ON THE SIZE AND LOCATION OF THE YOLK IN SPATIAL VOTING GAMES: RESULTS USING *CYBERSENATE* SOFTWARE

Nicholas R. Miller
Department of Political Science
University of Maryland Baltimore County (UMBC)
Baltimore, Maryland 21250
nmiller@umbc.edu

with

Joseph Godfrey
WinSet Group, LLC
4031 University Drive. Suite 200
Fairfax, VA 22030
jgodfrey@winset.com

July 2008

Abstract

The yolk is the set of points bounded by the smallest circle that intersects every median line in a two-dimensional spatial voting game. While the character of such voting games clearly depends critically on the location and size of the yolk, until recently it has been difficult make useful generalizations about these matters. Tovey has shown that, if voter ideal points are randomly drawn from a "centered" continuous distribution, the expected yolk radius decreases as the number of voters increases, approaching zero as the number of voters increases without limit. This result leaves open two questions: (1) the rate at which the yolk shrinks as the number of voters increases, and (2) the impact on the size and location of the yolk of the distinctive "non-random" clustering of ideal points typically seen in empirical data. Using *CyberSenate* spatial voting software, we can provide answers to these questions.

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In spatial voting games, the *yolk* is the smallest ball that intersects every median hyperplane. It has been clear on theoretical grounds that important characteristics of such games depend critically on the location and, especially, the size of the yolk, but until recently it has been difficult make useful generalizations about these matters. Most discussions of the yolk have been based on hand-constructed illustrative voting games with a small number (typically three or five) of voters, in which the yolk is typically large relative to the distribution of voters. In contrast, this paper uses *CyberSenate*, a computer program developed by one of us, to examine the location and size of the yolk in larger scale voting games with varying characteristics.[1]

Given the capabilities of the *CyberSenate* tool, we restrict our attention to two-dimensional spatial voting games — the lowest dimensionality in which yolk size is problematic. However, considerable empirical research indicates that two dimensions are often substantially sufficient to represent such choice situations as legislator preferences in parliamentary context (e.g., Poole and Rosenthal, 1997; Poole, 2005), or voter preferences over competing candidates or parties (e.g., Budge et al., eds, 1987; Schofield, 1995; Schofield et al., 1997; Lijphart, 1998).

1. The Yolk and Its Signifiance

We focus here on *two-dimensional spatial voting games* with an odd number n of *voters* having *Euclidean preferences* over points in the space — that is, each voter i has an ideal point x_i in the space, prefers any point closer to x_i to one more distant from it, and is indifferent between points equidistant from x_i. This implies that voter i's indifference curve through any point x, denoted $I_i(x)$, is the circle centered on x_i that passes through x. The set of points $P_i(x)$ that i prefers to x is the set of points bounded by $I_i(x)$.

If some majority of voters prefers x to y, we say "x beats y." The *win set* $W(x)$ of point x is the set of all points in X that beat x. The set of points that a particular majority of voters prefers to x is the intersection of all sets $P_i(x)$ such that i belongs to that majority. $W(x)$ is the union of such majority preference sets across all possible majorities. In a spatial context with n odd, x beats almost all points not in $W(x)$.[2]

Any line L through the space partitions the set of voter ideal points into three subsets: those that lie on one side of L, those that on the other side of L, and those that lie on L itself. A *median line* M partitions the ideal points in a special way — namely, so that no more than half of the ideal points lie on either side of M. If n is odd, every median line passes through at least one — and typically only one — ideal point, so fewer than half of the ideal points lie on either side of M. A *limiting* median line passes through (at least) two ideal points. Typically limiting median lines pass through a given ideal point in pairs, with non-limiting median sandwiched them.

[1] *CyberSenate* was developed by Joseph Godfrey of the WinSet Group, LLC [http://www.winset. com/ pages/ 5/index.htm].

[2] Even with n odd, some majority preference ties exist but, in order to simplify exposition, we overlook technical issues pertaining to points that lie on the boundaries of sets.

