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LIMITS ON AGENDA CONTROL IN SPATIAL VOTING GAMES†

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Abstract—A theorem due to McKelvey implies that, if a single agent controls the agenda of a spatial voting game, he can almost always design an agenda that yields whatever voting outcome he wishes. Here we make use of a geometrical construct called the "yolk" to demonstrate the existence of significant limits on such agenda control. We show that the feasibility of agenda control is inversely related to the size of the yolk. In general, there are strong centripetal forces in spatial voting games, which make it much easier to move voting processes in a centrist direction than in non-centrist one. Thus, outcomes of plausible agenda processes will probably be found in the central area of the space.

1. INTRODUCTION

The "chaos theorems" of Plott (1967), McKelvey (1976, 1979), Schofield (1978, 1986), and others have produced considerable pessimism about the possibility of meaningful social choice based on majority rule. More specifically, actual social choices appear to be highly agenda-dependent and subject, under appropriate institutional arrangements, to almost unlimited agenda control. In particular, McKelvey demonstrated that, in an issue space of two or more dimensions, it is almost always possible to create a majority rule cycle including any two points. Thus, it is almost always possible to design an agenda of sequential pairwise majority votes that, with sincere voting, generates a voting trajectory—i.e. a sequence of proposals each of which is chosen in preference to the preceding proposal—leading from any arbitrary point in the space to any other. This in turn implies that a single agent, if he controls the agenda, can almost always design an agenda that yields whatever voting outcome he wishes.

In this paper, we demonstrate the existence of significant limitations on such agenda control. In doing this, we apply a number of theorems—many, but not all, of which have been previously established—concerning properties of majority rule in spatial voting games. In particular, we focus on a geometrical construct introduced by Ferejohn, McKelvey and Packel (1984) and dubbed the *yolk* by McKelvey (1986). The yolk is the ball of minimum radius that intersects all median hyperplanes. The location of the yolk indicates the generalized center of the voter distribution, and the size to the yolk indicates how much the voter distribution deviates from one that would generate a majority rule equilibrium or core outcome.

McKelvey (1986) showed that the maximum size of the "uncovered set" is a function of the size of the yolk. He thereby showed that the size of the yolk sets limits on agenda control, given voting processes that produce outcomes in the uncovered set. Here we make the more general argument that agenda control is essentially *always* constrained by the size of the yolk, regardless of the nature of the voting process—the smaller the yolk, the greater the constraint. In particular, while it may in principle be possible, in the absence of a majority rule equilibrium, to design an agenda that generates a voting trajectory leading from any point to any other, we show that the feasibility of designing an agenda that generates a trajectory leading in a noncentrist direction is in practice (with an agenda of reasonable size and form) a function of the size of the yolk. If the yolk is large, the agenda setter indeed has considerable freedom of maneuver; but if the yolk is small, he operates under significant constraints. More specifically, the smaller the yolk, the longer and more elaborate a trajectory must be if it is to lead from one point to another point more distant from the yolk. In addition, if the agenda setter must follow a rule of "incrementalism", in that a voting trajectory

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can move only a limited distance at each step, outward movement is especially severely constrained as the size of the yolk decreases; moreover, the further away from the yolk a voting trajectory has already moved, the harder it is for the agenda setter to extend it still further outward. In general, there are strong centripetal forces in spatial voting games, and the strength of these forces is inversely related to the size of the yolk.

For purposes of this discussion, we make several simplifying assumptions.

First, for analytical convenience, we deal only with the case in which the number of voters n is odd, as special complexities in defining median lines and related constructions—indeed in defining majority rule itself—arise in the even number case.

Second, for purposes of exposition, our discussion focuses on the case of a two-dimensional space. (Thus, for example, we refer to median lines, rather than median hyperplanes.) However, the discussion extends straightforwardly to the multidimensional case, with some special provisos as noted in footnotes.

Third, we assume that all voters have "Euclidean" (or Type 1) preferences; this means that individual preference is based on simple Euclidean distance, i.e. each voter has an *ideal point* (point of highest preference) in the space and, in comparing any two points in the space, prefers the point closer to his ideal to the point more distant from his ideal, and is indifferent between points equidistant from his ideal. Thus, each voter's indifference curves are concentric circles centered on his ideal point. While this assumption is restrictive, we have the strong intuition that the general thrust of the results presented here extends to spatial voting games with more general preferences.

Finally, to avoid certain bothersome complexities, we assume that no two voters have identical ideal points.

Point x *beats* point y under majority rule iff more voters prefer x to y than prefer y to x; thus, given Euclidean preferences, x beats y iff, of all ideal points not equidistant from x and y, a majority are closer to x. (Point x *ties* y iff, of all ideal points not equidistant from x and y, exactly half are closer to x and half to y.) The set of points that beat x is called the *win set* of x and we label it $W(x)$.

A *voting trajectory* from x to y is a sequence of points beginning with x and ending with y such that each point in the sequence (except the first) beats the preceding point.

