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The structure of the Banks set*

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Abstract. We consider a new solution set for majority voting tournaments recently proposed by Banks (1985), and we examine its internal structure. In particular, we demonstrate that, in the absence of a Condorcet winner, there is always a cycle including precisely the points in the Banks set. We introduce the concept of "external stability" in order to facilitate analysis.

1. Introduction

In a recent paper, Banks (1985) provides an exact characterization of the set of alternatives that a committee may adopt operating under the standard amendment procedure when its members vote in a sophisticated fashion. Banks demonstrates that this set of alternatives – which we dub the *Banks set*, the elements of which are defined in terms of a construction we call a *Banks trajectory* – is a subset, sometimes proper, of the uncovered set, which as Miller (1980) previously demonstrates, contains the set of possible sophisticated outcomes under the amendment procedure.

Elsewhere (Miller, Grofman and Feld, forthcoming) we argue that Banks trajectories have more general relevance for committee voting and that the Banks set constitutes a solution set of broad significance for voting processes. In our other paper, we examine four apparently disparate voting processes and show that each generates, either analytically or behaviorally, Banks trajectories in the set of available alternatives and that each produces as outcomes alternatives in the Banks set. One of the four processes is sophisticated voting under the amendment procedure. The second is what we dub the "cycle avoiding sincere process", which we view as a quasi-cooperative model of voting in a small committee. The third is an open, "backwards built" (cf. Shepsle and

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Weingast, 1984), and strategic agenda building process. The last is the cooperative voting model presented in Miller (1980).

Here we consider the properties of Banks trajectories in more detail and then investigate the internal structure of the Banks set. In particular, we demonstrate that, while the top cycle of the uncovered set may be a proper subset of the uncovered set (Miller, 1983), the top cycle of the Banks set is always the Banks set itself; put otherwise, in the absence of a Condorcet winner, there is always a cycle including precisely the points in the Banks set. Along the way, we introduce the notion of "external stability" and exploit its relationship to Banks trajectories and the Banks set.

2. Preliminaries

We follow Banks by working within the conventional setup for finite voting games established by Black (1958) and Farquharson (1969) and followed also by Miller (1977, 1980), McKelvey and Niemi (1978), Bjurulf and Niemi (1982), and others. In particular, we assume that we can represent by a tournament (that is, a complete asymmetric digraph) the set of alternatives available for choice, together with the majority preference relation over the alternatives. This means that the set of alternatives – which we will henceforth refer to as "points" – is finite, and that there are no ties in majority preference between distinct alternatives. (This would be the case, for example, if an odd number of voters all had strong preferences.) Otherwise we assume no structure concerning voter preferences.

Let X designate the finite set of all points, m in number. Let x, y, and so forth, or x_1, x_2, and so forth, designate individual points, and let P designate the complete asymmetric majority preference relation. Thus, x P y means that x is majority preferred to y; we say "x *beats* y." Let W(x) designate the *win set* of x, that is, the set of points each of which beats x. Let D(x) designate the *dominion* of x, that is, the set of points each of which x beats.

We briefly review some elementary concepts and results.

The *top cycle* set, $X^* \subseteq X$, is a non-empty subset of X such that: (i) for all $x \in X^*$ and $y \in [X-X^*]$, x P y; and (ii) no proper subset of X^* meets condition (i). The set X^* always exists, and in a tournament it is unique. If there is a *Condorcet winner* (a point x^* that beats every other point in X), it is the unique element of X^*. Otherwise, X^* includes at least three points and there is a cycle of majority preference including precisely the points in X^*. (See Miller, 1977.) Let $m^* \leq m$ designate the number of points in X^*.

It turns out that Banks trajectories and the Banks set are closely related to the covering relation and the uncovered set introduced in Miller (1980). There Miller defines the *covering relation* in this way: x C y if and only if $D(y) \subseteq D(x)$;

in words, x covers y if and only if x beats everything y beats.[1] Let $UC(X) \subseteq X$ designate the *uncovered set* of X – that is, $x \in UC(X)$ if and only if there is no $y \in X$ such that $y\,C\,x$. Always $UC(X) \subseteq X^*$. If there is a Condorcet winner, it is the unique element of UC(X). Otherwise, UC(X) includes at least three points in a cycle (Miller, 1980, 1983).

