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In Search of the Uncovered Set

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This paper pursues a number of theoretical explorations and conjectures pertaining to the uncovered set in spatial voting games. It was stimulated by the article "The Uncovered Set and the Limits of Legislative Action" by W. T. Bianco, I. Jeliazkov, and I. Sened (2004, *Political Analysis* 12:256–78) that employed a grid-search computational algorithm for estimating the size, shape, and location of the uncovered set, and it has been greatly facilitated by access to the *CyberSenate* spatial voting software being developed by Joseph Godfrey. I bring to light theoretical considerations that account for important features of the Bianco, Jeliazkov, and Sened results (e.g., the straight-line boundaries of uncovered sets displayed in some of their figures, the "unexpectedly large" uncovered sets displayed in other figures, and the apparent sensitivity of the location of uncovered sets to small shifts in the relative sizes of party caucuses) and present theoretical insights of more general relevance to spatial voting theory.

1 Introduction

The cyclical and seemingly "chaotic" nature of majority rule revealed by the theoretical work on voting and social choice of Plott (1967), McKelvey (1976, 1979), Schofield (1978), and others suggested that political processes rarely achieve equilibrium and may "wander all over the place." But this theoretical conclusion was anomalous because actual political choice processes appear to be considerably more stable than the theory suggested. In the face of this anomaly, formal political theorists pursued two different, though not mutually exclusive, lines of inquiry. The first, exemplified most notably by Shepsle (1979), recognized that political choice is always embedded in some kind of institutional structure, which may constrain processes so as to create (perhaps rather arbitrary) equilibria that would not otherwise exist. The second, in contrast, focused directly on pure majority rule and sought to find some deeper structure and coherence within the system of majority preference that may constrain or guide political choice processes,

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even in the face of apparent chaos and independently of particular institutional arrangements. The *uncovered set* (Miller 1980; McKelvey 1986) was perhaps the leading contribution of the latter line of theorizing.

The basic idea is simple. Alternative x *beats* alternative y if some majority of voters prefer x to y. Alternative x *covers* y if x beats y and also beats every alternative that y beats. As it is based on set inclusion, the covering relation is transitive and has maximal (uncovered) elements. Thus, the uncovered set, comprising all alternatives not covered by other alternatives, always exists. An uncovered alternative x has this strategically important property—x beats any other alternative y in no more than two steps, i.e., either (1) x beats y or (2) there is some third alternative z such that x beats z and z beats y. The uncovered set collapses to the *Condorcet winner* (an alternative that beats every other alternative) if majority rule happens to be nonchaotic. But the uncovered set exists even if majority rule is chaotic, and it has a variety of nice properties: in particular, it is a subset of both the Pareto set and the top cycle set and it may be quite small relative to the set of all alternatives. Moreover, a variety of distinct competitive choice processes (electoral competition between power-oriented parties or candidates, sophisticated voting under standard amendment procedure, cooperative voting with free coalition formation, open agenda formation, etc.) appear to produce outcomes in the uncovered set.

But until recently one major problem remained. In the context of spatial voting games of two or more dimensions and in the absence of the "Plott symmetry" required for a Condorcet winner, the top cycle set encompasses the entire space, and theorists have had only incomplete or rough knowledge concerning the location, size, and shape of the uncovered set. This problem motivated the recent paper "The Uncovered Set and the Limits of Legislative Action" by Bianco, Jeliazkov, and Sened (2004 and henceforth BJS), who employed a grid-search computational algorithm to generate pictures of uncovered sets in a variety of spatial voting scenarios. These results were the first of their kind and of great significance for the theory of spatial voting and social choice. Here I pursue a number of theoretical explorations and conjectures initially stimulated by the BJS paper. This paper started out as, and in part remains, a commentary on BJS; in any event, it makes many references to BJS figures, so it may be advantageous to read it with the BJS paper at hand.

In pursuing these explorations, I have been enormously helped by access to early versions of *CyberSenate*, another computer program for analyzing spatial voting games developed by Joseph Godfrey. This software allows users to create configurations of ideal points by point-and-click methods, generate them by Monte Carlo methods, or derive them from empirical data. Indifference curves, median lines, Pareto sets, win sets, yolks, cardioid bounds on win sets, uncovered set approximations (based on an algorithm similar to that of BJS), and other constructions can be generated on-screen. *CyberSenate* produced all but one of the figures that follow.[1] More importantly, *CyberSenate* has been indispensable in developing and testing theoretical ideas.

Whereas the BJS paper is informative regarding the macrorelationship between ideal point configurations and uncovered sets, my aim here is to explore the microstructure that underlies the BJS findings. Consistent with BJS and the capabilities of the *CyberSenate*, I focus entirely on two-dimensional spatial voting games with Euclidean preferences, though points may generalize to higher dimensions. In general, I try to bring to light

[1]Labeling and other embellishments were subsequently added. Versions of these figures, plus many related figures that make advantageous use of color, may be found at http://research.umbc.edu/~nmiller/RESEARCH/UNCOVERED.htm. I am very much indebted to Joseph Godfrey of the WinSet Group LLC for making early versions of *CyberSenate* available to me. Further information about *CyberSenate* is available at http://www.winset.com.

theoretical considerations that account for important features of the BJS results—for example, the straight-line boundaries of the uncovered sets displayed in their Fig. 2, the "unexpectedly large" uncovered sets displayed in their Figs. 1 and 5, and the sensitivity of the location of uncovered sets to smalls shifts in the relative sizes of the party caucuses displayed in the first panel of their Fig. 4 and in their Fig. 5—as well as to present theoretical insights of more general relevance, focusing on the character of win sets, the nature of the covering relation in a spatial context, and the size and location of the yolk. I do not in these remarks address the relevance of the uncovered set for the empirical study of legislative politics and "the limits of legislative action."

As BJS note, three bounds on the uncovered set have been known for many years:

a. in the event that a Condorcet winner exists, the uncovered set coincides with it;
b. the uncovered set lies within the Pareto set; and
c. in a spatial context, the uncovered set lies within a circle centered on the yolk with a radius four times that of the yolk.

I show that bound (b) operates through *proximate covering* and is overgenerous in principle (the actual bound is the *visible Pareto set* after *invisible voters* have been deleted from the configuration of ideal points), though not in practice (invisible voters arise only in the presence of empirically unlikely collinearities in ideal points). On the other hand, bound (c) operates through *distant covering* and is overgenerous in both principle and practice, as the uncovered set is typically contained within a circle centered on the yolk with a radius only a bit larger than twice that of the yolk. Point (a) has long been recognized as a corollary of (c), since a yolk of zero radius implies a Condorcet winner. I show that bound (a) is also a corollary of bound (b) refined to refer to the visible Pareto set.

Section 2 provides a summary of earlier finding concerning the uncovered set in a spatial context. Section 3 presents some necessary theoretical preliminaries. Section 4 introduces the concepts of proximate covering and invisible voters to refine the Pareto bound on the uncovered set. Section 5 introduces the concept of distant covering, shows that the four-radius bound on the uncovered set results from distant as opposed to proximate covering, and, through a combination of theoretical deduction and induction from many *CyberSenate* experiments, further shows that this bound can be substantially reduced. Section 6 lays out new findings, based on *CyberSenate* simulations, concerning the size and location of the yolk (and the uncovered set). Section 7 concludes by drawing together the previous results to account for BJS's graphical results presented in their Figs. 1, 2, and 5 and by offering more speculative conjectures concerning the structure of the uncovered set in a spatial context.

2 Earlier Findings

Miller (1980) focused on discrete alternatives and unrestricted preferences but made several points relevant to determining the size and location of the uncovered set in a spatial context as well. First, I noted (p. 74) that a Condorcet winner, if it exists, is the unique uncovered point. Second, I noted (p. 80) that unanimity implies covering, from which it follows that the uncovered set must be a subset of the Pareto set. In a two-dimensional spatial voting game with Euclidean preferences, the Pareto set is the convex hull of voter-ideal points, so the uncovered set is contained within this set. Third, I noted (p. 74) that the uncovered set is a subset of the top cycle set. However, in a multidimensional spatial context, this bound loses its punch since, in the absence of Plott symmetry (and a Condorcet winner), the top cycle encompasses the entire space. Finally, I conjectured (footnote on p. 84) that in a spatial context the uncovered set "would be a relatively small subset of [the Pareto set], centrally located in the distribution of ideal points, and that it would shrink in size as the number and diversity of ideal points increase."

McKelvey (1986, drawing on Ferejohn, McKelvey, and Packel 1984) introduced the concept of the *yolk*, that is, the set of points bounded by the smallest circle that intersects every median line in a two-dimensional spatial voting game and showed that, if voters have Euclidean preferences, the uncovered set lies within the circle centered on c with a radius of $4r$, where c is the center of the yolk and r is its radius.

Shortly thereafter, Hartley and Kilgour (1987) established the precise boundaries of the uncovered set for configurations of three voters with Euclidean preferences in a two-dimensional space. They showed that, in the event ideal points form the vertices of an equilateral triangle, the uncovered set coincides with the Pareto set and that otherwise the uncovered set excludes portions of the Pareto triangle in the vicinity of the one (if the Pareto triangle is acute) or two (if it is obtuse) relatively "extreme" ideal points. An implication of their analysis was that, at least in the three-voter case, McKelvey's $4r$ bound is overgenerous.

When the concept was first propounded, there was a widespread intuition that the yolk is centrally located relative to the configuration of ideal points and that it tends to shrink in size as the number and diversity of voters increases. However, it was difficult to confirm this intuition or even to state it in a theoretically precise fashion. Feld, Grofman, and Miller (1988) took a few very modest first steps. Tovey (1990) took a considerably larger step by showing that, if ideal point configurations are random samples drawn from a *centered* continuous distribution, the expected yolk radius approaches zero as the number of ideal points increases without limit.