Point x *covers* point y iff x beats y, x beats everything y beats, and x beats or ties everything y ties; this implies that $W(x)$ is properly contained in $W(y)$. The set of points not covered by point x is designated $UC(x)$. The *uncovered set* is the set of points of which none is covered by any point.

The remainder of this paper is divided into four sections. Section 2 reviews the literature on agenda control in spatial voting games as it has developed to date. Section 3 presents basic constructions and theorems pertaining to spatial voting games with Euclidean preferences. Section 4 presents a series of "Agenda Propositions" that derive from the theorems and that specify more precisely the possibilities of and limits on agenda control in spatial voting games. Section 5 summarizes our conclusions.

2. AGENDA CONTROL

As the term is used in this paper, *agenda control* refers to the influence a single agent may have over the outcome of a voting process if he unilaterally determines the *voting agenda*, viz. the alternatives to be voted on and the order in which they are to be voted on, subject to the proviso that there is some distinguished alternative ϕ—say, the "status quo" or "doing nothing"—which is automatically on the agenda and must always enter the voting. It is natural, therefore, to say that we are dealing with the case of a *monopoly agenda setter*. However, the limits that constrain a monopoly agenda setter apply *a fortiori* to more decentralized agenda institutions.

Thus, the monopoly agenda setter selects some finite number of points (always including ϕ) out of the alternative space, orders them (perhaps under some constraint concerning the place of ϕ in the ordering) and presents this agenda to the voters to vote on under some variant of *amendment procedure*—i.e. the first two alternatives on the agenda are paired for a simple majority vote; the loser is rejected and the winner survives to be paired with the third alternative; and so forth until but one alternative survives, which is the voting outcome. Thus, if there are m alternatives on the agenda, $m - 1$ votes are taken; we call this an *$(m - 1)$-step agenda*.

In two well-known papers, McKelvey (1976, 1979) demonstrated, for Euclidean and much more general preferences, respectively, that—in the almost certain event that there is no point in the multidimensional space unbeaten by any other point—any two points in the space are linked in a majority rule cycle. That is, a "chaotic" global cycle almost always encompasses the entire space; cycling, if it exists at all, cannot be confined to a small region of the space.

From this result, McKelvey (1976) drew some implications for agenda control that have to some extent bedeviled voting theorists (e.g. Riker 1982) ever since. McKelvey observed that, if the monopoly agenda setter knows the preferences of all voters and if voters always vote according to their known preferences (i.e. sincerely), the agenda setter would almost always (i.e. in the absence of an unbeaten point) have total control over the voting outcome. More specifically, the agenda setter could almost always design an agenda generating a voting trajectory leading from *any status quo* point ϕ to *any other* point ϕ^* in the space—even one outside the Pareto set or, more plausibly, his own ideal point—no matter how extreme that point might be relative to the overall distribution of ideal points.

There are a number of significant limitations on this picture of an omnipotent and possibly demonic agenda setter, most of which McKelvey (1976, p. 481) himself recognized but did not explore. First, the canny agenda setter must know the preferences of all other voters in order to design the appropriate agenda. Second, all voters must be able and willing to distinguish between alternatives concerning which they are "almost indifferent". Third, the agenda setter is free to use a *forward moving agenda* (cf. Wilson 1986), under which an alternative is introduced and placed against the *status quo* for a vote; only then is a second alternative introduced and placed against the new *status quo* (i.e. the winner of the first vote) for a second vote; and so forth. Under this arrangement, voters have no real alternative but to vote sincerely. Alternatively, if the agenda setter must fix and announce the whole agenda before voting begins, it is assumed that all voting is nevertheless sincere, with no sophistication on the part of voters and no collusion among them. Fourth, if the agenda must be fixed before voting begins, there is no constraint on the agenda setter with respect to the order in which voting takes place; in particular, it is not required that the *status quo* alternative ϕ enter the voting last. And fifth, there is no constraint on the agenda setter with respect to the number of alternatives he may place on the agenda or on how much they may differ from one another.

We do not further explore the first two limitations here, since our purpose is to indicate how the properties of pure majority rule in spatial voting games affect agenda control. It is worth noting, however, that these two assumptions are most favorable to the power of the agenda setter, and that the subsequent propositions demonstrating limits on agenda control hold *even if* these assumptions are true. If, more realistically, (a) the agenda setter is incompletely informed about other voters' preferences and/or (b) voters are unwilling to vote in favor of a new proposal they prefer only barely to the current *status quo*, limits on the power of the agenda setter would be significantly greater.