If point x lies off some median line M, x is beaten its projection on M. It follows that a point x is unbeaten only if it lies on every median line, which is possible if and only if all median lines intersect at the single point x. This in turn can hold only in the presence of a sufficient (and unlikely) degree of “Plott symmetry” in the configuration of ideal points (Plott, 1967; Enelow and Hinich, 1983).

McKelvey (1986, drawing on Ferejohn, McKelvey, and Packel, 1984) introduced the concept of the yolk. In a two-dimensional spatial voting game, the *yolk* is the set of points bounded by the smallest circle that intersects every median line. The *location* of the yolk is given by its center c and its *size* of the yolk by its radius r. The yolk circle is inscribed within the *yolk triangle* formed by three median lines to which the circle is tangent.[3]

The yolk is a distinctively important characteristic of spatial voting games, for a number of reasons.

1. The center c of the yolk indicates the generalized center of the configuration of ideal points in the sense the median. The importance of this generalized median point is in turn suggested by the role of Black’s (1948, 1958) Median Voter Theorem in one-dimensional spatial voting games.
2. If $r = 0$, there is sufficient “Plott symmetry” that all median lines intersect in a single point, which is a “total median” and Condorcet winner. Otherwise, the magnitude the yolk radius r indicates the extent to which the configuration of ideal points departs from one exhibiting Plott symmetry.
3. More generally, as r increases in magnitude, win sets become more irregular, majority rule becomes more “chaotic,” and voting outcomes can more readily be manipulated by an agenda setter.[4]
4. Every win set $W(x)$ intersects the yolk; if x lies at least $2r$ from c, c lies in $W(x)$; if x lies at least $3r$ from c, $W(x)$ contains the yolk.
5. More generally, any point x lying at distance d from the center of the yolk c beats all points more than $d + 2r$ from the center of the yolk, and x is beaten by all points closer than $d - 2r$ to the center of the yolk; put otherwise, the boundary of $W(x)$ everywhere falls between two circles centered on the yolk with radii of $d + 2r$ and $d - 2r$ respectively (the inner constraint disappears if $d < r$ and the two circles coincide if $r = 0$).[5]

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[3] These are typically, but not always, limiting median lines; see Stone and Tovey (1992) and Koehler (1992). It should be emphasized that this discussion assumes that n is odd; the definition of a yolk is more complicated if n is even.

[4] For a general discussion, see Feld et al., (1989).

[5] Tighter bounds on $W(x)$, especially in the vicinity of x itself, are provided by the outer and inner cardioids described in Ferejohn et al. (1984), McKelvey (1986), and Miller et. al. (1989). The eccentricity of this cardioid likewise depends on the size of the yolk.

6. In the event of Plott symmetry (and a yolk with zero radius), c is the Condorcet winner and no majority preference cycles exist anywhere in the space. But McKelvey's (1976, 1979) Global Cycling Theorem tells that, in the event Plott symmetry fails in even the slightest degree, a global cycle engulfs the entire space and, in particular, a path of majority preference can be constructed between any two points in the space, so that even a point y that lies far beyond the periphery of the ideal point distribution can, in some finite number of steps, indirectly beat a point x that is centrally located within the distribution. However, the length and complexity of the required path from y to x depend on the size of the yolk — the smaller yolk, the longer and more convoluted the majority preference path must be.

7. A point x lying at distance d from the center of the yolk c covers all points more than $d + 4r$ from the center of the yolk and is covered by all points closer than $d - 4r$.[6]

8. As a corollary, given a status quo x located at distance d from c, an amendment agenda, and sophisticated voting, an agenda setter can design an agenda that produces an outcome at most $d + 4r$ from the center of the yolk (Shepsle and Weingast, 1984; Feld et al., 1989).

