ELECTORAL SYSTEMS, PARTY SYSTEMS, AND GOVERNMENT SELECTION

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1. Overview

This module in the *Vote*Democracy course examines interconnections among electoral systems, party systems, and processes of government selection. Since these matters exhibit great diversity across both nations and time, and since the module draws primarily on concepts drawn from the theory of voting and social choice introduced in other modules, these interconnected topics are discussed schematically and theoretically, in contrast to the more descriptive treatments common in courses on comparative politics. The module examines electoral systems based on *plurality rule* in *single-member districts* (SMDs) — commonly referred to as *First-Past-the-Post* (FPTP) and traditionally used in English-speaking countries — and those based on (list) *proportional representation* (PR) in (typically large) *multi-member districts* (MMDs) — commonly used on the continent of Europe and elsewhere (and discussed in greater detail in Module [Pukelsheim]) — as well as those based on semi-proportional electoral rules in small MMDs (also discussed in Module [Kilgour]).

2. Origin and Nature of Political Parties

Our aim is primarily to examine how electoral systems allocate seats in national parliaments (or other assemblies) to political parties, rather how individual candidates are elected. This is because political parties are the dominant players in elections and assemblies and in process of government selection — indeed, some claim that political parties created democracy as it exists in practice and that such democracy in unworkable in their absence.

Both theory and evidence suggest that any system of free elections to fill seats in a large legislative body can be expected to lead to the formation of political parties and thus to a *party system* of some type. Large-scale elections and government formation thus revolve around political parties. The following schematic theory of political party formation, drawn from Schattschneider (1942, Chapter 3), probably most closely fits the history of political parties in early U.S. elections, since the newly ratified U.S. Constitution created from scratch a popularly elected representative assembly, i.e., the U.S. House of Representatives (together with an indirectly elected Senate and President), at a time when nationally organized political parties did not yet exist (whereas in Britain and elsewhere parliamentary institutions and political parties evolved slowly and contemporaneously over an extended period of time).

In Schattschneider's view, a *political party* forms as a result of collaboration among *politicians*, i.e., elective office holders or office-seekers, rather than among rank-and-file voters. This collaboration extends over geographical areas (e.g., parliamentary districts or U.S. states), over a wide range of issue areas (in contrast to the collaboration that forms an *interest group*), and over time (unlike individual politicians, a party is likely to endure over many elections), and aims at securing political power through electoral activity. The most fundamental kind of collaboration among party politicians is the *nominating* (or candidate selection) function as a means of coordinating the votes of the party's natural supporters.

Schattscheider's theory of party formation starts with a 'pristine' legislature unsullied by any kind of political organization (possibly approximated by the first Congress of the U.S. in 1789) that is *elective*, that is relatively large, and that operates under *majority rule* (as discussed in Module [Miller], especially Sections 11-12). In such a legislature, preferences and votes may be highly dispersed or quasi-random. In this circumstance, any small group of members can gain a great advantage by collaborating to form a legislative *caucus*, i.e., a group that, prior to any legislative vote, meets to agree on a common *position* and then votes as a *bloc* one way or the other. At the outset, such a caucus is likely to carry most votes. But such organization provokes counter-organization, in the form of other caucuses, followed by the merger of smaller caucuses into large ones. This process of organization and counter-organization is likely to continue until one caucus achieves majority status. At this point, the majority caucus has no incentive to expand further, because (given legislative majority rule) it is already all-powerful, and any other (minority) caucus may be totally shut out of power.

A minority caucus therefore has a strong incentive to upset this state of affairs. Possibly it can induce some members of the majority caucus to defect to its side and thereby achieve majority status within the legislature. But the fact that the legislature is elective — which implies that there is always an upcoming election — gives the minority caucus an incentive to turn itself into an electoral (as opposed to merely legislative) collaboration, i.e., a political party. Its members can pool their resources for electoral battle, both by securing their own re-election prospects and also by running candidates against members of the majority caucus. They can thereby attempt to convert themselves from a minority caucus into a (majority) political party. Once, the minority caucus becomes a political party, the majority caucus must do the same, and a party system thus emerges. The nature of this party system depends importantly on the nature of the 'electoral system', i.e., the rules used to elect members of the legislature.

3. Electoral Systems: Districts and Electoral Rules

An *electoral system* is a voting rule used to fill the seats in any kind representative assembly, be it a small local council or a large national parliament. An electoral system has two principal components: a system of *districts* and an *electoral rule* employed within each district. In addition, the nature of the electoral rule determines the *ballot structure* employed, and the entire system is shaped by overall *assembly size*.