With respect to the third point, the most straightforward interpretation of McKelvey's global cycling theorem assumes a forward moving agenda, so that the agenda setter can, by the global cycling theorem, design an agenda generating a trajectory from any initial *status quo* ϕ to any other point ϕ^* in the space. But if the setter is required to fix and announce the entire agenda before any voting takes place and if voters have adequate information concerning each other's preferences (specifically, if they know which alternatives on the agenda beat which other alternatives), rational voters will vote in a sophisticated [i.e. game-theoretically optimal (cf. Farquharson 1969; McKelvey and Niemi 1978)] fashion, and [as anticipated by McKelvey and more directly hypothesized by Miller (1980)] the power of the monopoly agenda setter may be considerably tamed. More specifically, Shepsle and Weingast (1984) have shown that, given an agenda fixed in advance and given the sophisticated voting behavior such an agenda makes possible, it is impossible for the agenda setter to design an agenda yielding any outcome outside of $UC(\phi)$, i.e. the set of points not covered by the *status quo* ϕ.

With respect to the fourth point, McKelvey's picture of agenda control assumes that, even if the agenda must be fixed before voting begins, ϕ enters the voting initially, so that the voting trajectory starts at ϕ and then proceeds to the setter's desired outcome ϕ^*, following the cycle that McKelvey's theorem assures us almost always links any two points. But Anglo-American parliamentary procedure normally requires an agenda in which ϕ enters the voting last. If this requirement is imposed, we may speak of a *standard agenda*. If the agenda setter is limited to such agendas, the strategic situation changes entirely. Restricted in this fashion, the setter cannot design an agenda giving any outcome outside of $W(\phi)$, the win set of the *status quo*, since the final vote will in any case be between ϕ and some other alternative, and—regardless of whether voting is otherwise sincere or sophisticated (sophisticated voters vote sincerely at the final vote)—ϕ will be the voting outcome unless that other alternative can beat ϕ. Thus, the agenda setter can do no better than merely pick his most preferred point out of $W(\phi)$ and offer it to the voters on a take-it-or-leave-it (or, more precisely, take-it-or-take-ϕ) basis as part of a two-alternative, one-step agenda.

With respect to the fifth limitation, while McKelvey shows that any two points in the space are almost always linked by a cycle with a finite number of steps, his theorem gives no sense of how large that finite number of steps may actually be. One of the principal arguments below is that, even in the case most favorable to the agenda setter, viz. a forward moving agenda (with sincere voting), for the setter to manipulate the voting process so as to move from a more centrally located *status quo* ϕ to a less centrally located outcome ϕ^* typically requires an impractically large agenda and/or an agenda that generates a trajectory that moves wildly back and forth across the space.

We conclude this section by observing that there is a particular sense in which the efficacy of monopoly agenda control might be considered an either/or proposition. It is true that the agenda setter either can design an agenda that yields his ideal point as the outcome, or cannot. However, the efficacy of agenda control in this sense is, in large measure, a function of the location of the agenda setter's ideal point, in relation to the center of the distribution. We are, in effect, factoring out this variable by focusing on the scope of possible outcomes that different agendas may yield, regardless of the agenda setter's preferences.

3. MAJORITY RULE IN SPATIAL VOTING GAMES

Any line L through a two-dimensional alternative space partitions the ideal points into three sets: those that lie on one side of the line; those that lie on the other side of the line; and those that lie on the line. A *median line* M partitions that ideal points so that no more than half of them lie on either side of M. It follows immediately that, if—as we assume throughout—the number of ideal points n is odd, any median line must pass through at least one ideal point and that no two median lines can be parallel.

Now let us take any two points x and y and erect the perpendicular bisector of the line connecting x and y. Point x is preferred by all voters whose ideal points lie on the x side of the bisector and point y is preferred by all voters whose ideal points lie on the y side. There must be some median line M (and, if n is odd, only one) perpendicular to the line through x and y; x beats y if M lies on the x side of the bisector, and y beats x if the reverse is true. Only if M is identical to the bisector may x and y tie.

Thus, given an arbitrary point x and a line L through x, we can determine what segment of L intersects $W(x)$ by determining where the median line M perpendicular of L intersects L. Point x is beaten by every point y on L between x and its *reflection* x^* through M, i.e. the point x^* on L on the opposite side of M from x and at the same distance from M as x is. This is true because M necessarily lies on the y side of the perpendicular bisector of the line connecting x and any such y. However, for any point z that lies on L beyond the reflection of x through M, or that lies on L on the side of x away from M, the reverse is true; thus, x is beaten only by points on L between x and its reflection through M (and possibly by the reflection itself).†

This discussion makes it clear why an unbeaten point almost never exists in two or more dimensions—i.e. why $W(x)$ is almost always nonempty for all points x.

Theorem 1 (Plott 1967; Davis, DeGroot and Hinich 1972; Hoyer and Mayer 1974).
A point c is unbeaten iff every median line passes through c.

†It follows that $W(x)$ is *starlike* about x, i.e. $W(x)$ includes all points lying on any straight line between x and a point in $W(x)$, and also *polarized* about x, i.e. if some points on a line through x on one side of x belong to $W(x)$, no points on the line on the other side of x belong to x.

Proof. Essentially immediate from the preceding discussion, but see Plott (1967), Davis, DeGroot and Hinich (1972) and Hoyer and Mayer (1974).