Now consider this construction. We pick an arbitrary point x_1 in the tournament. We next pick a point x_2 that beats x_1. We then pick a third point x_3 that beats both x_2 and x_1. Proceeding in this manner, we construct a "cycle avoiding trajectory." More formally, a *cycle avoiding trajectory* is an ordered set of points $\langle x_1, x_2, x_3, \ldots, x_k \rangle$ such that $x_h\,P\,x_g$ if and only if $1 \leq g < h \leq k$. Let H(x) designate any such trajectory with *top* element x, that is, such that x beats everything else in the trajectory.

Suppose that we continue to construct a cycle avoiding trajectory until we can proceed no further – that is, until the top element of the trajectory is x_k and there is no point in the set X that beats x_k and all the points below x_k in the trajectory. We call a cycle avoiding trajectory that we cannot expand upwards a *Banks trajectory.* Let $H^+(x)$ designate a Banks trajectory with top element x. We call the top element of a Banks trajectory a *Banks point.* The *Banks set* B(X) of points consists of all Banks points.[2]

A set of points $X' \subset X$ is *externally stable* in X if and only if, for every $y \in [X-X']$, there is some $x \in X'$ such that $x\,P\,y$; in words, for every point outside the set there is some point inside the set that beats it, or – perhaps putting the matter more straightforwardly – there is no point outside the set that beats every point inside the set.[3] We can thus characterize a Banks trajectory as an acyclic externally stable set.

A set of points $X' \subset X$ is a *minimal externally stable* set if and only if X' is externally stable and no proper subset $X'' \subset X'$ is externally stable – that is, if every point in X' is essential for the external stability of X', in the sense that the removal of *any* point z from X' allows some point in $[X-X'] \cup \{z\}$ to beat all points in $X'-\{z\}$.[4]

We now have this useful lemma.

Lemma 1. For every $y \in X'$, where X' is a minimal externally stable set, either
(1) $y\,P\,z$ for all $z \in [X'-\{y\}]$; or
(2) there is some $v \in [X-X']$ such that $y\,P\,v$ and $v\,P\,z$ for all $z \in [X'-\{y\}]$.

In words, each point in a minimal externally stable set X' either (1) beats every other point in X' (obviously at most one point in X' can do this) or (2) beats some point outside X' that in turn beats every other point in X'. Note that (2) can be restated as follows: each point in a minimal externally stable set X' uniquely (with respect to X') beats some point outside X'.

This lemma follows essentially immediately from the definitions. Consider any minimal external stable set X′ and suppose that, contrary to the lemma, X′ includes some point y that does not meet either condition (1) or (2). We show that this leads to a contradiction. Because X′ is externally stable, every point outside X′ is beaten by some point in X′. Because y does not meet condition (2), every point outside X′ beaten by y is also beaten by some other point in X′. And because y does not meet condition (1), some point in $X' - \{y\}$ beats y. But then $X' - \{y\}$ is externally stable, so X′ cannot be a minimal externally stable set.

A *minimal Banks trajectory* is a Banks trajectory that is a minimal externally stable set. Let $H^+_{min}(x)$ designate a minimal Banks trajectory with top element x. Notice that x meets condition (1) in Lemma 1, so every other point in $H^+_{min}(x)$ meets condition (2). Thus, we can restate Lemma 1 as it applies to a minimal Banks trajectory.

Lemma 1′. For every $y \in [H^+_{min}(x) - \{x\}]$, there is some $z \in [X - H^+_{min}(x)]$ such that y P z and z P v for all $v \in [H^+_{min}(x) - \{y\}]$.

In words, every point in a minimal Banks trajectory, with the possible exception of its top point, uniquely (with respect to elements of the trajectory) beats some point outside the trajectory.

Clearly, if x C y, y cannot be a Banks point. Suppose that y were at the top of some cycle avoiding trajectory. On the one hand, x cannot be in the trajectory below y, because x P y. So x must be outside of the trajectory. On the other hand, x beats y and everything below y in the trajectory. So the trajectory is not externally stable, and it is not a Banks trajectory. Thus we get the result that Banks (1985) first provides:

Theorem 1. $B(X) \subseteq UC(X)$.