9. The *uncovered set* (which may constitute a plausible bound on the likely outcomes of many types of spatial voting games) lies within a circle centered on c and with a radius of $4r$ (McKelvey, 1986). Feld et al. (1987) trimmed this bound a bit. Other recent research using *CyberSenate* indicates that the uncovered set typically is compactly shaped, approximately centered on the yolk, and has a "radius" of about $2r$ to $2.5r$ (Miller, 2007).

When the concept was first propounded in the early 1980s, there was a widespread intuition that the yolk would be centrally located and would tend to shrink in size as the number of voters increases. However, it was difficult to confirm these intuitions or even to state them in a theoretically precise fashion. Feld et al. (1988) took a few very modest first steps. Koehler (1990) took more substantial steps in a paper that is discussed below). Tovey (1990) took a considerably larger step by showing that, if (and only if) ideal point configurations are random samples drawn from any "centered" probability distribution[7], the expected yolk radius approaches zero as the number of ideal points increases without limit. More intuitively, if the underlying distribution has a well-defined center, finite random samples drawn from it have imperfectly defined centers that become more perfectly defined as sample size increases. But Tovey's theoretical result left two important questions open.

The first question concerns the *rate* at which the yolk shrinks as the number of voters increases. For example, does a yolk with $n = 101$ or $n = 435$ (to pick two n's of political relevance in the U.S.) look more like a yolk in a committee-sized configuration of $n = 9$ to $n = 25$ ideal points

[6] Point x covers y if x beats y and every point that y beats.

[7] A (two-dimensional) probability distribution is *centered* around a point c if every line through c has half the probability mass on either side. Bivariate uniform and normal distributions (both considered in the next section) are centered, as are nonstandard (e.g., correlated) normal distributions (not considered here).

or in an electorate-sized configuration of $n = 100{,}000$ to $n = 100{,}000{,}000$ ideal points?

The second question concerns the impact on the location and size of the yolk of various patterns of "non-random" *clustering* within configurations of ideal points of varying sizes. Such clustering is typically seen in empirical ideal point data — for example, the Congressional ideal point configurations generated by the NOMINATE procedure of Poole and Rosenthal (1997), several of which are reproduced in Bianco et al., 2004).

We take up the first question in Section 2. We take up and the second question with respect to bimodal clusters in Section 3 and with respect to multi-modal clusters in Section 4. Some generalizations and conjectures are presented in the concluding Section 5.

2. Random Ideal Point Configurations[8]

Some years ago, Koehler and Binder (1990) developed a computer program to compute yolks in two dimensions.[9] With this tool, Koehler (1990) calculated yolk locations and sizes in ideal point configurations randomly drawn from an underlying *uniform* distribution over a 10×10 square. He drew twenty five configurations for each of $n = 25$, $n = 51$, and $n = 75$, plus one configuration for every n from 3 through 101.[10] This produced two main findings. First, the average yolk size was fairly small, relative to that found in small hand-constructed configurations three or five voters. Second, within the range of configurations studied, the average yolk radius clearly declined as n increased. Mean yolk radii were 0.66 for $n = 25$, 0.52 for $n = 50$, and 0.44 for $n = 75$. Thus the mean yolk radius for $n = 75$ was about 66% (and the mean yolk area is about 45%) of that for $n = 25$.

Subsequently, Hug (1999) used Koehler's program to extend his estimates to larger-scale ideal point configurations. Hug drew samples of five configurations for each of a variety of n's ranging up $n = 1001$ plus a single configuration of $n = 2001$. For the twenty configurations with $n = 701$ through 1001, the mean radius was about 0.175, or about 26.5% (and the mean yolk area about 7%) of that for $n = 25$. (The one yolk with $n = 2001$ had a radius of about 0.10.)

More recently, Bräuninger (2007, Figure 5) used his own computer program to randomly select 4000 voter configurations from an underlying uniform distribution over a 10×10 square, for odd n's running from 3 to 101 and calculate the yolk radius for each configuration and the mean radius for each n.