Not much more was learned about either the yolk or the uncovered set in a multidimensional spatial context until BJS provided the first pictures of the uncovered set in their Fig. 2 (for contrived five-voter configurations) and Figs. 1, 4, and 5 (for empirical U.S. House data) in their recent article. Based on the computational results displayed in their figures, BJS (pp. 270–1) made three theoretical claims concerning the location and size of the uncovered set.

1. "The uncovered set can be much larger than our expectations based on conventional wisdom and previous work," as all their figures seem to illustrate.
2. The uncovered set is not necessarily "centrally located." If ideal points are polarized (as in the contemporary House), "the uncovered set does not lie in the center of the distribution of legislators' ideal points but is skewed toward the majority caucus," as illustrated by BJS Fig. 1, the first panel of Fig. 4, and all panels of Fig. 5.
3. "The size, shape, and location of the uncovered set are very sensitive to the distribution of ideal points." With respect to size, this sensitivity is quite dramatically illustrated by their Fig. 2 and is less dramatically illustrated by comparing panels in Fig. 5. With respect to location, such sensitivity is illustrated by the first panel of their Fig. 4 and by a comparison of the last two panels of their Fig. 5.

We may also observe that BJS Fig. 2 is distinctive in that the uncovered sets appear to have straight-line boundaries that coincide with certain median lines. Furthermore, in several of the panels, the uncovered set appears to be similar to the Hartley–Kilgour construction for the three-voter case—in some way, the two additional ideal points (to the left and right) have no effect on the size and location of the uncovered set.

3 Theoretical Preliminaries

I follow BJS by focusing on two-dimensional majority-rule spatial voting games. Some degree of familiarity with standard terminology and notation is assumed. In this spatial context, I refer to alternatives as *points*. The set X of all alternatives is the set of all points in the space. I assume a finite odd number $n \geq 3$ of voters. Each voter i has Euclidean preferences—that is, i has an *ideal point* x_i in the space and prefers any point closer to this ideal point to one that is more distant, so that i's indifference curve through x, denoted $I_i(x)$, is the circle centered on x_i that passes through x. The set of points $P_i(x)$ that i prefers to x is the set enclosed by $I_i(x)$.

If some majority of $m = (n + 1)/2$ voters prefers x to y, I say x beats y. The *win set* $W(x)$ is the set of all points in X that beat x. The set of points that a particular majority of voters prefers to x is the intersection of all sets $P_i(x)$ such that i belongs to that majority. $W(x)$ is the union of such majority preference sets across all majorities. Thus, the boundary of a win set is everywhere demarcated by segments of individual voter indifference curves (segments of circles in the Euclidean context). In a spatial context with n odd, x beats almost all points not in $W(x)$.[2]

I call a configuration of ideal points *diverse* if no two ideal points precisely coincide. A key feature of a spatial voting game is whether the configuration of ideal points exhibits *collinearities*—that is, whether three or more ideal points lie precisely on the same straight line. Collinearity always exists when ideal points coincide but may be found in diverse configurations as well.[3] Collinearity produces a variety of peculiarities—in particular, the invisible voter phenomena discussed in Section 4.

A straight line L partitions the set of voter-ideal points into three subsets: those that lie on one side of L, those that lie on the other side of L, and those that lie on L itself. If it partitions the ideal points so that no more than half of the ideal points lie on either side, L is a *median line*, which we henceforth label M. Every ideal point lies on some median line, and if n is odd, every median line M passes through some ideal point such that fewer than half of the ideal points lie on either side of M and no other median line is parallel to M.

If n is odd, a typical median passes through just one ideal point, but a *limiting* median line passes through two (or more) ideal points. Typically, pairs of limiting median lines pass through a given ideal point, with *nonlimiting* median lines sandwiched between them. A median line that passes through the three or more (necessarily collinear) ideal points is a *stand-alone* limiting median line in which the "sandwich" of nonlimiting median lines is reduced to zero thickness.

Each voter i has an *induced ideal point*, that is, a most preferred point, on any line L. Given Euclidean preferences, voter i's induced ideal point is the point on L closest to x_i, that is, the intersection of L with the line through x_i perpendicular to L. The n induced ideal points appear on L in some (possibly weak) order, and (since n is odd) we can identify the median induced ideal point on L.[4] Note that the perpendicular line through the median

[2]Even with n odd, some majority preference ties exist but, in order to simplify exposition, I overlook technical issues pertaining to points that lie on the boundaries of win sets. This also sidesteps issues pertaining to alternate definitions of the uncovered set (see Penn 2006). (With n even, "tie sets" become substantial and alternate definitions of the uncovered set more consequential.)

[3]Nondiversity and collinearity may both be deemed "exceptional" in the sense that, if hypothetical ideal points were "randomly thrown" into a policy space, nondiversity and collinearity would almost never occur. Of course, we can deliberately contrive nondiverse and collinear configurations (as BJS do in their Fig. 2). In empirical work, where ideal point locations estimated from interest group rating scales or similar data are typically expressed in whole numbers, it is likely that several legislators have identical scores on a given dimension, producing nondiversity and other collinearities.

[4]Note that, even if ideal points are diverse, two (or more) distinct ideal points may lie on the same line perpendicular to L, so induced ideal points on L may coincide.

induced point on L is itself the unique median line perpendicular to L. By standard median voter logic, a point x on L is beaten by another point y on L if and only if y lies in the interval between x and its *reflection point* x'' such that x and x'' are equidistant from the median induced ideal point on L. In the event that x coincides with the median induced ideal point, x beats every other point on L.

This last consideration implies that, if a point x lies off any median line M, x is beaten by points on M. It follows that a point x in the space is unbeaten (and a Condorcet winner) if and only if it lies on every median line, which is possible if and only if all median lines intersect at the single point x (which itself must be an ideal point). This in turn can hold only in the presence of a sufficient (and unlikely) degree of Plott symmetry in the configuration of ideal points (Plott 1967; Enelow and Hinich 1983).

The *yolk* is the set of points bounded by the smallest circle that intersects every median line. The *location* of the yolk is given by its center c, which indicates the generalized center (in the sense of the median) of the configuration of ideal points. The *size* of the yolk is given by its radius r, which indicates the extent to which the configuration of ideal points departs from one exhibiting a degree of Plott symmetry sufficient for the existence of a Condorcet winner. The yolk circle is inscribed within the *yolk triangle* formed by three median lines to which the circle is tangent.[5]

To get a preliminary sense of the size, shape, and location of a win set $W(x)$ in the spatial context, consider the special case in which there is only one voter i. In this event, the center of the yolk is x_i, the yolk radius is zero, and $W(x)$ coincides with $P_i(x)$, which (given Euclidean preferences) is the circle with a center at c and a radius of d, where d is the distance from x to c.

Given the general case of multiple voters with diverse ideal points and for a point x outside the yolk, the boundary of $W(x)$ is approximated by the same circle centered on c with a radius of d. The accuracy of this approximation depends on the size of the yolk, as given by its radius r, according to the *2r rule*: point x beats all points more than $d + 2r$ from the center of the yolk and is beaten by all points closer than $d - 2r$ to the center of the yolk. Put otherwise, the boundary of $W(x)$ everywhere falls between two circles centered on the yolk with radii of $d + 2r$ and $d - 2r$, respectively (the inner circle disappears if $d < 2r$ and the two circles coincide if $r = 0$).[6]

It is useful to make a distinction between *orderly* and *disorderly* win sets. In two dimensions, a win set $W(x)$ is orderly if it is a subset of some open half-space about x. This implies that there is some voter i such that, within the vicinity of x, $W(x)$ is a subset of $P_i(x)$ and likewise for the win sets of other points in the vicinity of x. This guarantees that majority preference is transitive (being consistent with i's preferences) in the vicinity of x. On the other hand, a win set $W(x)$ is disorderly if it is not a subset of any half-space about x but rather has multiple small "petals" that "point in all directions" from x. This implies that there is no voter i such that, within the vicinity of x, $W(x)$ is a subset of $P_i(x)$ and likewise for the win sets of other points in the vicinity of x, which in turn implies that majority preference is cyclical in the vicinity of x.

Whether $W(x)$ is orderly or disorderly depends on the location of x relative to all median lines. A win set is orderly if it is not surrounded by (limiting) median lines—that is (to state the matter informally), if there is an escape path from x to outer reaches of the space that

[5]These are typically, but not always, limiting median lines; see Stone and Tovey (1992) and Koehler (1992). It should again be emphasized that this discussion assumes that n is odd; the definition of a yolk is more complicated if n is even.

[6]Tighter bounds on $W(x)$, especially in the vicinity of x itself, are provided by the outer and inner cardioids described in Ferejohn, McKelvey, and Packel (1984), McKelvey (1986), and Miller, Grofman, and Feld (1989).

does not cross any (limiting) median line.[7] A win set is disorderly if it is surrounded by (limiting) median lines. In this context, the set of all points with disorderly win sets corresponds to what Schofield (1999) calls the *heart*.

4 Proximate Covering, Invisible Voters, and the Visible Pareto Set

To say that x covers y is to say that x beats y and also every point that y beats. Put otherwise, y is beaten by x and by every point that beats x; thus, $W(x)$ is a proper subset of $W(y)$. So in the spatial context, the covering relationship manifests itself geometrically, with the boundary of $W(y)$ literally enclosing $W(x)$. However, there are two modes of covering in the spatial context, which we refer to as proximate covering and distant covering.

In the proximate mode, if point x covers point y, x also covers (and y is also covered by) every point on the line between x and y, so *covering operates between neighboring points*. If y is covered by a neighboring point x in this manner, it must be that *both win sets have essentially the same shape and W(x) is simply a slightly shrunken replica of W(y) with the latter enclosing the former*. It is evident that x covers y in this manner only if x is (slightly) closer to the center of the yolk than y is.