In words, every Banks point is uncovered. We further can show that the Banks set is identical to the uncovered set whenever $m^* \leq 6$ (Miller, Grofman and Feld, 1986) but the inclusion may be proper if $m^* \geq 7$. (See Figure 1, adapted from Moulin, 1986, where $UC(X) = \{x_1, x_2, x_3, x_4, x_5\}$ and $B(X) = \{x_1, x_2, x_3, x_4\}$.) Further, Banks (1985) showed that B(X) must be contained within the top cycle of UC(X). The top cycle UC(X) can be a proper subset of UC(X), however, only if $m^* \geq 7$ (Miller, Grofman and Feld, 1986). And in turn B(X) can be a proper subset of the top cycle of UC(X) only if the top cycle of UC(X) contains five or more points (Miller, Grofman and Feld, 1986).

3. The structure of the Banks set

We turn now to consider the internal structure of the Banks set.

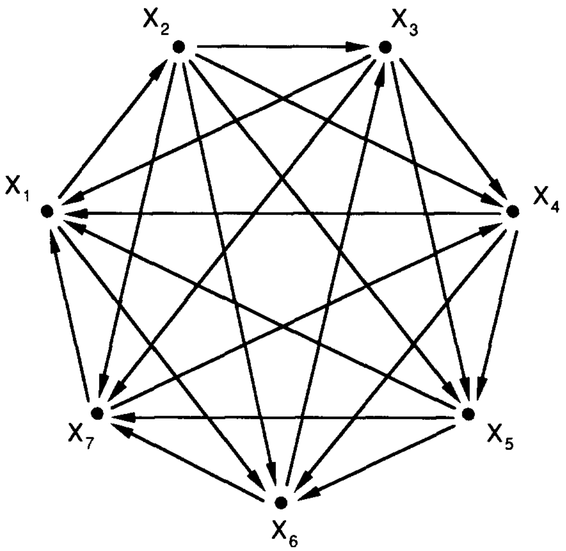

Figure 1. An illustrative tournament.

Lemma 2. If $|B(X)| > 1$, for every $x \in X$ there is some $y \in B(X)$ such that $y \, P \, x$.

In words, if the Banks set contains more than one point (which is to say, if there is no Condorcet winner), every point in X is beaten by some point in the Banks set. One important implication of this lemma is that, in the absence of a Condorcet winner, every point in the Banks set is beaten by some other point in the Banks set.

Consider any $x \in X$. Because there is no Condorcet winner, $W(x) \neq \Phi$. Take any $z \in W(x)$ and form the cycle avoiding trajectory $\langle x, z \rangle$. If $\langle x, z \rangle$ is not a Banks trajectory, extend the trajectory upwards until we create a Banks trajectory with top element y. (If $\langle x, z \rangle$ is a Banks trajectory, then $y = z$.) By definition of the Banks set, $y \in B(X)$. And by acyclicity, $y \, P \, x$.

From this, it follows that the Banks set as a whole shares one property of any Banks trajectory:

Theorem 2. B(X) is an externally stable set.

If there is a Condorcet winner x^*, then x^* is the unique Banks point and by definition x^* beats every other point. If there is no Condorcet winner, Theorem 2 follows immediately from Lemma 2.

While the Banks set is externally stable, it cannot be a minimal externally stable set in the absence of a Condorcet winner.

Theorem 3. If $|B(X)| > 1$, B(X) is never a minimal externally stable set.

Suppose that we remove any point z from B(X); by Lemma 2 we know that there is some $y \epsilon B(X)$ such that y P z. Suppose that $[B(X)-\{z\}]$ is not externally stable. This means that there must be some $w \notin B(X)$ such that, while z P w, w P v for all $v \epsilon [B(X)-\{z\}]$ and, in particular, w P y. Consider the cycle avoiding trajectory $\langle y, w \rangle$; this cannot be a Banks trajectory because the top point is w and $w \notin B(X)$. Extend the cycle avoiding trajectory upwards until it become a Banks trajectory with top element u. By definition $u \epsilon B(X)$, and certainly $u \neq z$ because y P z; but by acyclicity u P w, contradicting the supposition that w P v for all $v \epsilon [B(X)-\{z\}]$ and contradicting also the supposition that $[B(X)-\{z\}]$ is not externally stable. Because the argument holds for any $z \epsilon B(X)$, B(X) cannot be a minimal externally stable set.