Given an unbeaten point c and an arbitrary point x, we can demarcate the win set $W(x)$ by examining every line L through x and determining where the median line M perpendicular to L intersects L. Since every median line must pass through c, we can establish the following theorem:

Theorem 2 (Davis, DeGroot and Hinich 1972). If there is an unbeaten point c, for any point x, point y beats x iff y is closer to c than x is.

Proof. Follows directly from the preceding discussion, but see Davis, DeGroot and Hinich (1972).

Thus, if there is an unbeaten point c, any win set $W(x)$ is the set of points enclosed by the circle centered on c and passing through x.

Suppose, however, that there is no unbeaten point, i.e. median lines do not all intersect at a common point c. Following Ferejohn, McKelvey and Packel (1984) and McKelvey (1986), we define the *yolk* as the circle of minimum radius that intersects every median line. The location of the yolk, given by its *center* c, indicates the generalized center (in the sense of the median) of the distribution of ideal points. The yolk can be a circle with zero radius, i.e. the single point c; this is the special case to which Theorems 1 and 2 pertain. In the general case, the size of the yolk, given by its *radius* r, measures the extent to which the configuration of ideal points departs from one that generates an unbeaten point.

Lemma 1 In the absence of an unbeaten point, at least three median lines are tangent to the yolk.†

Proof. If this were not so, a smaller circle would touch all median lines.

Theorem 3 If $r > 0$, for any point x there is some other point y that both beats x and is further from the center of the yolk than x is.

Proof. From Lemma 1, it follows that there is always some median line M (in any event, one of the three tangent to the yolk) such that c and any arbitrary point x both lie on the same side of M. Consider the line L through x that is perpendicular to M; x is beaten by every point on L between x and its reflection x^* through M. It may be checked that the reflection of x through M is further from c than x is.

Thus, in the absence of an unbeaten point, $W(x)$ at some places extends beyond the circle centered on c and passing through x (and, by a parallel argument, at other places falls short of that circle).

Now consider any line L through x. We know that x is beaten by all points on L between x and its reflection through the median line M perpendicular to L. Of course, if the only information we have concerning the configuration of ideal points is that conveyed by yolk—i.e. by the parameters c and r—and if $r \neq 0$, we do not know exactly where the median line M perpendicular to L lies. But we do know that it lies between the two lines perpendicular to L and tangent to opposite sides of the yolk, for by definition every median line passes through the yolk. If both such *tangent lines* intersect L on the same side of x, the median line M perpendicular to L must lie on that side of x, so (regardless of the particular configuration of ideal points) x must be beaten by points on L on that side of x and cannot be beaten by any points on L on the other side of x. If the tangent lines intersect L on opposite sides of x (as must be the case if x is inside the yolk), we cannot say on which side of x the perpendicular median line M lies, but (unless it happens that M passes precisely through x, in which event no point on L beats x) x is beaten by points on L on one or other side (depending on the particular configuration of ideal points) of x, but not on both sides.

Thus, given c and r, we can partition all lines L through x into two classes, according to whether the tangent lines intersect L on the same side of x or not.

†This is a strictly two-dimensional result. But in a similar manner, in three dimensions there are always four median planes tangent to the (spherical) yolk, and in w dimensions $w + 1$ median hyperplanes tangent to the yolk, from which parallel results follow.

In turn, we can partition all *rays* from x, i.e. half lines that lie on one or other side of x, into three classes:

(1) *dominating* rays, which must intersect $W(x)$ regardless of the particular configuration of ideal points, because they strictly intersect both tangent lines (i.e. each tangent line passes through a point on the ray other than x);
(2) *dominated* rays, which cannot intersect $W(x)$ regardless of the particular configuration of ideal points, because they strictly intersect neither tangent line;
(3) *contingent* rays, which may or may not intersect $W(x)$ depending on the particular configuration of ideal points, because they strictly intersect one or other tangent line but not both.

We call a ray *undominated* if it is either dominating or contingent. We call two rays *opposites* if they lie on the same line pointing in opposite directions. Clearly, if a ray from x is dominated, its opposite is dominated, and vice versa; and if a ray is contingent, so is its opposite.†

The next matter is to specify which rays are of which type. First, if point x is inside the yolk, no ray from x can strictly intersect both tangent lines, so all rays from x are contingent. Otherwise, if x lies at a distance d from c (where $d > r$), we may specify rays from x in terms of the angle θ ($\leqslant 180°$) between the ray in question and the ray from x through c. We can determine which rays are of which type by computing the critical angles θ^* and θ^{**} that separate dominating from contingent rays and contingent from dominated rays. These critical angles specify the rays perpendicular to each of the two tangent lines when one tangent line passes through x. It may be checked that $\cos\theta^* = r/d$ and $\cos\theta^{**} = -r/d$.

We can summarize this discussion in the following theorem:

Theorem 4 For any point x at a distance d from the center of the yolk c, and for any ray from x specified by θ:
(1) if $1 \geqslant \cos\theta > r/d$, the ray is dominating;
(2) if $-r/d \geqslant \cos\theta \geqslant -1$, the ray is dominated;
(3) if $r/d \geqslant \cos\theta > -r/d$ or if $r/d \geqslant 1$, the ray is contingent.