It is clear that $W(x)$ cannot be a subset of $W(y)$ if these neighboring points have disorderly win sets, so the reach of proximate covering is limited in that it can operate only with respect to orderly win sets.[8] But beyond this, a kind of unanimity among voters is required—specifically, *a point y is covered by neighboring point x if and only if x is closer to the ideal points of every voter whose indifference curve through y demarcates part of the boundary of W(y).*

Proximate covering most obviously operates if point y lies outside the Pareto set, so y is covered by neighboring Pareto-superior points. If and only if x is Pareto superior to y, x is closer to *every* ideal point than y is, so $W(x)$ is an everywhere shrunken replica of $W(y)$ and lies entirely inside it. Moreover, x can be arbitrarily close to y. Figure 1 illustrates proximate covering of a point outside the Pareto set. As BJS point out (p. 260), it has long been known that x covers y if x is unanimously preferred to y. We now see that, in the spatial context, such covering operates proximately.

There is one well-recognized circumstance in which proximate covering operates within the Pareto set—namely, if the ideal point configuration exhibits Plott symmetry and therefore has a Condorcet winner (as in the first panel of BJS Fig. 2). In this event, all win sets are perfect circles centered on a yolk with zero radius (being the indifference curves of the voter i around whose ideal point Plott symmetry exists) and are therefore ordered by inclusion. This implies that, if y is beaten by a (neighboring) point x, y is also covered by y, even if both x and y lie within the Pareto set. But how can this be, given that x cannot be closer to all ideal points than y is? The answer is that (as noted parenthetically just above) all win sets are demarcated exclusively by the indifference curves of the single voter whose ideal point defines the yolk. With respect to median lines and win set boundaries, Plott symmetry creates collinearities that render all other voters *invisible* in that their indifference curves nowhere demarcate the boundaries of win sets.[9]

[7] The parenthetical qualifier "limiting" is not logically necessary but facilitates visualization of the stated condition.

[8] However, if and only if $W(y)$ is orderly, y is "locally covered" by a neighboring point x that beats y, in that x beats everything that y beats in the vicinity of y, even though x may fail to beat distant points that y beats and x may therefore fail to (globally) cover y. Accordingly, Schofield's (1999) heart can be characterized as the "locally uncovered set."

[9] In the standard one-dimensional Euclidean spatial model, all voters other than the median voter are invisible in the same sense, regardless of the particular configuration of ideal points. Precisely for this reason, a standard shortcut is to let the median voter stand in analytically for a full committee, legislature, or electorate. (It should be noted, however, that voters who are invisible with respect to majority rule may become visible under more demanding decision rules.)

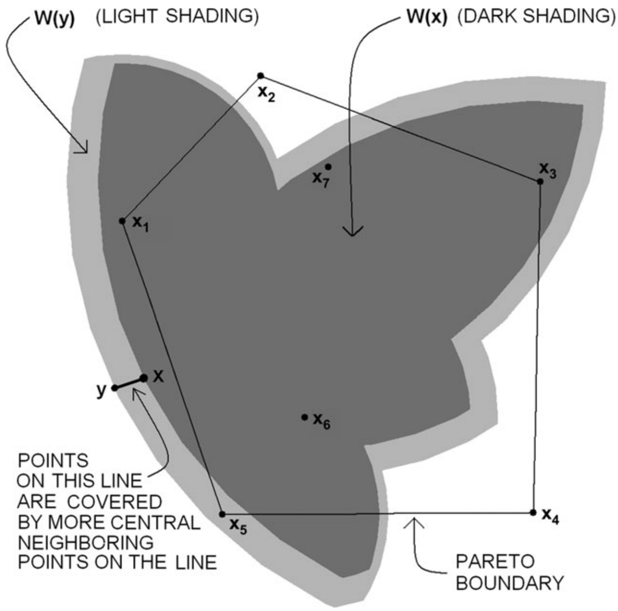

Fig. 1 Proximate covering outside the Pareto set.

However, even in the absence of full Plott symmetry, collinearities of ideal points may occur, which render some voters invisible with respect to the demarcation of win sets.

Let us first identify the circumstance in which a voter is *visible*. Consider a nonlimiting median line M passing through ideal point x_i (but no other ideal point) and also the line L through x perpendicular to M, as shown in Fig. 2. (For the moment, suppose ideal points x_j

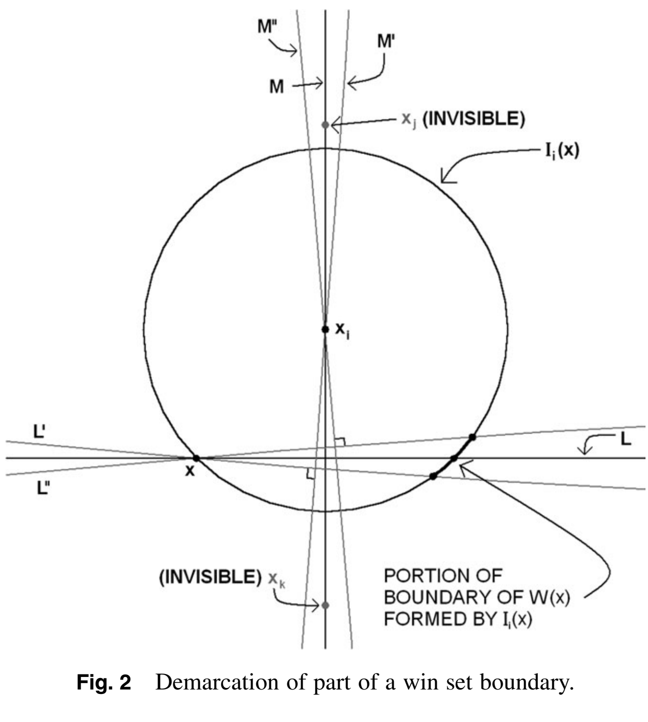

Fig. 2 Demarcation of part of a win set boundary.

and x_k do not exist.) We know that $W(x)$ extends from x along L to its reflection point x'' through the median induced point lying at the intersection of L and M. Given that M is a nonlimiting median line, we can rotate M about x_i just a bit (e.g., to M' or M'' in Fig. 2) while allowing the perpendicular line L to rotate just a bit around x (to L' and L'') while still passing through x. We thereby trace out a bit of the boundary of $W(x)$ that follows i's indifference curve through x from its reflection point on L' to its reflection point L''.[10] Therefore, voter i, with nonlimiting median lines passing through x_i, "controls" part of the boundary of $W(x)$ and in that sense is visible.

Now let ideal points x_j and x_k (collinear with x_i) in Fig. 2 appear. It can be verified that neither j nor k ever *uniquely* occupies the median induced ideal point position on any line through x between L' and L''. (Voters j and k instantaneously share the median induced ideal point status with voter i on line L itself.) Thus, neither j nor k controls any of this part of the boundary of $W(x)$. Furthermore, by the same considerations, if no nonlimiting median lines pass through x_j or x_k, neither j nor k controls any portion of the boundary of $W(x)$ or any other win set. In this sense, voters j and k are rendered invisible.

More generally, a voter i is invisible if and only if the following (equivalent) conditions hold:

a. voter i does not occupy a unique median induced ideal point position on any line L,

b. no nonlimiting median lines pass through voter i's ideal point, or

c. voter i has a *Shapley–Owen spatial voting power index* value of zero.[11]

Figure 3A (which reproduces the essential feature of the intermediate panels of BJS Fig. 2) illustrates the phenomenon of invisible voters. Inspection of Fig. 3A further shows that points immediately to the right of y (in the four-fifths majority area) are closer to the ideal points of all visible voters 2, 4, and 5 (even though they are more distant from invisible voter 1), so even though it lies within the Pareto set, point y is proximately covered by x, as is shown in Fig. 3B.

Let us call the convex hull of visible voter ideal points the visible Pareto set—that is, it is the Pareto set after invisible voters have been removed from the configuration. Points are proximately covered whenever they lie outside of the visible Pareto set, so proximate covering not only pares the uncovered set down to the Pareto set but (in the presence of invisible voters) further pares it down to the visible Pareto set. This has two implications.

1. If an ideal point configuration exhibits Plott symmetry, all voters except one are invisible, so proximate covering pares the uncovered set down to the ideal point of the one visible voter.
2. Otherwise, insofar as the demarcation of the uncovered set results from proximate covering, it has straight-line boundaries, since the (visible) Pareto set has straight-line boundaries.[12]

[10]Indeed, we can rotate L about x a full 180° and trace out the entire boundary of $W(x)$. At the instant the median line perpendicular to L is limiting (passing through two ideal points), a kink in the win set boundary occurs as control of the boundary shifts from one voter to another.

[11]See Grofman et al. (1987), Owen and Shapley (1989), Godfrey (2005), and additional citations therein. (*CyberSenate* can compute Shapley–Owen index values as approximations based on a sample of lines L.)

[12]Whenever it is a proper subset of the Pareto set, the visible Pareto set is "structurally unstable"—that is, if the ideal point configuration is slightly perturbed, collinearities are destroyed and the size of the visible Pareto set "jumps" in a discontinuous manner to fill the full Pareto set and all proximate covering within the Pareto set disappears. This raises the question whether the size of the uncovered set likewise jumps in a discontinuous manner. In fact, covering relationships, and the boundary of the uncovered set, still change in a smooth manner, as covering at a very small distance (as discussed in Section 5) takes over from proximate covering.