Districting guarantees a measure of geographic representation. For example, a small assembly — for example, a local council with five members — council members may be elected from *single-member districts* (SMDs), so that the territory represented by the assembly is divided into five approximately equally populated districts and one council member is elected from each district. Alternatively, if geographic representation is deemed less important, the members may be elected *at-large*, so that all five members are elected from a single area-wide *multi-member district* (MMD). A third possibility is a *mixed system* in which some members are elected from SMDs and others are elected at-large. In a larger assembly representing a larger population spread over a larger area, members may be elected from several or many (MMDs, so that the region or nation is divided into larger approximately equally populated districts and each district elects the same fixed number of two or more members; alternatively, it may be divided into unequally populated districts (perhaps defined by the pre-existing boundaries of states, provinces, counties, or administrative regions), each of which elects a number of members (approximately) proportional to its population.[1] Election at-large may be deemed election from a single multi-member district.

Next, some voting rule must be used to elect members from each (single-member or multi-member) district. Modules [Miller, Section 5; and Nurmi] describe various voting rules that may be used to elect a single candidate, and that therefore may be used as electoral rules for SMDs. The most commonly used is *plurality rule*. The electoral system that combines single-member districts with plurality rule is commonly called *First-Past-the-Post* (FPTP). Others rules that are used in conjunction with SMDs include *plurality runoff*, the *Alternative Vote* (or *Instant Runoff Voting*), and *Two-Round Majority-Plurality* (see Module [Miller], Section 5).[2]

Many electoral rules are *candidate-oriented*, in that a voter is presented with a ballot that lists a number of candidates (often with their party affiliations indicated) and invites the voter to express some kind of preference with respect to these candidates. In practice, SMD systems always use candidate-oriented ballots. Plurality rule, plurality runoff rule, and Two-Round Majority-Plurality (as well as Approval Voting) use *nominal ballots* on which a voter either votes 'for' a candidate or not. use nominal ballots. In contrast, the Alternative Vote requires an *ordinal ballot* on which a voter ranks the candidates in order of preference, typically by putting a '1', '2', etc. beside the names of candidates to indicate order of preference. Many MMD also use candidate-oriented ballots, which may be either nominal or ordinal.

But other MMD electoral rules are *party-oriented*, in that a voter is presented with a ballot that lists a number of parties and invites the voter to express a preference with respect to these parties. In practice, such ballots are nominal, in that each voter simply votes for one party in the manner of plurality voting for candidates. Proportional representation electoral rules are party-oriented in this sense, as their purpose is to give each party a share of seats in parliament that matches its shares of the vote (as discussed in Module [Pukelsheim]). However, some proportional representation ballots also allow voters to express certain candidate preferences as well.

Total assembly size is a significant feature of electoral systems, in particular as large assemblies may allow representation for small parties and more generally can better realize a goal that parties have representation proportionate to their electoral support. Typically larger population are represented in larger assemblies; state legislatures and provincial assemblies are typically larger than city or county councils but smaller than national parliaments. Moreover, populous nations typically

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1 Districting is sometimes based on electorate size rather than total population; see Module [Bickerstaff] for a full discussion of electoral districts and issues related to them..

2 Module [Miller] also discusses Approval Voting and the Borda and Copeland rules, but these are rarely if ever used as electoral rules.

have larger parliaments than less populous ones. Indeed as a rough rule, assembly size, where S is the total number of seats, bears a definable (if approximate) relationship to population (or electorate) size P such that S is approximately the cube root of P.[3]

3.1 Candidate-Oriented Electoral Systems

We first discuss electoral rules that use candidate oriented nominal ballots, are typically applied in small MMDs, and the top m candidates are elected. We will use the following notation pertaining to nominal candidate-oriented ballots:

n = the number of candidates on the ballot
m = the number of candidates to be elected, commonly called the *district magnitude* (presumably $m < n$);
k = the maximum number of votes each voter can cast (typically the maximum number of candidates each voter can vote for);
v = the number of votes each voter actually casts (so $v \leq k$ and ballots with $v > k$ are 'spoiled' and disqualified);
N = the total number of voters casting ballots;

Under FPTP, $v = k = m = 1$. Plurality rule can be generalized to MMDs (for which $m > 1$) in two ways. First, k can be increased to match the larger m. This gives us either *plurality at-large rule* with $1 \leq v \leq k = m$, or *Block Voting* with $v = k = m$. In either case, each voter *may* vote for as many candidates as are to be elected, but in the latter case each voter *must* do so. Plurality at-large is rather commonly (and Block Voting occasionally) used in local council elections and some state legislative elections in the U.S. Moreover, a party-oriented *general ticket system*, under which each party runs a slate of m candidates and voters choose among party slates, rather than individual candidates, on the basis of plurality rule turns an MMD election into one that is logically equivalent to an SMD single-winner election.[4] We shall refer to this family of rules as *MMD plurality*.