Note that the union of dominated rays forms what we may call the *dominated cone*, defined by a vertex at x, an axis along the line through c and x, and a generating angle of $180 - \theta$.

It is important to bear in mind that, while x beats *every* point in the dominated cone, x certainly is *not* beaten by every point on a undominated ray but only by "nearby" points. The question naturally arises of how "nearby" these points must be. The answer follows directly from previous considerations.

Consider any point x at distance d from the center of the yolk and any dominating ray from x specified by θ. By definition both tangent lines strictly intersect the ray. The median line perpendicular to the ray cannot be closer to x than the closer tangent line nor more distant from x than the further tangent line. Thus, x must be beaten by all points on the ray between x and its reflection through the closer tangent the closer tangent line and x must beat all points beyond its reflection through the further tangent line.

Now consider any undominated ray from x. By definition, one tangent line strictly intersects the ray and the median line perpendicular to the ray cannot be further from x than this tangent line. Thus, x must beat all points on the ray beyond its reflection through this tangent line.

Appropriate calculations leads to the following conclusion:

Theorem 5 (Ferejohn, McKelvey and Packel 1984). For any point x at a distance d from the center of the yolk c, and for any ray from x specified by θ:
(1) x is beaten by all points on a dominating ray up to a distance of $2d\cos\theta - 2r$ from x;
(2) x beats all points on an undominated ray beyond a distance of $2d\cos\theta + 2r$ from x.

†One "boundary condition" constitutes an exception to these statements: if the tangent line passes exactly through x, one ray from x is contingent and its opposite is dominated.

Proof. Follows directly from preceding discussion, but see Ferejohn, McKelvey and Packel (1984).

Ferejohn, McKelvey and Packel (1984; McKelvey 1986) state the theorem in the following manner. The locus of points at a distance of $2d \cos \theta$ from x is simply the circle centered on c and passing through x. The inner and outer bounds on $W(x)$ are given by the locus of points at distances of $2d \cos \theta - 2r$ and $2d \cos \theta + 2r$, respectively, from x. Each locus is a *cardioid* with center c, underlying radius d, and its *cusp* at x. The *inner* cardioid has a (*negative*) *eccentricity* of $-2r$, and the *outer* cardioid has a (*positive*) *eccentricity* of $+2r$. Note that the inner cardioid does not exist if $d < r$—i.e. if x is inside the yolk. Ferejohn, McKelvey and Packel state their theorem in this way: the region enclosed by the inner cardioid is contained in $W(x)$ and $W(x)$ is contained in the region enclosed by the outer cardioid.

Corollary 5.1. If point y is more than $2r$ further away from the center of the yolk than point x is, x beats y.

It may be noted that Corollary 5.1 subsumes Theorem 2, for the special case of $r = 0$.

Corollary 5.2. If point y is more than $4r$ further away from the center of the yolk than point x is, x covers y.

Theorem 6 If x covers y, x is closer to the center of the yolk than y is.

Proof. The theorem says that, if $W(x)$ is properly contained in $W(y)$, x is closer to the center of the yolk than y is. To demonstrate this, consider any two points x and y. Draw the two parallel lines, L_1 and L_1', through y and x, respectively, that are perpendicular to the line through x and y. Wherever the boundaries of $W(x)$ and $W(y)$ intersect these lines, they do so in the same direction and at the same distance from x and y, respectively, since L_1 and L_1' are perpendicular to the same median line M_1. Call the reflections through M_1 x_1^* and y_1^*, respectively. Thus, the points x, y, x_1^* and y_1^* define a rectangle. Let c^* designate the center of the rectangle. Construct the perpendicular bisector B of the line connecting x and y; thus, M_1 and B intersect at c^*. Now consider the line L_2 through y and x_1^*. Note that L_2 passes through c^*. Given the starlike character of win sets,† if $W(x)$ is contained in $W(y)$, the intersection of L_2 and the boundary of $W(y)$ lies at or beyond x_1^*, which implies that the median line M_2 perpendicular to L_2 intersects L_2 at c^* [if the boundaries of $W(y)$ and $W(x)$ coincide along L_2] or beyond c^* from y [if $W(y)$ extends beyond $W(x)$ along L_2]. In any event, the intersection of M_2 and M_1 is not closer to y than to x. Now consider the line L_2' from x that is parallel to L_2. It also is perpendicular to M_2 so we can fix the boundary of $W(x)$ along this line, i.e. at the reflection x_2^* through M_2. Now consider the line L_3 through y and x_2^*. Given the starlike character of win sets, if $W(x)$ is contained in $W(y)$, the intersection of L_3 and the boundary of $W(y)$ lies at or beyond x_2^*, which implies that the median line M_3 perpendicular to L_3 passes through c^* or lies beyond c^*, so that x and c^* are on the same side of M_3. In any event, the intersection of M_3 and M_1 is not closer to y than to x. We can construct a similar argument for any line L through y and parallel line L′ through x. Thus, we must conclude that, if $W(x)$ is properly contained in $W(y)$, no other median line intersects M_1 closer to y than x and at least one intersects M_1 closer to x than y. Thus, the center of the yolk must lie on the x side of the bisector B, i.e. closer to x than y.