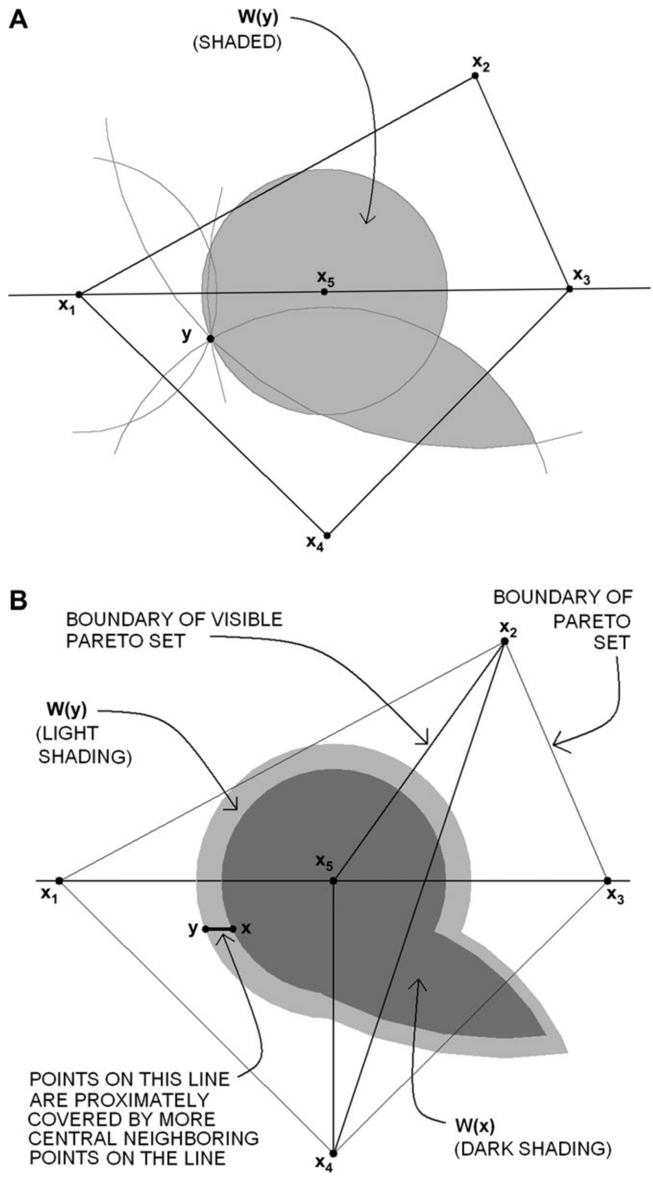

Fig. 3 (A) Ideal points x_1, x_5, and x_3 are collinear, no nonlimiting median lines pass through x_1 and x_3, and x_1 and x_3 are invisible. (B) Proximate covering inside the Pareto set but outside the visible Pareto set.

5 Distant Covering, the 4*r* Bound, and the Uncovered Set

We have seen that proximate covering entails covering between neighboring points and operates only outside the visible Pareto set. A second mode of covering entails only "action at a distance" in that, if point *x* covers point *y*, *x* and *y* cannot be neighboring—rather, they must be some substantial distance apart (*x* being closer to the center of the yolk than *y* is). In practice, the most relevant covering relationships operate at a distance, as only covering at a distance can operate within the visible Pareto set.

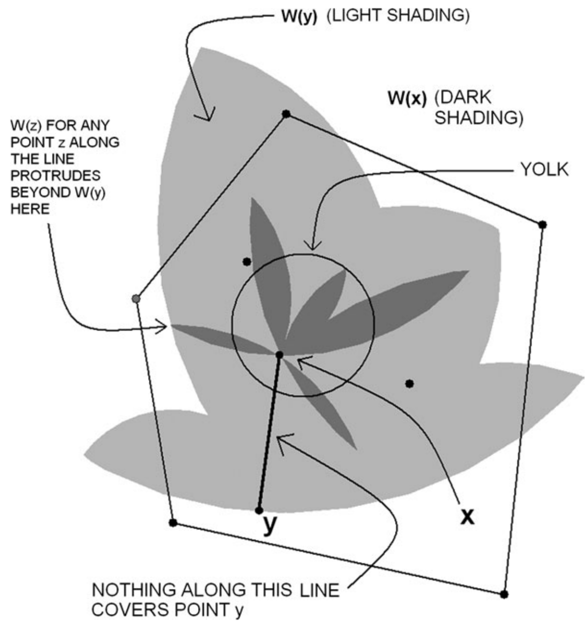

Fig. 4 Typical covering at a distance.

If point x covers point y at a distance, x covers y but does not cover points on the line between x and y, and $W(x)$ is not simply a shrunken replica of $W(y)$. Typically, $W(x)$ is somewhat (or totally) differently shaped from $W(y)$, but at the same time, $W(x)$ is sufficiently *smaller* than $W(y)$ (because x is sufficiently closer to the center of the yolk than y is) that *its differently shaped (and perhaps very disorderly) boundary is nevertheless enclosed within the boundary of W(y)*. A typical example of covering at a distance is displayed in Fig. 4.

A natural question to ask about such covering is how great a distance is required. What really matters is not the distance between x and y per se but rather that x must be substantially closer to the center of the yolk than y is. The $2r$ rule allows us to specify the minimum distance *sufficient* for covering at a distance. Like so much else, this distance is a function of the size of the yolk. Point x beats every point z that is at least $2r$ further from the center of the yolk than x is, and z beats every point y that is $2r$ further from the center of the yolk than z is. Applying the $2r$ rule twice gives us this *4r rule*: x covers at a distance any point y more than $4r$ further from the center of the yolk than x is, and x is covered by any point z more than $4r$ closer to the center of the yolk than x is.

However, x may cover y at a distance even if y is considerably less that $4r$ more distant from c than x is (as Fig. 4 illustrates). Indeed, we cannot specify a minimum distance *necessary* for covering at a distance, since this distance may be arbitrarily small, as is illustrated by Fig. 5 depicting three ideal points forming an elongated Pareto set. A point y within the Pareto set but near the extreme ideal point x_3 cannot be proximately covered (since y is in the yolk triangle and $W(y)$ is disorderly), but y is covered at a distance by point x that is only slightly closer to the yolk than y is. Indeed, it is evident that, as y moves toward ideal point 3, the distance from x to y sufficient for distant covering further

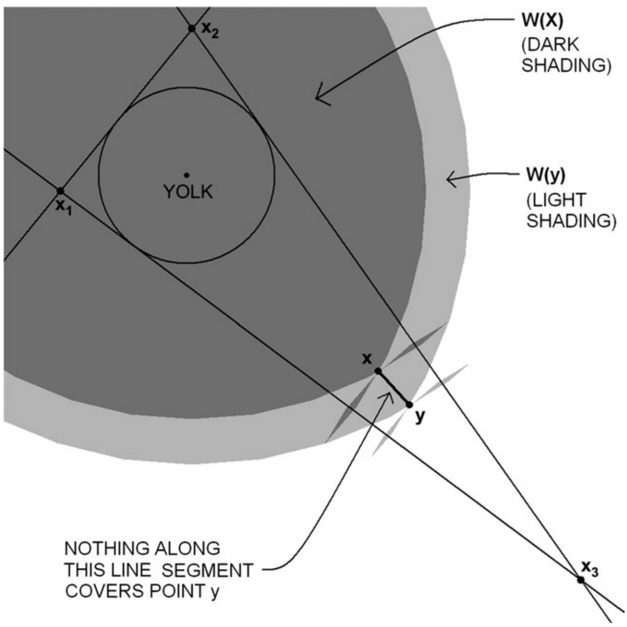

Fig. 5 Atypical covering at a distance.

diminishes and converges on zero as x and y converge on the vertex.[13] Typical covering at a distance operates at intermediate distances. Given configurations with more than a few ideal points distributed in a more or less random manner, a point x typically covers *all* points y that are about $3.5r$ further from the center of the yolk than x is and *most* points z that are about $2.5r$ further from the center of the yolk than x is.

The *uncovered set of point x*, UC(x), is the set of all points not covered by x. $W(x)$ is the set of all points that beat x in one step, and UC(x) is the set of all points that beat x in one or two steps. Clearly, UC(x) is a superset of $W(x)$. In principle, UC(x) may be demarcated by (1) forming $W(x)$, (2) forming $W(z)$ for all points z in $W(x)$, and (3) forming the union of $W(x)$ and all such $W(z)$.[14]

For configurations with a fair number (e.g., $n > 15$) of ideal points, theoretical considerations and much *CyberSenate* experimentation support the following observations. For any point x,

a. UC(x) contains the yolk and is approximately centered on it;
b. by definition, UC(x) encompasses $W(x)$ and is larger than $W(x)$ to the extent that r is greater than zero; and

[13]Figure 4 also demonstrates that points in the yolk triangle (and the heart) may be covered. Of course, point y is also covered by more distant points in the vicinity of the center of the yolk. More generally, points just inside the Pareto boundary cover nearby points just outside not proximately but at a very small distance. In this sense, proximate covering by points on the Pareto boundary itself is the limiting case of distant covering by points just inside the boundary.

[14]Step (2) in principle requires an infinite number of operations, but in small and/or symmetric ideal point configurations, it is sufficient to find $W(z)$ only for points z that lie at the tips of $W(x)$. In such cases, UC(x) is formed entirely out of win sets and is thus everywhere demarcated by individual indifference curves. More generally, it appears that only points on the boundary of $W(x)$ need be considered and that portions of the boundary of UC(x) may not be formed out of win set boundaries at all but are produced by a continuous mapping from parts of the boundary of $W(x)$.

c. UC(x) is far from convex but, like $W(x)$, is *starlike* relative to x (which is to say, if x fails to cover y, x also fails to covers every point on the straight line between x and y).

Of course, point x lies in the interior of UC(x) unless x lies on or outside the boundary of the visible Pareto set so there are points that x covers proximately. If x lies on the boundary of the visible Pareto set, UC(x) has a cusp with an angle of essentially 0° at x. As x moves further outside the visible Pareto frontier, the angle of the cusp widens.

