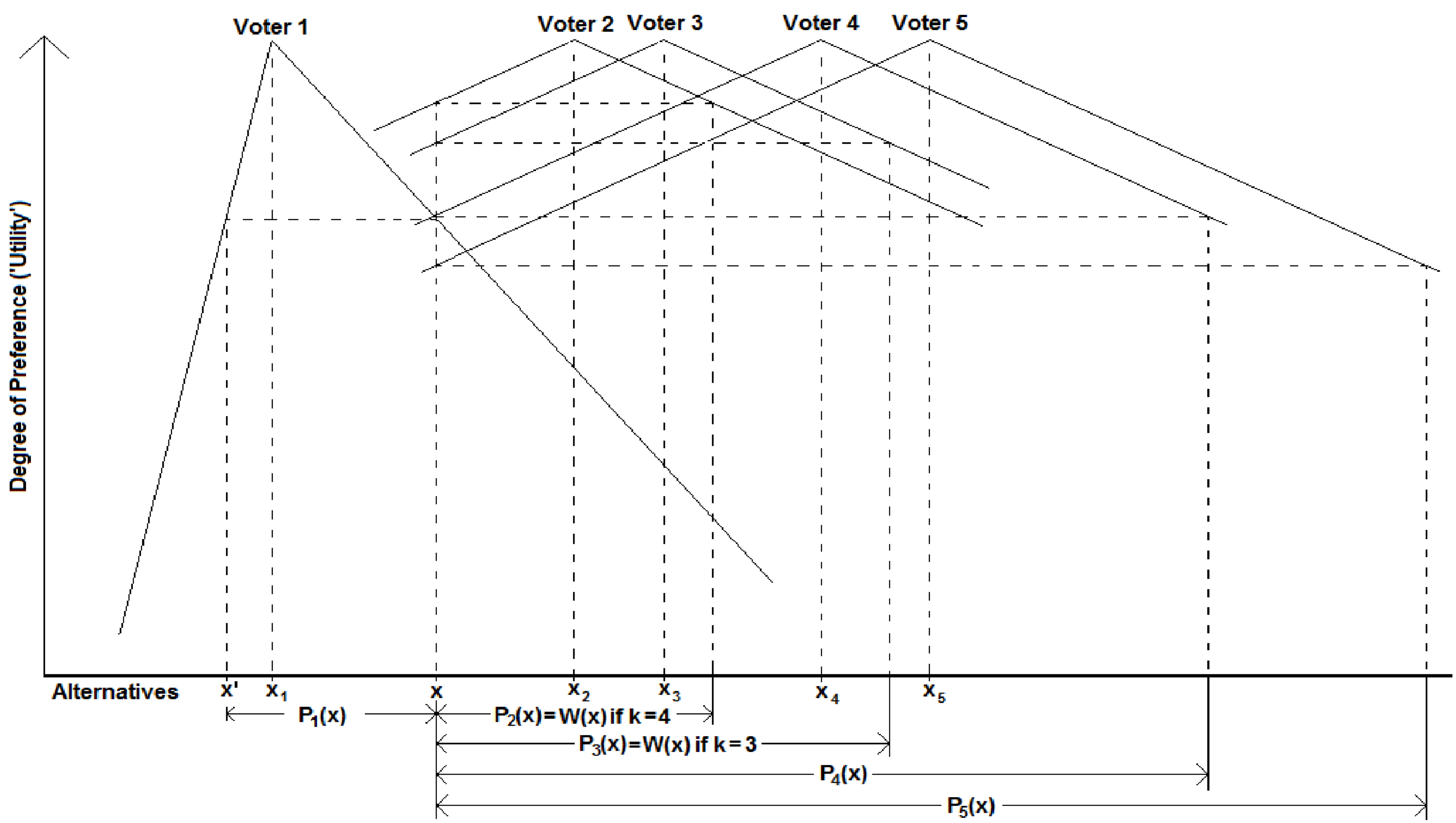
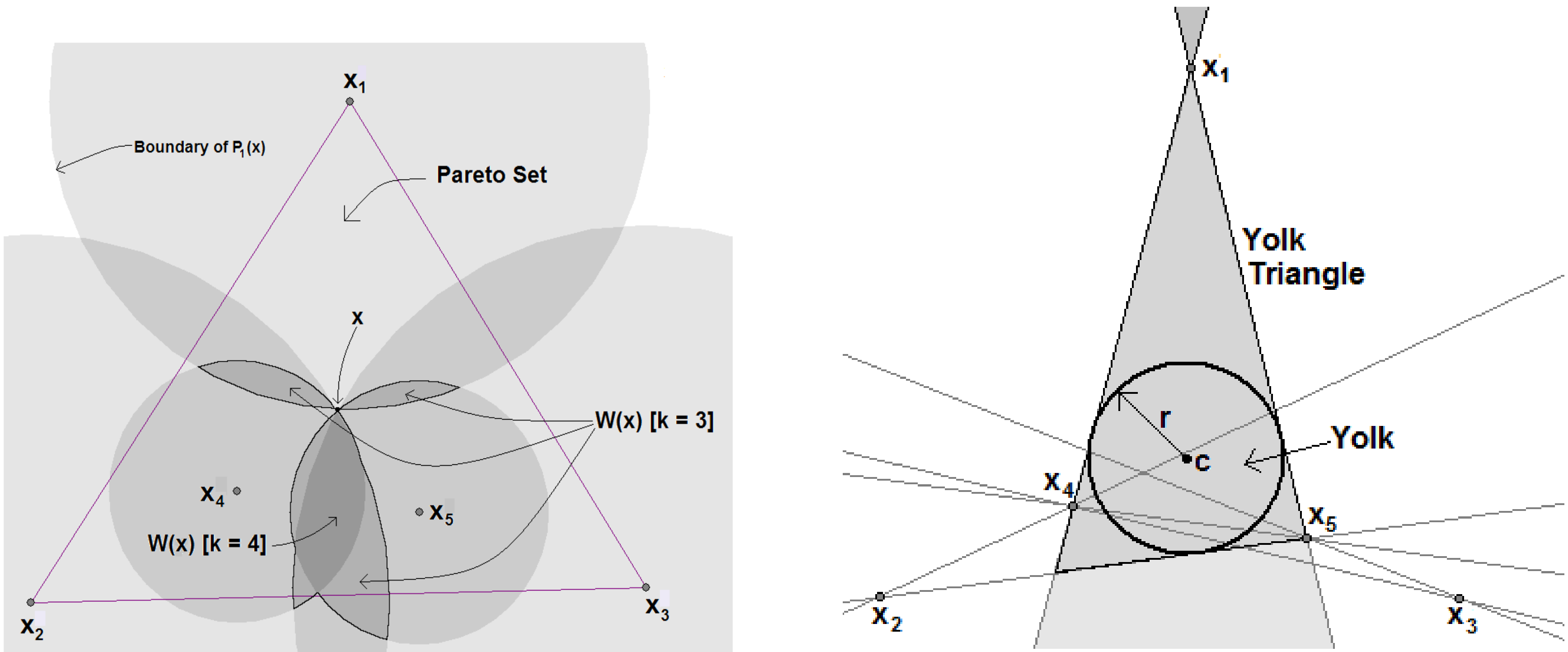
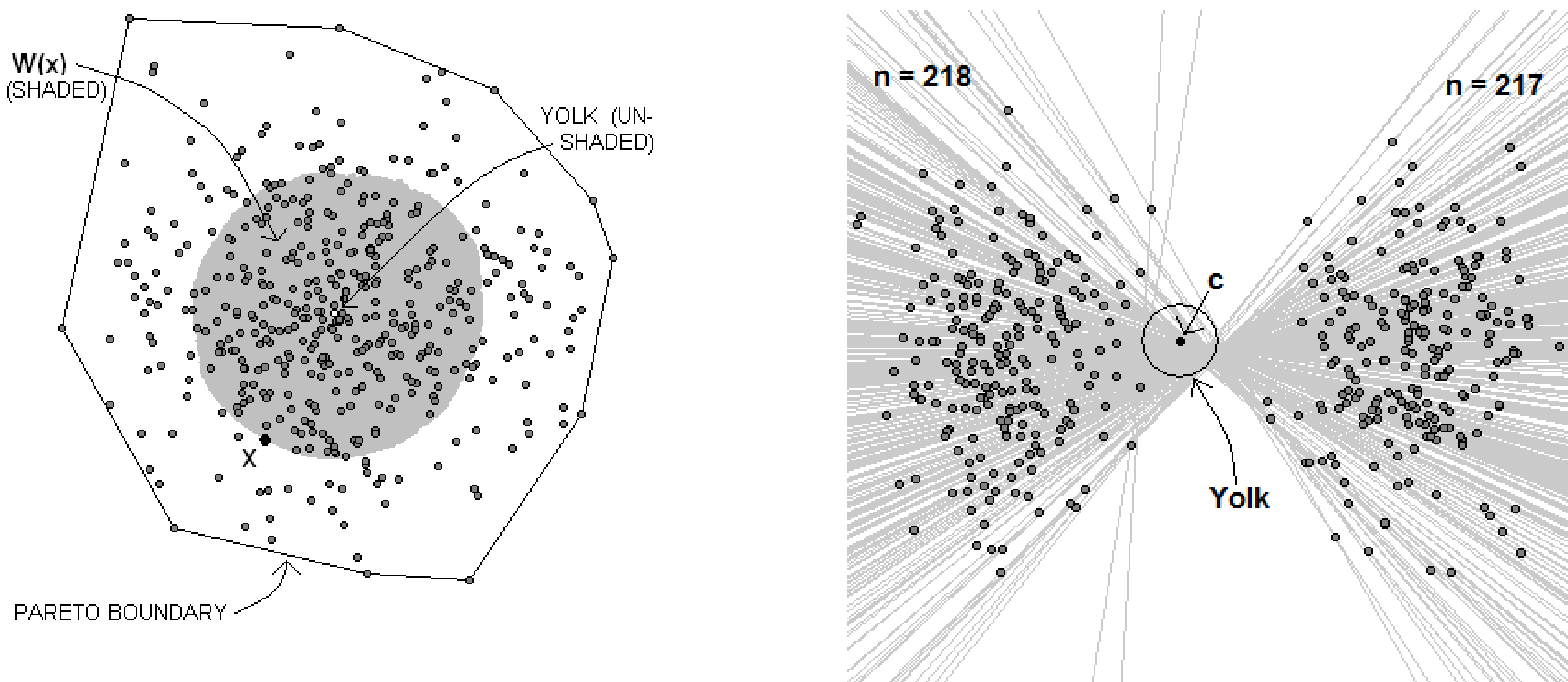


Figure 1. Five Voters with Single-Peaked (and Euclidean) Preferences and Win Sets in One Dimension



(a) Preferred-to Sets and a Win Set **(b) Median Lines and the Yolk**

Figure 2. Five Voters with Euclidean Preferences in Two Dimensions



(a) Unclustered **(b) Clustered**

Figure 3. Large (n = 435) Ideal Point Configurations