4. AGENDA PROPOSITIONS

Having reviewed the literature on agenda control in spatial voting games, and having summarized or proved some theorems pertaining to majority rule in spatial voting games, we can now present a series of "Agenda Propositions" deriving from these theorems that specify more precisely the possibilities of and limits on agenda control.

†See the footnote immediately preceding Theorem 1 (p. 408).

In all the Agenda Propositions, the size of the yolk, as given by its radius r, is the critical variable. Most of the propositions assert, in one way or other, that the agenda setter has more freedom of maneuver if the yolk is large than if the yolk is small.

Most of the following propositions pertain to the ease or difficulty with which the agenda setter can design an agenda yielding an outcome ϕ^* more extreme, i.e. further from the center of the yolk, than the *status quo* ϕ. It is worth stating explicitly at the outset that designing an agenda that yields an outcome less extreme, i.e. closer to the center of the yolk, than the *status quo* is not problematic and is hardly dependent on the size of the yolk.

Agenda Proposition 1. Regardless of whether voting is sincere or sophisticated:

(a) From any *status quo* ϕ outside the yolk, the agenda setter can propose a one-step agenda that yields *some* point ϕ^* inside the yolk as the voting outcome.
(b) From any *status quo* ϕ at a distance $>2r$ from the center of the yolk, the agenda setter can propose a one-step agenda that yields the *center* of the yolk c as the voting outcome.
(c) From any *status quo* ϕ at a distance $>3r$ from the center of the yolk, the agenda setter can propose a one-step agenda that yields *any* point ϕ^* inside the yolk as the voting outcome.
(d) In general, from any *status quo* at a distance $>kr$ $(k > 2)$ from the center of the yolk, the agenda can propose a one-step agenda that yields any point ϕ^* at a distance $\leqslant (k-2)r$ from the center of the yolk as the voting outcome.

Given a one-step agenda, sophisticated and sincere voting are equivalent. Thus, the question is simply whether $W(\phi)$ includes the point ϕ^* in question. For part (a), consider the line L through ϕ and c. The median line perpendicular to L intersects the yolk and ϕ is beaten by all points on L between ϕ and its reflection through M and, thus, by points in the yolk. Parts (b)–(d) follow directly from Corollary 5.1.

Agenda Proposition 2. Regardless of whether voting is sincere or sophisticated, from any *status quo* ϕ the agenda setter can propose an agenda of no more than two steps that yields any point ϕ^* closer to the center of the yolk than ϕ is as the voting outcome.

If ϕ^* belongs to $W(\phi)$, a one-step agenda will do. But in any event, by Theorem 6, ϕ^* is not covered by ϕ, so there is some point z such that ϕ^* beats z and z beats ϕ. Thus, a forward moving agenda (which implies sincere voting) that pits z against ϕ and then x against the winner of the first vote (i.e. z) against ϕ^*, yields ϕ^* as the outcome. If voting is sophisticated, an agenda that pits ϕ^* against ϕ and the winner against z yields ϕ^* as the outcome.

Thus, regardless of the size of the yolk, the setter can readily design a simple agenda that yields a point more centrist than the *status quo*. But, as the subsequent propositions indicate, it is more difficult to design an agenda that yields a point more extreme than the *status quo*. Indeed, if the yolk is of minimum size. i.e. if $r = 0$, it is *impossible* to design such an agenda.

Agenda Proposition 3. If the yolk has a zero radius, for any *status quo* ϕ, the agenda setter cannot propose an agenda that yields an outcome ϕ^* more distant from the center of the yolk than ϕ is.

This proposition is an immediate consequence of Theorem 2.

Of course, almost always the yolk has a positive radius. Agenda Proposition 4 deals with this complementary and far more likely case.

Agenda Proposition 4. If the yolk has a positive radius, for any *status quo* ϕ the agenda setter can always propose a forward moving agenda that yields any outcome ϕ^* in the alternative space as the outcome.

This proposition follows directly from Theorem 3; it is, in effect, the Agenda Proposition that is associated with McKelvey's (1976) name. However, this proposition, like the theorem on which it is based, distinguishes only between the case in which an unbeaten point exists and the case in which no such point exists; it may suggest that, once the yolk has any positive radius at all, the setter has (subject to the provisos discussed in Section 2) effectively unlimited agenda control, and that any further increase in the size of the yolk has no further effect on agenda control. In contrast, a principal theme of this paper is that, in relevant respects, agenda control is a continuous function of the size of the yolk; in particular, if the radius of the yolk is small but not zero, agenda control is effectively very limited. Most of the remaining Agenda Propositions indicate different ways in which this is so.