Given $r > 0$, the set of points UC(c) not covered by the center of the yolk is of particular significance. Point c is beaten by points at a distance of $2r$ from c (i.e., the reflections of c through each of the median lines forming the yolk triangle) but none that are more distant (every other median line passes closer to c), and UC(c) in turn extends outward to a distance of no more $4r$ from c. Figure 6 illustrates these points for a configuration of five ideal points at the vertices of a regular pentagon.

A similar $d + 4r$ bound applies to UC(x) for any point x at distance d from c. Thus, the boundary of UC(x) always lies between circles centered on c with radii of d and $d + 4r$. But, given the generalization made above (and in the absence of proximate covering), the boundary of UC(x) typically lies between circles centered on c with a radii of about $d + 2.5r$ and $d + 3.5r$.

Recall that X is the set of all points x in the space. The *uncovered set* UC(X) is the set of all points not covered by any other point—put otherwise, UC(X) is the intersection of the sets UC(x) for all points x in X. Equivalently, UC(X) is the set of all points each of which beats every other point in one or two steps.

It is practically impossible to demarcate the precise boundary of the uncovered set in a spatial context, since this requires forming the intersection of an infinite number of sets, each of which is the union of many, and perhaps an infinite number of, other sets. This is why both the BJS grid-search algorithm and the similar procedure built into *CyberSenate* must use approximation methods.

However, the uncovered set UC(X) is by definition a subset of UC(c), from which it follows that UC(X) also lies within the circle centered on c with a radius of $4r$—indeed, the $4r$ bound on the uncovered set first identified by McKelvey is actually the $4r$ bound on UC(c) previously noted, which necessarily becomes a bound on UC(X) also. This consideration raises two questions. First, is the $4r$ bound on UC(c) itself overgenerous? And second, is the uncovered set itself only a bit smaller than UC(c) or is it considerably smaller? (A third question also arises, which we take up in Section 6: how large typically is the $4r$ bound relative to the configuration of ideal points?)

Consider again the regular pentagon configuration of ideal points in Fig. 6. Note that UC(c) falls short of the five ideal points but elsewhere protrudes somewhat beyond the edges of the Pareto pentagon. Of course, we know already that proximate covering by points on the Pareto frontier pares the uncovered set back to the edges of the pentagon. But, in fact, this proximate covering is redundant in that covering at a distance by other points in the vicinity of c pares down the uncovered set further, so that its boundary everywhere falls just short of the edges of the pentagon, as is also displayed in Fig. 6.[15]

Given configurations of a modestly large number (e.g., $n > 15$) of diverse ideal points, the boundaries of the uncovered set are determined entirely by covering at a distance, and UC(c) is substantially pared down by intersection with UC(x) for points x distinct from but in the vicinity of c. An example is provided in Fig. 7, which mimics the ideal point

[15]This and subsequent pictures of uncovered sets are *CyberSenate* approximations. The BJS algorithm gives a picture essentially identical to Fig. 6.

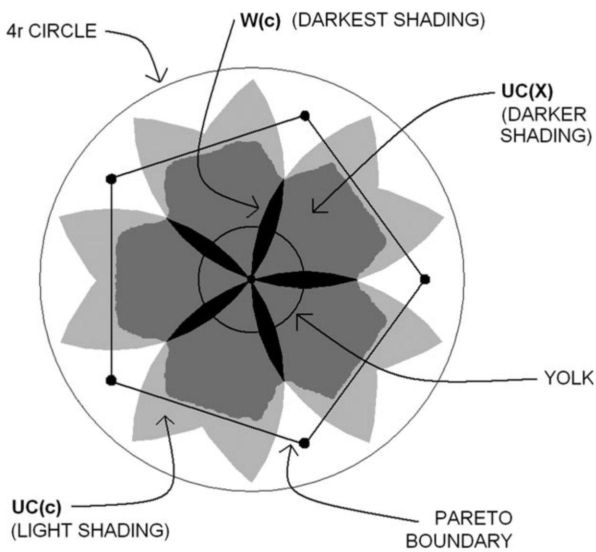

Fig. 6 *W*(*c*), UC(*c*), UC(*X*), and the 4*r* circle in a regular pentagon ideal point configuration.

configuration displayed in BJS Fig. 1 with the number of voters scaled-down to $n = 25$. Perhaps the most striking thing about the figure—and also (as they note) BJS Fig. 1—is that the uncovered set is considerably smaller than required by the 4*r* bound.[16] By showing the approximate boundary of UC(*c*), Fig. 7 suggests how the reduction from UC(*c*) to UC(*X*) occurs. It is evident that the 4*r* circular bound on UC(*c*) is not overly generous, in that UC(*c*) approaches the 4*r* circle at a number of points. But it is also evident that UC(*c*) is quite irregularly shaped, with points emanating from its central core that approach the 4*r* circle, although the central core itself has a radius of only about 2.5*r*. And, most significantly, the uncovered set UC(*X*) nowhere pushes beyond the relatively compact central core of UC(*c*).

Examination of this and many other configurations using *CyberSenate* indicates that what we see in Fig. 7 is typical and consistent with the following generalizations.

1. For points *x* close to *c*, the UC(*x*) sets, like UC(*c*), are irregularly shaped with points emanating from a central core.
2. The central cores of all such UC(*x*) sets substantially coincide, but their points emanate in offsetting directions.
3. Therefore, as the uncovered set UC(*X*) is formed from the intersection of such UC(*x*) sets, the points emanating from the cores of the individual UC(*x*) sets are snipped off, leaving an uncovered set that is the common core of all sets UC(*x*), which in turn is essentially the central core of UC(*c*).
4. Though not convex, the uncovered set is therefore considerably more compact than the individual UC(*x*) sets, with a boundary generally lying about 2*r* to 2.5*r* from *c*.

[16] The match with respect to the location, size, and even the polygon-like shape of the uncovered sets displayed in Fig. 7 and BJS Fig. 1 is also striking. UC(*c*) in Fig. 7 is an approximation created by forming the union of *W*(*z*) for a strategically selected sample of about 20 points *z* on the boundary of *W*(*c*).

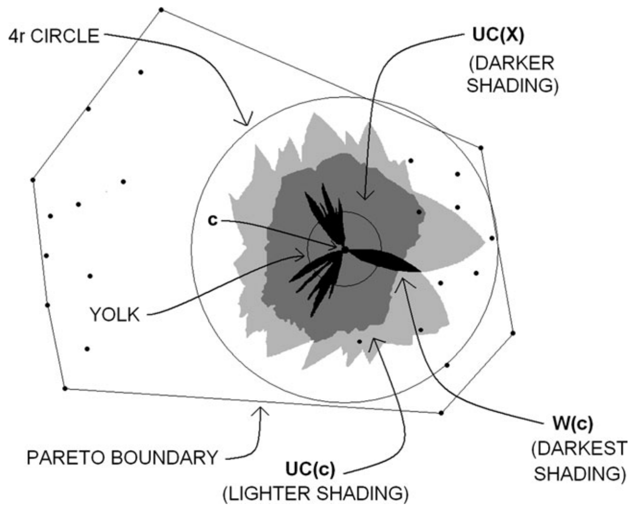

Fig. 7 The yolk, $W(c)$, UC(c), UC(X), and the $4r$ circle in scaled-down ($n = 25$) version of BJS Figure 1.

5. However, if most ideal points lie on or near the Pareto frontier, the "radius" UC(X) expands to about $3r$. This is most apparent in configurations in which ideal points form the vertices of a polygon, as in Fig. 6 and (more conspicuously) in Fig. 8.
6. Moreover, except where the boundary of the uncovered set is determined by proximate covering, points covered by a point x beyond about $0.5r$ from c are also covered by points closer to c, and therefore such sets UC(x) are irrelevant to demarcating the boundary of the uncovered set.

6 The Size and Location of the Yolk

We have just seen that the long established $4r$ bound on the uncovered set is overly generous. But a further question pertains to the typical size of the yolk (and thus of the $4r$ circle), relative to the Pareto set or some other measure of the span of the ideal point configuration. If the yolk is typically very small, the $4r$ circle is also quite small, but if the yolk is quite large, the $4r$ circle may be very large (perhaps larger than the Pareto set). Note also that the $4r$ bound makes a claim not only about the size of the uncovered set but also about its location. Thus, it is also important to know whether the yolk typically occupies a "central" location (e.g., relative to the Pareto boundary) or whether it may be substantially "off-center."