Second, we may keep $v = k = 1$ even as m increases beyond 1. This gives us the *Single Non-Transferable Vote* (SNTV), which has commonly been used in parliamentary elections in Japan, Korea, Taiwan, and elsewhere. Between these two possibilities lie a range of electoral rules that fall within the rubric of *Limited Voting* with $1 \leq v \leq k < m$, under which each voter may vote for several candidates but fewer than the number to be elected. (SNTV may be deemed a special case of Limited Voting.)

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[3] While the active electorate is typically only about one half of the total population, it can be checked that, given the cube root relationship, it makes little difference for predicted assembly size which version of P is used. This relationship can be deduced logically as the one that produces an optimal balance between the 'ratio of representation', i.e., representatives per constituent (in terms of which a larger S is desirable) and a manageable assembly (in terms of which a smaller S is desirable). Moreover the cube root relationship is broadly supported empirically.

[4] A notable example is provided by the system used by almost all U.S. states to select Presidential electors (which produces the state 'winner-take-all' feature of the Electoral College.

Approval Voting and the Borda rule also can be readily generalized to MMDs, simply by electing the top m candidates ranked by approval votes or Borda scores.[5]

One other multi-winner electoral rule using nominal ballot may be mentioned. Under *Cumulative Voting*, each voter can cast as many votes as there are candidates to be elected (so $v \leq k = m$) but, instead of being spread equally over m candidates, they may be 'cumulated' on fewer than m candidates (or even 'plumped' a single candidate).

The principal electoral rule that uses ordinal ballots in MMDs is the *Single Transferable Vote*, a rather complicated rule used in Ireland, Malta, Australian Senate elections, and in a few other nations and localities. In the U.K., STV has been passionately advocated by the Electoral Reform Society (ERS) and by the Liberal (now Liberal Democrat) Party. It works in the following manner.[6]

(1) The total number of ballots cast (N) is determined.

(2) The *Droop quota* Q_D is calculated, where Q_D is the next integer above $N / (m + 1)$; Q_D is the smallest number of votes such that no more than m candidates can get Q_D votes. A candidate with Q_D votes is said to 'meet quota'.

(3) The ballots are then sorted into piles according their first preferences. Any candidate who meets quota is elected. If m candidates meet quota, counting stops.

(4) Otherwise 'surplus' ballots are 'transferred' from elected to non-elected candidates. Ballots in excess of Q_D in the piles of elected candidates are transferred to the piles non-unelected candidates according to the second preferences expressed on these ballots.[7] As a result of the transfer of surplus votes, additional candidates may now meet quota and be elected. If m candidates now meet quota, counting stops.

(5) Otherwise the candidate with the *fewest* (first preference plus transferred) ballots is eliminated and all of his or her votes are transferred according to the highest preference expressed on the ballots for any remaining (non-elected and non-eliminated) candidates. As a result of this transfer, additional candidates may meet quota and be elected, with their surplus votes transferred as above. If m candidate now meet the quota, counting stops.

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[5] Several ministates in the South Pacific actually use variants of the Borda rule. Approval voting is not used in any public elections.

[6] The following describes the ERS rules for STV, which are designed for hand counting of ballots. More sophisticated rules (that can produce different winners) have been proposed that require computer processing of ballots.

[7] However, it is arbitrary *which* specific ballots are deemed to be surplus and different ballots have different lower preferences. Thus ERS recommends that ballots be transferred in proportion to *all* the second (or lower) preference; this means that ballots transferred from elected to unelected candidate typically have fractional values.

(6) Otherwise, candidates are eliminated one at a time, their votes are transferred, and the counting continues until m candidates have met quota and are elected.[8]

Note that if $m = 1$, Q_D is a simple majority of votes cast and ballots are transferred only from eliminated candidates, so the Alternative Vote is the special case of STV applied to SMDs.