What Agenda Proposition 4 does not indicate is how many steps a forward moving agenda must include to yield an outcome a given distance further out from the center of the yolk than ϕ is.

Agenda Proposition 5. For any *status quo* ϕ at a distance d from the center of the yolk, the agenda setter can design a one-step agenda that yields an outcome ϕ^* *at most* $d + 2r$ from the center of the yolk.

This proposition follows directly from Corollary 5.1. Note that the proposition does not say that the agenda setter can design a one-step agenda that yields any point within $d + 2r$ of the center of the yolk as the outcome but only that any such outcome must lie within this distance from c.

From this it follows that, if the agenda setter is for any reason restricted to an agenda of some particular length, he is always limited in the outcomes he can bring about. Further, for any given agenda length, this limit is a function of the size of the yolk, as indicated by Agenda Proposition 6.

Agenda Proposition 6. For any *status quo* ϕ at a distance d from the center of the yolk, the agenda setter can design a k-step forward moving agenda that yields an outcome ϕ^* that is *at most* $d + 2kr$ from the center of the yolk.

This proposition follows from a k-fold application of Agenda Proposition 5.

Agenda Proposition 7. If the *status quo* is ϕ at a distance d from the center of the yolk, if the agenda must be fixed in advance of voting and if voting is sophisticated, the agenda setter can design and agenda that yields an outcome ϕ^* that is *at most* $d + 4r$ from the center of the yolk.

Recall that Shepsle and Weingast (1984) demonstrate that, under the specified conditions, the outcome ϕ^* must belong to $UC(\phi)$. The limit given by Agenda Proposition 7 [previously noted by McKelvey (1986, p. 302)] then follows by applying Corollary 5.2.†

Agenda Proposition 5 has the following further implication if, as under normal parliamentary procedure, the *status quo* ϕ must enter the voting last.

Agenda Proposition 8. If the *status quo* is ϕ at a distance d from the center of the yolk and if a standard agenda must be used, the agenda setter can design an agenda yielding an outcome ϕ^* *at most* $d + 2r$ from the center of the yolk.

Recall that, under the specified condition, ϕ^* must belong to $W(\phi)$. The proposition then follows directly from Agenda Proposition 5.

The bounds established by Agenda Propositions 5–8 overstate the agenda setter's freedom of maneuver in an important way, in that they derive from the corollaries to Theorem 5 and not from Theorems 4 and 5 themselves. Consider a *status quo* ϕ at a distance d from the center of the yolk. The distance $d + 2r$ from the center of the yolk has been taken to establish the agenda setter's one-step freedom of maneuver in all directions from c. But, in fact, the constraints given by the outer cardioid are tighter, in particular in the direction of ϕ itself. Indeed, Theorem 4 tells us that, for a point ϕ lying outside of the yolk, $W(\phi)$ within the vicinity of ϕ does not extend much outward from the yolk beyond ϕ, especially if the distance from the center of the yolk to ϕ is large relative to the size of the yolk.

†In particular, the setter must be able to find—as in the manner of sophisticated voting in Agenda Proposition 2—some point z that beats ϕ and is beaten by ϕ^*. The thrust of Shepsle and Weingast's (1984) substantive interpretation, based on particular examples, is that this is but a loose constraint and that the setter could typically get an outcome at or near his own ideal point. However, their examples involve just three voters. With more voters, the yolk would—as we note in the concluding section—typically be smaller relative to the distribution of ideal points, making the constraint implied by Agenda Proposition 7 relatively severe.

Table 1

d	D	$(D-d)/2r$
2	4.000	1.000
3	4.583	0.792
5	6.083	0.542
7	7.810	0.405
10	10.583	0.292
15	15.395	0.198
30	30.199	0.100

Agenda Proposition 9. From any point x outside the yolk, the agenda setter can extend a voting trajectory outward from the yolk only to points outside the dominated cone, i.e. to points on rays such that $\cos\theta > -r/d$.

Since the trajectory can be extended only to points in $W(x)$, the proposition follows directly from Theorem 4.

The dominated cone is thin when d is only slightly greater than r but it opens wider and wider as d increases relative to r. Thus, the agenda setter is more and more restricted as the trajectory moves outward.

Clearly the greatest distance a voting trajectory can move from point x (at a distance d from the center of the yolk) outward in a single step is to (just short of) the reflection of x through the median line most distant from x. This median line will be on the far side of the yolk from x and the reflection will be close to maximum distance of $d + 2r$ from c specified by Agenda Proposition 5. Thus, the kind of k-step forward moving agenda that leads from a *status quo* ϕ at a distance d from the center of the yolk to an outcome ϕ^* at a distance approaching $d + 2kr$, i.e. the maximum specified by Agenda Proposition 6, from the center of the yolk requires a trajectory that bounces wildly, and with increasing amplitude, back and forth across the alternative space.

One potential institutional constraint on the agenda setter, however, may preclude such an agenda. This is the imposition—by formal rules or informal norms—of what we may call an *incrementalism constraint* on a forward moving agenda, i.e. a rule that a new proposal cannot be "too far" from the current *status quo*.