Corollary 3.1. If $|B(X)| > 1$, for every $z \notin B(X)$, $|W(z) \cap B(X)| \geq 2$.

In words, every point outside of the Banks set is beaten by at least two Banks points.

Lemma 3. If $|B(X)| > 1$, for every $x \epsilon B(X)$ there is some other $y \epsilon B(X)$ such that x P y.

In words, if the Banks set contains more than one point, every point in the Banks set beats some other point in the Banks set. Thus, Lemma 3 parallels the implication of Lemma 2 that, if the Banks set contains more than one point, every point in the Banks set is beaten by some other point in the Banks set.

Given $|B(X)| > 1$, consider any $x \epsilon B(X)$ and some minimal Banks trajectory $H^+_{min}(x) = \langle y_1, \ldots, y_k, x \rangle$. We show that x beats some other Banks point. Let $z \epsilon [X-H^+_{min}(x)]$ be a point outside of $H^+_{min}(x)$ uniquely (with respect to elements of $H^+_{min}(x)$) beaten by y_k. We know from Lemma 1′ that such a point exists. Now form the ordered set $\langle y_1, \ldots, y_{k-1}, z, y_k \rangle$. Clearly this is a cycle avoiding trajectory; we consider whether it is also a Banks trajectory. Given its tournament structure, the whole set X of points may be partitioned thus: $\{x\}$, W(x), D(x). Whatever point we might place on top of this cycle avoiding trajectory we must draw from one of these three subsets of X. But we cannot place x on top, because y_k is the only element of $H^+_{min}(x)$ that beats z, so z P x. And we can place on top of the cycle avoiding trajectory no element v of W(x) outside of the trajectory, because $H^+_{min}(x) = \langle y_1, \ldots, y_k, x \rangle$ is a Banks trajectory and thus is externally stable, which means for any such v there is some $y \epsilon \{y_1, \ldots, y_k\}$ such that y P v. So either $\langle y_1, \ldots, y_{k-1}, z, y_k \rangle$ is a Banks trajectory or we can place some other element(s) of D(x) on top of y_k. But in either event x beats the top element of a Banks trajectory. So x beats something else in the Banks set.

If the Banks set shares one property of a Banks trajectory – namely, external stability – the other defining property of a Banks trajectory – namely, acyclicity – certainly is not shared by the Banks set. Let $B^*(X)$ designate the top cycle of the Banks set, that is, the minimal non-empty subset of $B(X)$ such that, for all $x \in B^*(X)$ and $y \in [B(X)-B^*(X)]$, $x \, P \, y$. We can now state the principal result.

Theorem 4. Always $B^*(X) = B(X)$.

In words, the top cycle of the Banks set is always the Banks set itself. Equivalently, if the Banks set has more than one point (that is, in the absence of a Condorcet winner), there is always a cycle containing precisely the points in the Banks set.

If there is a Condorcet winner, so $|B(X)| = 1$, the proposition is immediate. Suppose, however, that $|B(X)| > 1$.

By Lemma 2 the Banks set can have no "top point," that is, no single point that beats everything else in the Banks set. Thus, it must have a top cycle of three or more points. Likewise, by Lemma 3 the Banks set can have no "bottom point," that is, no single point beaten by everything else in the Banks set. Thus, it must have a "bottom cycle" of three or more points, that is, a minimal subset $B^-(X) \subseteq B(X)$ such that for every $x \in [B(X)-B^-(X)]$ and every $y \in B^-(X)$, $x \, P \, y$.

Now, there are two possibilities: the top and bottom cycles intersect, or they are disjoint. If they intersect, then by Black's (1958: 48–49; also see Miller, 1977: 777) Lemma – which states that in a tournament, if two cycles intersect, there is a cycle including precisely the points in both – the top cycle and the bottom cycle are the same cycle, and we have established that there is a single cycle including all the points in B(X).

Suppose, however, that the top and bottom cycles are disjoint. We show that this leads to a contradiction.