3.2 Party-Oriented Electoral Systems: List Proportional Representation

Under (party list) *proportional representation* electoral rules, candidates are elected from large MMDs (possibly a single nationwide MMD) by voters who indicate their preferred party using a simple nominal party-oriented ballot. Once each party's vote is tallied, some mathematical *apportionment formula* is used to allocate the m seats among the parties in proportion to their respective vote shares.[9] Prior to the election, each party draws up an *ordered list* of m candidates. If the apportionment formula awards a party p seats, the top p candidates on its list are elected. However, proportionality can (almost) never be perfect, because perfect proportionality would require that parties be awarded fractional seats, whereas seats must be awarded in whole numbers. Different apportionment formulas resolve this problem in different ways. Some additional notation is useful in discussing apportionment formulas:

n_i = vote for party i (where $\Sigma n_i = N$); and
q_i = party i's 'ideal quota' of seats, i.e., $q_i = m \times (n_i / N)$.

List PR is used in most continental European countries and in many others elsewhere. However, PR rules can be extremely complex and a great many variations exists. No two national PR systems are identical; moreover, individual PR countries frequently modify the details of their PR systems. First of all, different systems use different apportionment formulas, which may apportion seats in somewhat different ways. These formulas are of two main types.

Quota methods. These methods award each party i one seat for every 'quota' contained in its vote n_i and then assign any remaining seats to the parties with the 'largest remainders', where a party's *remainder* is the difference between its vote and the whole number of quotas contained therein. Different methods use different quotas, including the *simple* or *Hare* quota $Q_H = N/m$, the *Droop* (or *Hagenbach-Bischoff*) quota $Q_D = N/(m+1)$, which was discussed in connection with STV, the *Imperiali* quota $Q_I = N/(m+2)$, and the 'enhanced' Imperiali quota $Q_{H+} = N/(m+3)$, where each ratio is rounded up to the next whole number. Notice that, taking them in the order listed, the quotas become smaller and therefore allocate more seats, leaving fewer to be allocated on the basis of remainders; the effect is to make the allocation of seats more favorable to larger parties. The operation of these formulas and their effects on seat allocation are illustrated in Table 1 for the case of $m = 5$. The Hare quota awards two seats to Party A and one to Party B; parties C and B have

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[8] Though complicated, this description does not address all the complexities (resulting from incomplete ordinal ballots, various types of ties, etc.) that can arise in the STV vote counting process

[9] Similarly such a formula is used to fulfill the requirement of the U.S. Constitution that 'Representatives . . . shall be apportioned among the states according to their respective numbers [populations]', as noted in Module [Bickerstaff]).

largest remainders and awarded to two remaining seats. The Droop quota awards seats the same way but now parties A and C have the largest remainders. The Imperiali quota awards three seats to A and one to B and B has the largest remainder and gets the extra seat. The (rarely used) 'enhanced' Imperiali quota awards too many (six) seats, so the size of the district must (temporarily) be increased in magnitude (thereby also increasing the size of parliament). The most common quota apportionment method, *Largest Remainder-Hare* (LR-H), uses the Hare quota. LR-H may also be characterized in this way. First give each party *i* its ideal quota rounded down to the nearest whole number of seats and award any remaining seats to the parties with the 'largest remainders'.

Divisor Methods. While quotas must be calculated for each district magnitude (m = 5 in Table), the standard way of applying divisor methods allocates seats sequentially from m =1 upwasrds, i.e., they determine which party gets the first available seat, which the second, and so forth. For each seat it has been awarded, a party's vote is adjusted by dividing it by a sequence of divisors. The *D'Hondt* method uses the divisor sequence 1, 2, 3, 4, . . ., the *Sainte-Laguë* method uses the sequence 1, 3, 5, 7 . . ., and the *Modified Sainte-Laguë* uses the sequence 1, 2.14, 3.57, 5, ...[10] At each point in the sequence, the next seat is awarded to the party with the highest adjusted vote. The operation of these methods is illustrated up to m = 9 in Table 2 for a profile of votes for five parties. Since the initial divisor is always 1, votes are initially unadjusted and the largest party A is always awarded the first seat. D'Hondt also awards the second seat to A, since its vote divided by 2 is larger than B's vote; however, both Sainte-Laguë and its modified variant award the second seat to party B, since its vote exceeds A's vote divided either 3 or 2.14. However, Sainte-Laguë awards the third seat to party C, while Modified Sainte-Laguë awards it to A. It is evident from the table that D'Hondt treats large parties more favorably than Sainte-Laguë does, while Modified Sainte-Laguë stands between them in this respect. LR-H and (unmodified) Sainte-Laguë allocate seats in very similar (often identical) ways; they exhibit no bias between large and small parties and are generally deemed to be the 'most proportional' apportionment formulas.[11]

Other variations in PR systems include the following matters.