CyberSenate can generate configurations of ideal points drawn randomly from either bivariate

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[8] Some of the results in this section were previewed in Miller (2007).

[9] This program was an ancient ancestor of *CyberSenate*.

[10] As the title of his paper indicates, Koehler also analyzed even-sized committees. With n even, distinct "inner" and "outer" yolks appear. We do not take up the n-even case here.

uniform or bivariate normal distribution, compute and display all limiting median lines and the yolk, display the yolk, and compute *c* and *r* (see Figure 1 for an example of bivariate normal configuration with $n = 51$).[11] Using *CyberSenate*, we have computed yolk sizes for 670 ideal point configurations, half drawn from each type of distribution, with various *n*'s (all odd) ranging from 3 to 2001.[12]

The results for the 335 configurations drawn from a uniform distribution over a 10 × 10 square are displayed in Figure 2, which plots each individual configuration and shows the mean yolk radius for each *n*. It is evident (and unsurprising) that yolk sizes are quite stable from sample to sample in large configurations but highly variable in small configurations.[13] Nevertheless, it is clear that, once a low threshold of about $n = 7$ is crossed, the expected yolk radius shrinks as the number of voters increases and, given configurations of several hundred voters, the expected yolk radius is about one quarter (and yolk area about 6%) of that for configurations with $n = 3$ to 15.

Figure 2 combines data from Koehler, Hug (for $n \geq 101$ only), Bräuninger, and Figure 1 into a single chart. Koehler's data for all *n* from 3 to 101 is shown individually. Otherwise only the mean for a given n is shown (with simple interpolation between data points). Figure 4 combines the Lowess curves for the same data. It is evident that there is good agreement among all results from $n = 15$ upwards.

Configurations with ideal points uniformly distributed over a square look very artificial, while configurations drawn from a bivariate *normal* distribution (like that in Figure 1) appear considerably more natural. Figure 3 displays *CyberSenate* results for the 228 configurations (such as that shown in Figure 1) drawn from a bivariate normal distribution centered on (50,50) with a standard deviation of 15 each in dimension.

3. Ideal Point Configurations with Two Clusters

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[11] *CyberSenate* also identifies "Tovey anomalies" (when the smallest circle intersecting every limiting median does not intersect some non-limiting median line) and adjusts yolks radii accordingly. Causal impressions based on extensive *CyberSenate* usage suggest that such anomalies occur more frequently than Koehler (1992) anticipated but also that (with *n* odd) the required adjustments in the yolk are small. See footnote 3 and McKelvey and Tovey (2005).

[12] More samples were drawn for smaller configurations than for larger ones, both because computations take much more time for the large ones (hours for $n = 2001$) and also because there is much more variability in yolk sizes in the small configurations. ***As might be expected, for a given *n*, expected yolk size is a somewhat smaller when ideal points are drawn from a normal distribution rather than a uniform distribution with the same standard deviation.

[13] SD vs. variation ratio.

4. Ideal Point Configurations with More Than two Clusters

References

Bianco, William T., Ivan Jeliazkov, and Itai Sened. 2004. "The Uncovered Set and the Limits of Legislative Action." *Political Analysis* 12: 256-278.

Black, Duncan (1948). "On the Rationale of Group Decision-Making." *Journal of Political Economy*, 56: 23-34.

Black, Duncan (1958). *The Theory of Committees and Elections*. Cambridge: Cambridge University Press.

Bräuninger, Thomas. (2007). "Stability with Restricted Preference Maximizing." *Journal of Theoretical Politics*. 19: 173-191.

Budge, Ian, Hans-Dieter Klingemann, Andrea Volkens, Judith Bara, and Eric Tanenbaum (2001). *Mapping Policy Preferences: Estimates for Parties, Electors, and Governments, 1945-1998*. Oxford University Press.