Consider any $x \in B^-(X)$. Because $x \in B(X)$, we can form some Banks trajectory with top element x and, in particular, some minimal Banks trajectory $H^+_{min}(x)$. Let $B = [B(X)-B^-(X)]$, that is, B is the set of points in the top cycle of B(X) (disjoint from $B^-(X)$) plus any other Banks points "between" the top and bottom cycles (each beaten by every point in the top cycle and beating every point in the bottom cycle). Clearly, $H^+_{min}(x)$ is disjoint from B, because every point in B beats x. At the same time $H^+_{min}(x)$ cannot be contained in $B^-(X)$, because, if it were, we could place any element of B on top of the cycle avoiding trajectory, and it could not be a Banks trajectory. Let A designate the set of points in $H^+_{min}(x)$ disjoint from $B^-(X)$ (and, of course, disjoint also from B). Let $A = \{a_1, \ldots, a_k\}$, where the subscripts are consistent with the ordering given by the cycle avoiding trajectory, that is, $a_h \, P \, a_g$ if and only if $g < h$.

Because $H^+_{min}(x)$ is a Banks trajectory, it is externally stable; that is, everything not in $H^+_{min}(x)$ is beaten by something in $H^+_{min}(x)$. But because every point in B beats every point in $B^-(X)$, the external stability of $H^+_{min}(x)$ *vis-à-vis* B must be provided entirely by A; that is, everything in B must be beaten by something in A.

Let a_h be the highest element in the cycle avoiding (sub)trajectory A that beats some element $z \in B$ that is not also beaten by any lower element of A, that is, any a_g where $g < h$. Put otherwise, we select a_h so that $\{a_1, \ldots, a_h\}$ is minimal externally stable *vis-à-vis* B. We now form the ordered set $\langle a_1, \ldots, a_{h-1}, z, a_h \rangle$. Because a_h P z and z P a_g for all $g < h$, and because the elements of A, ordered by their subscripts, form a cycle avoiding trajectory, $\langle a_1, \ldots, a_{h-1}, z, a_h \rangle$ is also a cycle avoiding trajectory.

Now $\langle a_1, \ldots, a_{h-1}, z, a_h \rangle$ most likely is not a Banks trajectory, but it is externally stable *vis-à-vis* B(X). By the criterion we used to identify a_h, $\{a_1, \ldots, a_h\}$ provides external stability *vis-à-vis* B, and $z \in B$ by itself provides external stability *vis-à-vis* $B^-(x)$. Thus, either $(a_1, \ldots, a_{h-1}, z, a_h)$ is a Banks trajectory with top element a_h, or the cycle avoiding trajectory is not externally stable (*vis-à-vis* the whole set X) and we can place some point(s), necessarily from outside B(X), on top of it. But in either event, we get a Banks trajectory with the top element outside of B(X), which is a contradiction, since we assumed B(X) to be the Banks set.

Thus, the top cycle and the bottom cycle of B(X) must intersect and, by Black's Lemma, the theorem is established.

To restate the theorem, the top cycle of the Banks set is always the Banks set iself. But other "reductions" of the Banks set are possible. In particular, the uncovered set of the Banks set may be a proper subset of the Banks set, which implies that the Banks set of the Banks set may be a proper subset of the Banks set (Miller, Grofman and Feld, 1987). Figure 1 provides an example; it may be checked that $B(X) = \{x_1, x_2, x_3, x_4\}$ but $UC[B(X)] = B[B(X)] = \{x_1, x_2, x_3\}$. Schwartz (1986), Miller, Grofman and Feld (1987), and Dutta (1988) have proposed further reductions.

Notes

1. In a tournament this condition implies that x P y and thus that the defining set inclusion is strict, i.e., $D(y) \subset D(x)$.
2. Banks (1985) demonstrates that an alternative can be a sophisticated voting outcome under the standard amendment procedure if and only if it is the top element of some Banks trajectory. He does this by observing that the elements of a "sophisticated agenda" in the sense of Shepsle and Weingast (1984) constitute a Banks trajectory.
3. An externally stable set thus possesses one of the two defining properties of a von Neumann-Morgenstern "stable set." The other property of a stable set is *internal stability*, that is, no point in the set is beaten by any other point in the set. But in a tournament no nonsingleton set can possess internal stability.

4. Notice that if there is some $z \in X'$ that beats all other points in X', z is essential for the external stability of X' even if z beats no points outside of X'.

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