MMD magnitudes and tiers. Different PR systems use different and often varied magnitudes of MMDs. Moreover, many two or more tiers of MMDs, often including a tier of 'national adjustment seats' designed so that, with respect to the overall allocation of seats in parliament, there

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[10] The Modified Sainte-Laguë divisor sequence is customarily stated as 1.4, 3, 5, 7, which makes more apparent its similarity with unmodified Sainte-Laguë. The sequence given in the text simply divides each of these numbers by 1.4, so that the sequence begins with 1 like the others.

[11] Divisor methods for a fixed district magnitude m may also be implemented by dividing each party's vote by a common divisor such that, when the resulting quotients are rounded to whole numbers according to some rule, they add up to m. Divisor methods differ according to the rounding rule used: D'Hondt rounds all quotients down to the next whole number, while Sainte-Laguë rounds quotients up or down in the normal manner. For the example in Table with m = 9, the common divisor 8,875 produces the D'Hondt apportionment and the common divisor 12,500 produces the Sainte-Laguë apportionment. Such a common divisor must be discovered by trial and error, but the 'ratio of representation" N/m is a good starting point. Despite this, the common divisor approach may by the most efficient mode a calculation in large (especially nationwide) MMDs.

is effectively a single nationwide MMD, even while most members are elected from smaller (regional or local) MMDs. In addition, different apportionment formulas may be used in different tiers.

Thresholds. Given a MMD of magnitude m and a profile of party vote shares, small parties supported by no more than about N/m voters fail to win any seats under any apportionment formula. However, it is not possible to specify a fixed threshold below which a party fails to win even one seat, because (if there are three or more parties) the number of seats a party wins depends not only on its own vote share but also on the vote shares of all parties. However, most national PR systems impose a higher and fixed vote share *threshold* such 1.5%, 3%, 5%, etc.) that a party must meet before it qualify for any seats, even though the normal operation of the apportionment formula might award it one or more seats. The apportionment of seats among parties that do win seats is then based, not on their shares of the total vote, but on their shares of the total vote cast for parties that meet the threshold.

Party Alliances. Some PR systems allow two or more parties to form a pre-election alliance (so-called *apparentement*), by pooling their lists and votes in order to (possibly) increase the number of seats they jointly win.

Candidate Preferences. Various types of 'open-list' PR systems allow voters to express preferences for candidates and thereby (possibly) influence which particular candidates are elected, e.g., to change the order of candidates on a party's list. Some systems allow a voter to vote directly for a candidate; this counts both a vote for the party's list and a vote for the particular candidate on the list.

3.3 Mixed Electoral Systems

Some countries use 'mixed' electoral systems that combine FPTP and PR in various ways, for example by electing about half the members from SMDs and the other half by list PR applied on a national basis. These two components may operate in either a *parallel* or a *compensating* fashion. In the first case, the PR apportionment formula is applied only to the seats elected by PR; the effect is that the overall distribution of seats is a compromise between FPTP and PR results. In the second case, the PR apportionment formula is applied to the whole number of parliamentary seats and the PR seats are allocated so as to bring the overall distribution of seats in line with the PR results (subject to any threshold requirement) compensating for any disproportionality in the FPTP results; the effect is that, even as about half the members are tied to local SMDs, the system is equivalent to PR so far as the overall allocation of seats is concerned.[12] A further question is whether each voter casts only a single vote for the local SMD candidate listed on the ballot by party affiliation, which is also counted as the party-oriented vote in the PR election, or whether voters can 'split' their votes by supporting a candidate of one party in the SMD election and a different party in the PR election.

The post-war German constitution notably provided for a mixed electoral system with compensating PR (and since 1953 German voters have been able to 'split' their votes); this is referred to as 'personalized PR'. More recently New Zealand and has adopted a similar system

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[12] Thus the PR seats become in effect 'national adjustment seats'.

(referred to as the 'additional member system') for it national elections, as has Scotland for election of its devolved parliament.

4. Properties of Electoral Systems at the District Level

This section considers properties of electoral systems as they operate at the district level, while Section 6 considers the relationship between votes and seats at the national level.