An incrementalism constraint restricts centrist movement only in the obvious fashion, i.e. regardless of distance from the center of the yolk, each inward step is limited simply by the magnitude of the constraint.

Movement outward, however, is restricted dramatically, as the incrementalism constraint limits each step to nearby points, and, as distance from the yolk increases, an increasing proportion of these points fall within the dominated cone. The impact of the constraint on outward movement, moreover, increases essentially with the square of distance from the center of the yolk. The numerical example shown in Table 1, for the case in which $r = 1$ and the magnitude of the constraint is 6, is illustrative. For sample values of d (the distance from the center of the yolk to the current *status quo* x), the table shows the corresponding values of D, the maximum distance from the center of the yolk that the agenda setter can extend the trajectory outward in one step from x, given the incrementalism constraint.† The third column shows $D - d$ increments in relation to the $2r$ maximum that the agenda setter could move the trajectory outward in the absence of any constraint.

Thus, at $d = 2$ the constraint has no effect, but beyond that such a constraint reduces potential outward movement, the more so as d increases. At $d = 10$, potential outward movement has been reduced to less than a third of what it would be in the absence of a constraint, and at $d = 30$ it has been reduced to one-tenth.‡

†D is the distance from c to the intersection of the circle about x defining the incrementalism constraint and the outer cardioid with cusp at x.

‡Results due to Schofield (1986) indicate that if voting trajectories are continuous—in effect, if an infinitely severe incrementalism constraint were imposed, there would be significant differences between the two-dimensional and higher-dimensional cases. In fact, it appears that even a finite incrementalism constraint will, in two but not higher dimensions, absolutely block further outward movement of a voting trajectory at some distance from the center of the yolk, but we have not tried to establish this point here.

5. CONCLUSIONS

While the "chaos theorems" pertaining to majority rule in spatial voting games are important and technically elegant results, their practical significance for political choice processes can be overstated. On the one hand, McKelvey (1986) has shown that several choice processes driven by competition among agents lead to generally centrist outcomes—in particular, to outcomes in the uncovered set. Thus, the "chaos theorems" appear to have the greatest practical relevance for essentially noncompetitive choice processes, such as those controlled by a monopoly agenda setter. But we have shown that the fundamental structure of majority rule in spatial voting games (at least those with Euclidean preferences) creates significant centripetal forces that make it intrinsically much harder to design agendas that generate trajectories leading outward, as opposed to inward, and which, as a result, impose significant limits even on a monopoly agenda setter. These limits become especially significant in conjunction with what might appear to be merely technical procedural rules pertaining to voting, which the agenda setter may be obliged to follow. These include rules that limit the size of agendas, that require that the agenda be fixed and announced in advance of any voting (permitting sophisticated voting), that require that the *status quo* be voted on last or that require that new proposals cannot differ too much from the *status quo*.

The strength of these centripetal forces is inversely related to the size of the yolk—put otherwise, they are directly related to how close the distribution of voter ideal points comes to generating a majority rule equilibrium. It has been known for a long time that these centripetal forces are dominant in the unlikely event a majority rule equilibrium exists. But the "chaos theorems" may suggest that, if the distribution is perturbed even slightly so that the majority rule equilibrium is destroyed and a global cycle appears, these centripetal forces disappear entirely. In contrast, we show that the strength of these forces is a continuous function of how closely the voter distribution approaches one that would generate a majority rule equilibrium. If the yolk is very small, the centripetal forces are very strong. If the yolk is very large, majority rule is indeed rather chaotic.

In concluding, two further observations are in order. First, it is worthwhile to point out explicitly that most theorems and propositions in this paper assume all that is known about the distribution of voter ideal points is what is given by the location and size of the yolk. As a necessary result, the indicated limits on agenda control are if anything understated, especially if the number of voters is small. Suppose, for example, that the *status quo* ϕ is at the center of the yolk and the agenda setter wants to produce some outcome ϕ^* at a distance a bit under $4r$ from the center of the yolk. By Agenda Proposition 6, this will require a forward moving agenda of at least two steps. But if we examine an actual configuration of three voter ideal points at the vertices of an equilateral triangle, it turns out that at least three steps will be required.

Our final observation is that the import of our Agenda Propositions obviously depends on whether the yolk is typically small or large relative to the distribution of ideal points. If the yolk is typically large—so that most ideal points lie within or near the yolk—majority rule would typically be quite chaotic and the limits on agenda control identified here would be very weak. However, we expect the yolk typically to be small relative to the distribution of voter ideal points. Certainly, the yolk is contained within the Pareto set; it can contain more than a very small proportion of the ideal points only if the remaining ideal points are very oddly distributed; and the yolk is unlikely to expand in size, and probably shrinks in size, as new ideal points are added to the distribution.† Thus, we believe that the limits identified here, stated relative to the size of the yolk, are typically quite severe, relative to the distribution of voter ideal points.

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