Moreover, the location and size of the yolk are important parameters of spatial voting games in their own right. It is worth remarking that, although BJS (and others) suggest that the work of Shepsle and Weingast (1984) highlights the role of the uncovered set in demarcating the boundaries of "enactability" in a sophisticated voting body, what Shepsle and Weingast actually focused on was not the uncovered set per se but on the (larger) set of points not covered by some status quo point q, that is, UC(q). Subsequently, Feld, Grofman, and Miller (1989) showed that the size of UC(q) depends on the size of the yolk. Indeed, almost all the "agenda propositions" in Feld, Grofman, and Miller refer to the size of the yolk and the set of points not covered by the status quo, but none refers to the uncovered set per se. Thus, an important side benefit of the BJS computational procedure

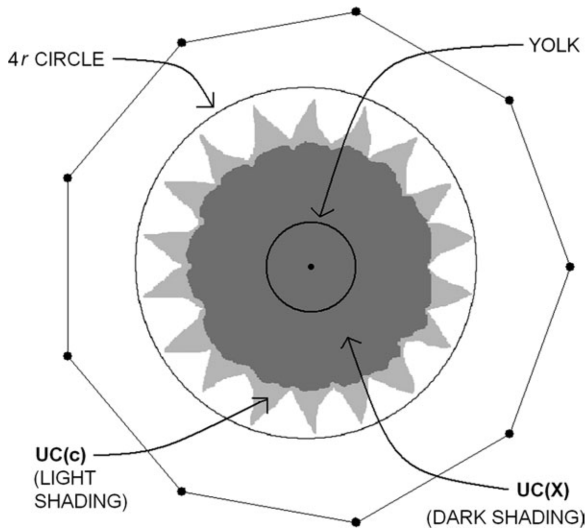

Fig. 8 The yolk, UC(c), UC(X), and the $4r$ circle in a regular polygon configuration ($n = 9$).

is that it can provide pictures of the yolk in large ideal point configurations. BJS claim (I believe accurately) in their footnote 14 that their Fig. 1 showing the yolk for a large configuration of ideal points is the first of its kind.[17]

As noted in Section 2, it is reasonably clear that the size of the yolk declines as the number and diversity of ideal points increase. We know from Tovey (1990) that, if ideal point configurations are random samples (with n odd) drawn from any centered (e.g., bivariate uniform or normal) continuous distribution, the expected size of the yolk approaches zero as the number of ideal points increases without limit. More intuitively, if the underlying distribution has a well-defined center, finite random samples drawn from it have imperfectly defined centers that become more perfectly defined as sample size increases.

But Tovey's theoretical result leaves two important questions open. The first concerns the *rate* at which the yolk shrinks as the number of voters increases. The second concerns the impact of "nonrandom" *clustering* within configurations of ideal points, such as we might expect to see in empirical ideal point data (and certainly do see in BJS Figures 1, 4, and 5), on the size and location of the yolk.

Previous simulations by Koehler (1990), subsequently extended by Hug (1999), together with more recent simulations by Bräuninger (2007), show that the expected size of the yolk declines quite rapidly as larger samples of ideal points are drawn out of a bivariate *uniform* distribution. However, the resulting ideal point configurations, with ideal points more or less "evenly" distributed over a square, look very artificial. Configurations drawn from a bivariate *normal* distribution appear considerably more "natural." *CyberSenate* can generate configurations of ideal points drawn randomly from bivariate distributions that are either uniform or normal, display all limiting median lines and the

[17]Though BJS say in the text (p. 261) and in the figure caption that Fig. 1 displays median lines, it actually does not do so. However, I have seen other BJS figures that do display (limiting) median lines for large-n configurations, and *CyberSenate* can produce similar figures.

yolk, and compute c and r. Using *CyberSenate*, I have computed yolk sizes for 456 ideal point configurations, half drawn from each type of distribution, with various n's (all odd) ranging from 3 to 435.

My results for uniform distributions are very similar to those produced by Koehler, Hug, and Bräuninger and are not displayed here. Results for the 228 configurations drawn from a bivariate normal distribution with an SD of 1 each in dimension are displayed in Fig. 9.[18] It is evident (and unsurprising) that yolk sizes are quite stable from sample to sample in large configurations but highly variable in small configurations. Nevertheless, it is clear that, once a low threshold of about $n = 9$ is crossed, the expected yolk radius shrinks as the number of voters increases, and given configurations of several hundred voters, the expected yolk radius is about one-quarter (and yolk area about 6%) of that for most small configurations.

More specifically, for $n = 101$ (e.g., the U.S. Senate) and with configurations randomly drawn from a bivariate normal distribution with an SD of 1 in each dimension, the expected yolk radius is about 0.11, whereas for $n = 435$ (e.g., the U.S. House), the expected yolk radius is a bit under 0.05. We can convert these estimates into yolk *areas* relative to the area of what we may call the *pseudo-Pareto set*, defined as the area of a circle centered on the underlying distribution from which the sample configurations are drawn and with a radius three times its SD. (Such a circle can be expected to enclose more than 99% of the ideal points in each sample configuration.) For $n = 101$, the yolk can be expected to occupy about 0.135%, and for $n = 435$ about 0.025%, of the pseudo-Pareto set. For larger n's, the expected yolk *radius* appears to follow an inverse square root law with respect to sample size (in the manner of sampling error more generally), so that the expected yolk *area* follows a simple inverse law.[19]

Applying the $2r$ rule on win sets implies that, given large "random" ideal point configurations, win sets of points at some distance from the yolk come very close to forming perfect circles. Figure 10 provides a *CyberSenate*-generated example for $n = 435$. It is clear that in such a configuration, the $4r$ bound on the uncovered set confines it to a tiny area within the (pseudo-)Pareto set. (More generally, configurations like that depicted in Fig. 10 lead one to wonder whether excessive ink has been spilled decrying the chaotic nature of majority rule in typical two-dimensional voting games.)

With respect to the second question, it is evident that nonrandom clustering of ideal points can considerably increase the expected size of the yolk. The configuration of 106th U.S. House ideal points in BJS Fig. 1 is far from a (typical) random draw out of an underlying bivariate normal (or uniform) distribution—rather it displays two distinct "clusters" of ideal points (evidently party groups) and a yolk that is considerably larger than would be expected if ideal points were more normally distributed. Putting a ruler to the diagram as it appears on the printed page, we find that the yolk has a diameter of about

[18]Note that the horizontal dimension in Fig. 7 is a log scale; the trend is given by a Lowess fit line. More samples were drawn for smaller configurations than for larger ones, both because computations take much more time for the large ones (about 10 min for each configuration with $n = 435$) and also because there is much more variability in yolk sizes in the small configurations. As might be expected, for a given n, expected yolk size is somewhat smaller when ideal points are drawn from a normal distribution rather than from a uniform distribution with the same SD.

[19]Although it is beyond the practical computing power of *CyberSenate* to work with electorate-sized n's, such yolk sizes can be estimated on the basis of the inferred inverse law. For n greater than about 100 drawn from a bivariate normal distribution, this law appears to be approximately RATIO = $0.12/n$, where RATIO is the expected yolk area divided by the pseudo-Pareto area. In fact, given large n's, the true Pareto area will (almost certainly) be considerably larger than the pseudo-Pareto area because large samples of ideal points will (almost certainly) include extreme outliers (well beyond three SDs from the mean), so the ratio between expected yolk area and actual Pareto area will be even smaller.

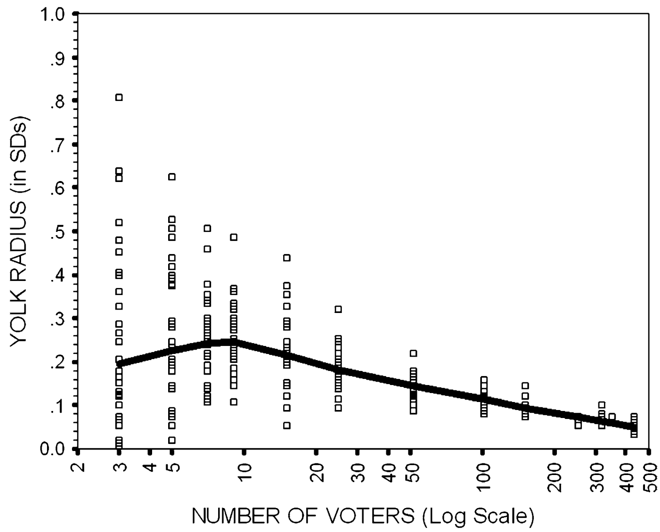

Fig. 9 Expected yolk radius by number of ideal points, with $n = 3$ through $n = 435$ drawn randomly from a bivariate normal distribution with SD = 1.

five-sixteenths of an inch and the (approximately circular) Pareto set has a diameter of about 3 inches, so the yolk occupies about 1.2% of the Pareto set—roughly 45 times the area it would be expected to occupy in the event ideal points were normally distributed. In the scaled-down version of BJS Fig. 1 presented earlier in Fig. 7, the yolk occupies a bit over 3% of the Pareto set, about eight times the area expected in a configuration of 25 normally distributed ideal points.

With respect to the location of the yolk, it is worth first noting that there is no reason to expect that the center of the yolk will coincide with either the "center" of the policy space

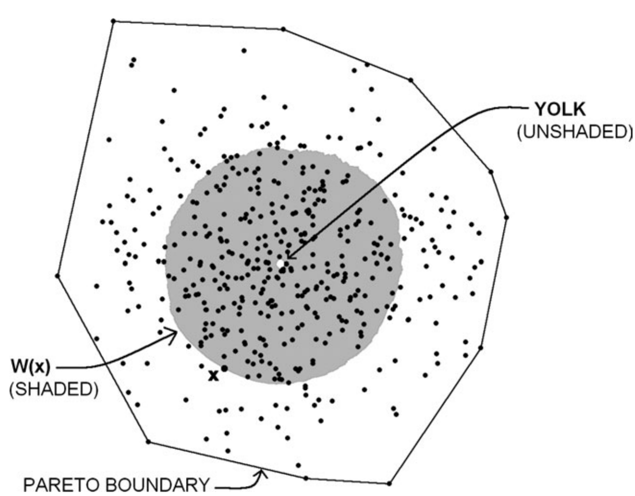

Fig. 10 A highly circular win set and tiny yolk with $n = 435$ drawn from a bivariate normal distribution.