Budge Ian, David Robertson, and Derek J. Hearl, eds. (1987). *Ideology, Strategy, and Partty Change: Spatial Analyses of Post-War Election Programmes in 19 Democracies*. Cambridge University Press.

Enelow, James M.; Melvin J. Hinich, (1983) "On Plott's Pairwise Symmetry Condition for Majority Rule Equilibrium." *Public Choice*, 40: 317-321.

Feld, Scott L.; Bernard Grofman, Richard Hartley, D. Marc Kilgour, Nicholas R.Miller, and Nicolas Noviello (1987) "The Uncovered Set in Spatial Voting Games." *Theory and Decision*, 23:129-155.

Feld, Scott L., Bernard Grofman, and Nicholas R. Miller (1989). "Limits on Agenda Control in Spatial Voting Games" *Mathematical and Computer Modelling*, 12: 405-416.

Ferejohn, John A., Richard D. McKelvey, and Edward W. Packel (1984). "Limiting Distributions for Continuous State Markov Voting Models," *Social Choice and Welfare*, 1: 45-67.

Hug, Simon. 1999. "Nonunitary Actors in Spatial Models." *Journal of Conflict Resolution* 43: 479-500.

Koehler, David H. (1990) "The Size of the Yolk: Computations for Odd and Even-Numbered Committees." *Social Choice and Welfare*, 7; 231-245.

Koehler, David H. (1992). "Limiting Median Lines Frequently Determine the Yolk: A Rejoinder." *Social Choice and Welfare*, 9:37-41.

Koehler, Davin H., and Michael G. Binder (1990). "A Program for Computing the Yolk in Two Dimensions: Description and Discussion," American University, mimeo.

Lijphart, Arend (1998). *Patterns of Democracy: Government Forms and Performance in Thirty-sic Countries*. Yale University Press.

McKelvey, Richard D. "Intransitivities in Multidimensional Voting Models and Some Implications for Agenda Control." *Journal of Economic Theory*, 12: 472-482.

McKelvey, Richard D. "General Conditions For Global Intransitivities in Formal Voting Models." *Econometrica*, 47: 1085-1112.

McKelvey, Richard D. (1986). "Covering, Dominance, and Institution Free Properties of Social Choice," *American Journal of Political Science*, 30: 283-314.

McKelvey, Richard D., and Craig A. Tovey (2008). "Approximation of the Yolk by the LP Yolk." Georgia Institute of Technology, mimeo.

Nicholas R. Miller (2007). "In Search of the Uncovered Set." *Political Analysis*, 15: 21-45.

Miller, Nicholas R., Bernard Grofman, and Scott L. Feld (1989) "The Geometry of Majority Rule." *Journal of Theoretical Politics*, 1: 379-406.

Plott, Charles R. (1967). "A Notion of Equilibrium and Its Possiblity Under Majority Rule." *American Economic Review*, 57: 787-806.

Poole, Keith T. (2005). *Spatial Models of Parliamentary Voting*. Cambridge University Press.

Poole, Keith T., and Howard Rosenthal (1997). *Congress: A Political-Economic History of Roll Call Voting*. Oxford University Press.

Schofield, Norman. (1996). "The Heart of a Polity," in Norman Schofield, ed., *Collective Decision-Making: Social Choice and Political Economy*. Kluwer Academic Publishers.

Schofield, Norman (2008). *The Spatial Model of Politics*. Routledge.

Schofield, Norman, Gary J. Miller, and Andrew D. Martin (2003). "Critical Elections and Political Realignment in the U.S.: 1860-2000." *Political Studies*. 51: 217-240.

Shepsle, Kenneth A., and Barry R. Weingast (1984). "Uncovered Sets and Sophisticated Voting Outcomes with Implications for Agenda Institutions." *American Journal of Political Science*, 28: 49-74.

Stone, Richard E., and Craig A. Tovey (1992). "Limiting Median Lines Do Not Suffice to Determine the Yolk." *Social Choice and Welfare*, 9:33-35.