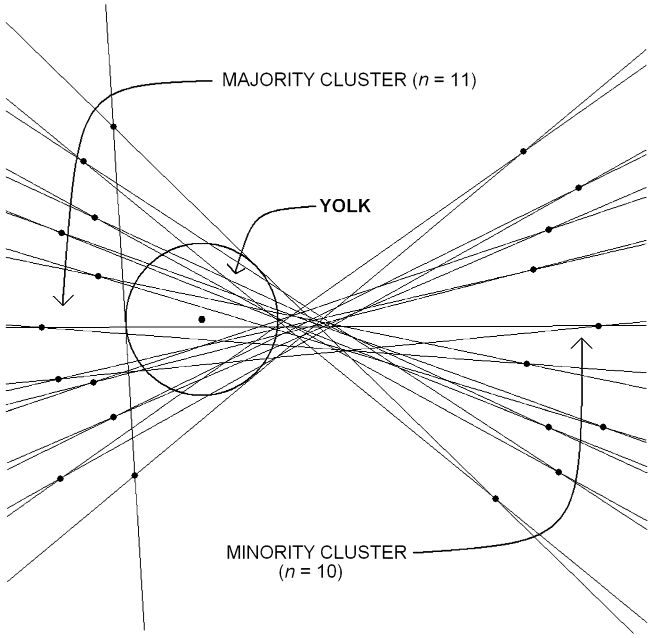

Fig. 11 Limiting median lines and the location and size of the yolk with two closely balanced clusters of ideal points.

(however that might be defined) or with the "center of gravity" (i.e., mean) of the configuration of ideal points. Rather the (center of the) yolk, defined with respect to median lines, indicates the generalized center of the configuration of voter-ideal points in the sense of the median. Although we generally think of the median as a "stable" measure of central tendency, we should also remember (as BJS do not in their footnote 28) that there is one circumstance in which the median is highly "unstable" and (unlike the mean in the same circumstance) shifts radically in response to small changes in the overall distribution. In the one-dimensional case, this occurs when data are polarized into two quite widely separated clusters of nearly equal size. The same phenomenon arises in two dimensions, as is illustrated in Fig. 11, which shows two clusters of $n = 11$ and $n = 10$ ideal points that overlap in the vertical dimension but are polarized in the horizontal dimension (much like BJS Figs. 1, 4, and 5). Note that almost all limiting median lines form a "bow tie" pattern, all passing through a small area about halfway between the two clusters. This might suggest that the yolk lies in this small central region, but there must be at least one additional median line that lies more or less vertically along the centrist face of the majority cluster, with the entire minority cluster and the empty space between them on the other side. Since it intersects all median lines, the yolk must lie within the majority side of the bow tie (which essentially forms the yolk triangle) and be nestled against the centrist face of the majority cluster. (If the vertical spread of ideal points is compressed relative to the horizontal polarization, the yolk is even more conspicuously pushed in the majority direction.) In such circumstances, therefore, the yolk is not centrally located within the configuration of ideal points. Moreover, it is evident that if just enough ideal points switch clusters to reverse their majority versus minority status, the yolk likewise flips from one side of the bow tie to the other.

Finally, we can combine these yolk size estimates with our previous conclusions concerning the size and location of the uncovered set—namely, that it is approximately centered on the yolk with a radius of somewhat over $2r$. For $n = 101$, with configurations randomly drawn from a bivariate normal distribution, the expected radius of the uncovered set is therefore about 0.25, where the SD (in each dimension) of the distribution is 1, so the uncovered set can be expected to occupy about 0.7% of the pseudo-Pareto set. For $n = 435$, the expected uncovered set radius is about 0.12, so UC(X) can be expected to occupy about 0.2% of the pseudo-Pareto set. The clustered configurations of Fig. 7 and BJS Fig. 1 produce uncovered sets that occupy about 17% and 5% of the (actual) Pareto sets, respectively.

7 Conclusion

I conclude by doing four things. First, I elaborate a bit on the interaction between proximate and distant covering. Second, I examine BJS Fig. 2 in light of theoretical considerations previously identified. Third, I assess the three theoretical claims made by BJS cited in Section 2. Fourth, I offer some conjectures concerning the structure of the uncovered set in the spatial context.

The relative importance of proximate versus distant covering in demarcating the uncovered set depends on the size and diversity of ideal point configurations. In a small-n configuration or a large-n configuration with a small number of tight clusters (that effectively take us back to the small-n case), the boundary of the uncovered set is determined largely (and in one limiting case entirely) by proximate covering.

Consider the three-voter case fully analyzed by Hartley and Kilgour (1987). In the event that the three ideal points form the vertices of an equilateral triangle (as shown in Fig. 12A), UC(c) everywhere extends beyond the Pareto triangle to a maximum of about $3.6r$ from c, so the $4r$ bound on UC(c) is not greatly overgenerous. But all covered points are proximately covered, so the boundary of the uncovered set nowhere results from covering at a distance by c or any other point, with the result that it coincides with the Pareto triangle and the $4r$ bound is greatly overgenerous. As the Pareto triangle departs from equilateral perfection, distant covering begins to exclude points near the more extreme vertex (Fig. 12B) or vertices (Fig. 12C) from the uncovered set. Indeed, as the Pareto triangle becomes sufficiently skewed, the $4r$ bound itself excludes a portion of the Pareto triangle from the uncovered set, but UC(c) by itself puts much tighter bounds on UC(X) within the Pareto set. On the other hand, distant covering by other points near c pares UC(c) down only slightly, so UC(X) is only slightly smaller than the intersection of UC(c) and the Pareto set, as is dramatically true in Fig. 12B and less dramatically true in Fig. 12C.

As the number and diversity of voters increase, the size of the yolk shrinks relative to the Pareto set, so our approximate $2.5r$ bound on the uncovered set quickly lies wholly within the Pareto set and the uncovered set boundary is determined entirely by distant covering (as is true in Figs. 6, 7, and 10). (Of course, as n further increases, even the $4r$ circle lies wholly within the Pareto set, as is true in BJS Fig. 1, almost true in Fig. 7, and dramatically true in Fig. 10.)

However, if collinearities render some voters invisible, proximate covering may again become relevant. This theoretical consideration accounts for the distinctive features of the uncovered sets displayed in the several panels of BJS Fig. 2—in particular, their predominant straight-line boundaries. Let us label the leftmost ideal point 1 and, proceeding clockwise, label the others 2 through 4 and label the central point 5.

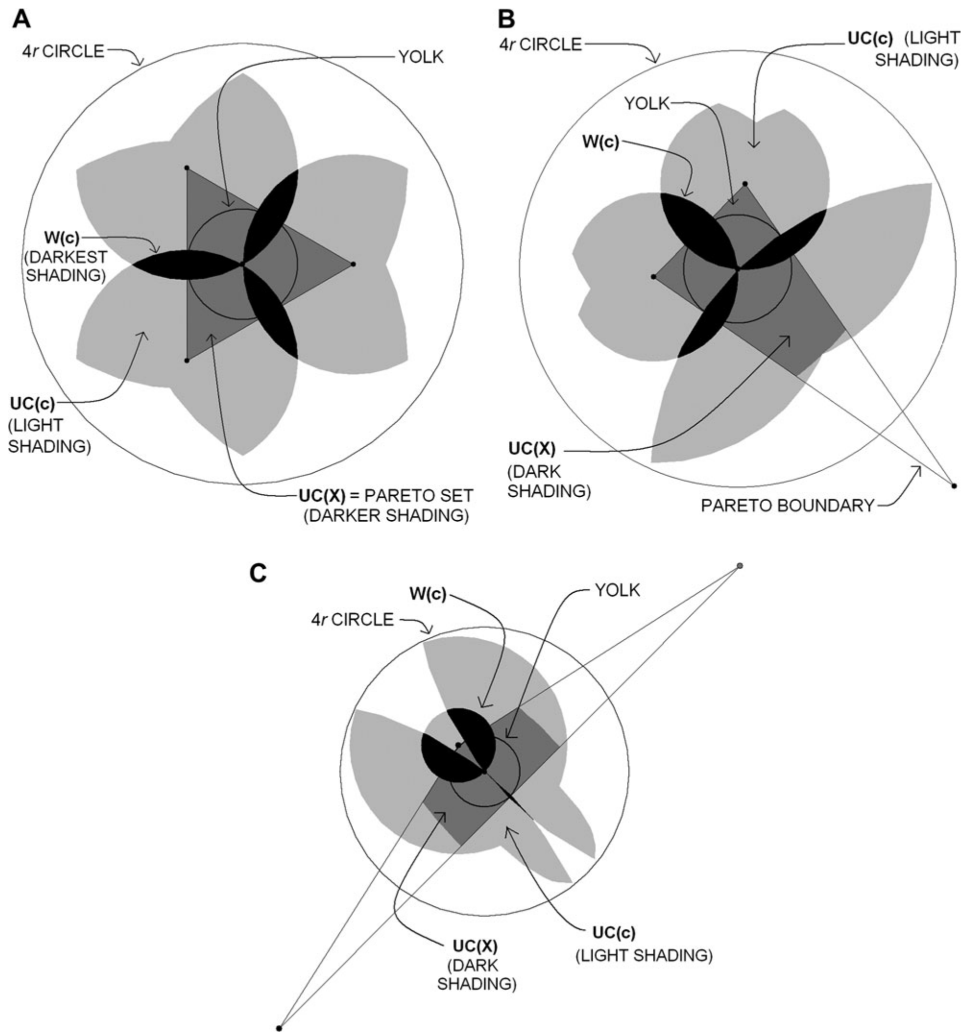

Fig. 12 (A) UC(X) and UC(c) versus the $4r$ circle in an equilateral Pareto triangle where the uncovered set boundary is demarcated entirely by proximate covering. (B) UC(X) and UC(c) versus the $4r$ circle in a highly acute Pareto triangle. (C) UC(X) and UC(c) versus the $4r$ circle in a highly obtuse Pareto triangle.

Note that, because of the contrived nature of the configurations, collinearities exist in every panel. The first panel exhibits full Plott symmetry. Voters 1, 5, and 3 lie on one stand-alone limiting median line, and voters 2, 5, and 4 lie on another. Since no other median lines pass through 1, 2, 3, or 4, all voters other than 5 are invisible, and the effective Pareto set coincides with ideal point 5, which is therefore the unique uncovered point.

In the second through fifth panels, the collinearity of 2, 5, and 4 is upset by voter 2's clockwise rotation toward voter 3, but voters 1 and 3 remain invisible and continue to play no role in demarcating win sets. (In the final panel, ideal points 2 and 3 coincide, so either 2 or 3 may be deemed invisible along with voter 1.) Figure 3A and 3B were constructed to duplicate these intermediate panels of BJS Fig. 2, in which the triangle with vertices at ideal points 2, 4, and 5 forms the visible Pareto set. It can be checked that in each of these panels the uncovered set displayed in BJS Fig. 2 fits within this triangle.[20]

Finally, in none of the panels of BJS Fig. 2 does the uncovered set, as demarcated by their grid-search algorithm, fill up the visible Pareto set triangle. This is because distant covering is also at play. Within the visible Pareto triangle in the various panels of BJS Fig. 2, we observe the pattern of distant covering in the three-voter case analyzed by Hartley and Kilgour (1987), reviewed just above, with an obtuse Pareto triangle as depicted in Fig. 12C.

Let us now review the three BJS (pp. 270–271) claims summarized in Section 2. These claims all appear to be accurate, and they follow from the theoretical first principles I have tried to elucidate here.

What leads BJS to their first claim concerning unexpectedly large uncovered sets is actually that the clustering of ideal points found in their data produces a yolk "much larger than our expectations based on conventional wisdom and previous work" (though it needs to be added that there was little relevant previous work). Indeed (and consistent with the findings outlined in Section 5), their work, if anything, indicates that the *size of the uncovered set relative to the yolk* is much smaller than expectations based on conventional wisdom and previous work suggested. In particular (and as BJS note on p. 261 with respect to their Fig. 1), the familiar McKelvey $4r$ circular bound on the uncovered set is overly generous (even when proximate covering plays no role in demarcating its boundary).

With respect to the second BJS claim that the uncovered set may not be centrally located, we have found that the uncovered set is (more or less) centered on the yolk but that the yolk, being a generalization of the median, need not coincide with the center of the ideal point configuration. In BJS Figs. 1, 4, and 5, the two-party clusters largely overlap with respect to the vertical dimension but are highly polarized with respect to the horizontal dimension.[21] As we saw in Fig. 11, given ideal points polarized into two clusters in this manner, almost all median lines form into a bow tie pattern intersecting approximately midway between the clusters; however, at least one median line lies along the centrist face of the majority cluster, and the yolk must intersect it. Thus, the yolk lies within the majority side of the bow tie pattern and is nestled against the centrist face of the majority cluster. We have also seen that the uncovered set is approximately centered on the yolk and has a radius of about twice that of the yolk. Thus, one side of the uncovered set penetrates

[20]However, the BJS algorithm entails approximation, which shows up in microfeatures of their Fig. 2. The uncovered set boundaries that actually coincide with the edges of the triangle formed by ideal points 2, 4, and 5 appear to bulge a bit beyond them. This results because points slightly outside the triangle are (in the underlying continuous policy space) proximately covered by points nearer the edge of the triangle, but the latter points may not appear in the finite grid even when some of the former ones do. (BJS Theorem 2 tells us that this problem will diminish as the grid is refined.) For the same reason, the uncovered set boundaries appear everywhere to be slightly irregular, and the true shape of the very small uncovered set in the second panel is pretty much hidden from view. We can also consider two panels that would be added to BJS Fig. 2 if we allow point 2 to rotate beyond 3 into the southeast quadrant. Whereas points 1, 5, and 3 remain collinear, and the line through them remains a stand-alone limiting median line, it may be checked that new nonlimiting median lines now pass through both 1 and 3. As a result, voters 1 and 3 are no longer invisible, and the visible Pareto set expands to coincide with the full Pareto set. Finally, if we allow the wandering point 2 to continue in its path until it coincides with point 4, it may be checked that this turns the line through 4 (and 2) and 5 into a second stand-alone limiting median line and that now no nonlimiting median lines pass through 5, so that the initially "all-powerful" voter 5 is rendered invisible (along with one of the two coinciding points 2 and 4). Though this configuration again presents us with two invisible voters, the visible Pareto set coincides with the full Pareto set because neither invisible voter uniquely lies at a vertex of the Pareto set.

[21]The overlap and polarization are almost perfect if the axis system is rotated about 20° counterclockwise. Note that all concepts and analyses presented here (and by BJS) are independent of the axis system, which can be rotated in any fashion without affecting any conclusions. I refer to the dimensions in the BJS figures as "horizontal" and "vertical" as if this rotation had taken place.

well into the majority cluster itself, whereas the other side extends approximately to the midpoint between the two clusters. This is pretty much what we observe in BJS Figs. 1, 4, and 5 and evidently accounts for the findings BJS cite in their footnote 26.

With respect to the third BJS claim concerning the sensitive dependence of the size and location of the uncovered set on particular features of the ideal point configuration, the underlying theoretical principles pertaining to BJS Fig. 2 have already been elucidated. With respect to BJS Figs. 4(a) and 5 and the "flipping" of the uncovered set from one cluster to the other, we need simply to recall from Fig. 11 that, if the two clusters are closely balanced in size and if a few ideal points are deleted from the former and/or added to the latter so that the clusters exchange majority and minority status, the yolk flips from one side of the bow tie pattern to the other. Since it tracks the yolk, the uncovered set does likewise.

We may note one additional point concerning the panels of BJS Fig. 5: the uncovered sets appear to be noticeably larger in the 1949–1970 period than in the 1979–2000 period. We can provide an explanation for this difference that combines theoretical principles with political context. In the earlier (pre-Southern realignment) period, the Democratic (majority) cluster is considerably more spread out in the (more or less) vertical dimension than the Republican minority cluster. This produces an imbalance in the bow tie pattern of median lines such that it is especially wide on the Democratic majority side, with the result that the size of the yolk (and uncovered set) increased. Alternatively, we may think of the 1949–1970 House as a three-party/cluster system of Northern Democrats, Southern Democrats, and Republicans, none of which was of majority size. Thus, many median lines pass through two clusters but entirely miss the third. The yolk therefore is very large, rather resembling the yolk in a typical small-scale three-voter configuration (such as Fig. 12A). On the other hand, if the more homogenous Republicans had ever constituted a majority in this era, the same logic would imply that the yolk and uncovered set would have been smaller and more distinctly skewed in the Republican direction.

I conclude with some conjectures concerning the structure of the uncovered set that are supported by considerable work with *CyberSenate* but remain relatively speculative.

7.1 *The Shape of the Uncovered Set*

The uncovered set is not convex, but it is starlike relative to c and other points in the vicinity of c. Given a random configuration of a modestly large number of ideal points, the uncovered set is approximately circular but with flattish segments on its boundary. Given nonrandom clustering of ideal points, the uncovered set seems to assume a more distinctively polygon shape, as seen in Fig. 6 and BJS Fig. 1. Of course, the boundary of the uncovered set is formed out of straight-line segments wherever it is determined by proximate covering.

7.2 *The Boundary of the Uncovered Set*

Boundaries of win sets are everywhere formed out of segments of individual indifference curves. In a small ideal point configuration, the boundary of a set UC(x) of points not covered by x is formed out of win sets (and individual indifference curves); it therefore has kinks but is otherwise smooth. In the general case, however, portions of the boundary of UC(x) do not correspond to win set boundaries; rather as a point z travels along some portion of the boundary of $W(x)$, the tip of $W(z)$ traces out a portion of the boundary of UC(x). The boundary of the uncovered set itself is nowhere formed by win set boundaries. As it is the intersection of an infinite number of sets UC(x), its boundary (within the visible

Pareto set and as demarcated by distant covering) appears to be created by a continuous mapping from points on a somewhat complex loop in the vicinity of the center of the yolk to points on the UC(X) boundary, such that each former point covers the corresponding latter point at a distance and is the least distant point to do so. Such a mapping probably produces a smooth boundary on the uncovered set (despite its somewhat ragged appearance in both BJS and *CyberSenate* approximations).[22] Once again, where the effective Pareto frontier forms the boundary of UC(X), the boundary is composed of straight-line segments, possibly producing kinks in the boundary.

7.3 *The Internal Structure of the Uncovered Set*

An uncovered set is composed of a "central nucleus" and an "outer shell." The central nucleus includes all points each of which uniquely covers some other point at a distance and, in particular, uniquely covers a point on the boundary of the uncovered set. This central nucleus appears to lie within about $0.5r$ of the center of the yolk and contains the loop described in the preceding paragraph.[23] The outer shell is composed of all other uncovered points, which themselves cover only points beyond the boundary of UC(X) also covered by points in the central nucleus. Thus, the points in the outer shell belong to the uncovered set, not because they do any "essential" covering but only because they are not themselves covered by points in the central nucleus. However, if the (visible) Pareto frontier forms part of the boundary of UC(X) through proximate covering, points outside but neighboring that part of the boundary of UC(X) may not be covered at a distance by points in the vicinity of the center of the yolk. Rather they are proximately covered by neighboring points lying on the boundary of UC(X) (and the visible Pareto frontier) and also are covered at a small distance by points just inside the boundary UC(X).

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[22]Hartley and Kilgour (1987) have pinned this down in the case of $n = 3$. They show that there is a continuous mapping between a sequence of points in the vicinity of the center of the yolk and the points they cover at a distance that form the boundary of the uncovered set within the Pareto triangle and that the resulting portion of the boundary takes the form of an ellipse with foci at two ideal points.

[23]See BJS's second approach for enhancing the efficiency of their algorithm (p. 275). This consideration suggests that the uncovered set is starlike with respect to all points in the "nucleus" and that it therefore is also "connected" in the sense conjectured by BJS in their footnote 17.

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