Electoral systems based on SMD are inherently majoritarian at the district level, in that only one candidate is to be elected and all SMD electoral rules in actual use are *strictly majoritarian*, i.e., any majority coalition of voters can elect its preferred candidate; moreover, they are *simply majoritarian*, i.e., it members can do this by casting all their votes for (or placing at the top of their ballot rankings) their preferred candidate. (Recall the discussion in Module [Miller], Section 5.1.) Thus the candidate of a party preferred by a majority of voters in the district is likely to be elected, though if several candidates affiliated with the majority party are running for the single seat, vote splitting may allow a candidate of a minority party to win the most votes and be elected under FPTP (as suggested by the discussion of 'clone candidates' in Module [Miller]).

In like manner, all variants of plurality rule applied to MMDs are majoritarian, in that a party supported by a majority of voters can nominate m candidates and expect to elect all of them. Conversely, even a large minority party cannot be assured of electing any candidates, though it may succeed in doing so as a result of vote splitting or coordination failure within the majority party. Block Voting somewhat mitigates such coordination problems, and a general ticket system preclude them entirely.

In contrast, other electoral rules based on MMDs do not allow a majority of voters to elect all their candidate but rather tend to produce roughly proportional results, though the degree of proportionality that it is logically possible is clearly limited in small MMDs. (Proportionality can improve as the magnitude of a district increases.) Moreover, strategic and coordination problems arise under some of these rules that mean that some parties may fail to win their roughly proportionate share of seats (and so others may win more).

In particular, under STV a party's success in electing candidates within a MMD depends on the number of Droop quotas contained in its electoral support contains, and this is more or less true under SNTV and CV as well. For example, in an MMD with $m = 5$, $Q_D \approx V/6 \approx 17\%$ of the vote, so a party with at least this much support should be able to win one seat in the MMD, one with about 34% should be able to win two seats, and so forth. Let's see how this works under specific electoral systems.

Under SNTV, a party with support clearly less than two Droop quotas should nominate a single candidate for whom its supporters would cast their single votes. If the party has a full quota of supporters, this candidate will be among the top m candidates, since no more than m candidates can receive such a quota of votes. However, a party that has greater electoral support faces strategic problems. The first is how many candidates it should nominate: too few and the party will certainly not receive its roughly proportionate share of seats; too many, and it risks winning fewer than its share and perhaps none at all, as its supporters spread their votes over too many candidates, many or all of whom may therefore fail to be among the top m. The second problem is that, if the party has two or more full quotas of support in the electorate, even if it nominates the appropriate number of candidates, the votes of its supporters must be coordinated so that they are spread as equally as possible over all of its candidates (thereby maximizing support for its least supported candidate).

Cumulative Voting has essentially the same properties as SNTV, except that the voter coordination problem is mitigated by the fact that party supporters can individually distribute their m votes more or less equally among the party's candidates.[13]

However, STV essentially eliminates these strategic problems in that, if every one of h voters ranks (in any order) a set of candidates higher than all candidates outside of the set, at least one of these candidates will be elected for every full quota contained in the set of k voters, i.e., at least h/Q_D candidates (rounded down to the nearest whole number) will be elected. Indeed, STV can in principle be characterized as a means of implementing the principle of *free association* (or 'self-defined constituencies'), according to which any group of voters of size Q_D is empowered to elect a candidate of their choice, regardless of their geographical distribution within the MMD (which could in principle be the whole nation). But since as a practical matter STV can be implemented only in small magnitude MMDs, realization of this principle is quite constrained in practice.

SNTV, CV, and STV in small MMDs are commonly referred to as *semi-proportional* electoral systems. The qualification 'semi' reflects the facts that (i) these systems are not based on any explicit apportionment formula, (ii) they (especially SNTV and CV) do not achieve proportional results entirely reliably, and (iii) they can in practice be used only in small MMDs that do not permit a high degree of proportionality. Recall that Limited Voting is equivalent to SNTV if $k = 1$ and to Plurality At-Large if $k = m$. Thus LV with $2 < k < m - 1$ ranges between these two extremes and tends to produce 'supraproportional' results, such that the leading party probably wins fewer than all the seats but more than its proportionate share.[14]

Partly because list PR systems use explicit apportionment formulas, but mostly because they can be used in high magnitude MMDs and because they entail no strategic or coordination problems for parties of any size, such systems can produce highly proportional results. But just as single-winner voting rules run into problems once there are three or more candidacies, PR apportionment formulas may run into various types problems once there are three of more parties.

First, we identify a number of 'monotonicity' and related conditions that we might expect proportional representation systems to satisfy but which in fact they may not.

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[13] Some variants of CV allow voters to cast fractional votes, so that each voter can always distribute them precisely equally among the party candidates.

[14] In practice SNTV tends to produce 'subproportional' results, such that small parties tend win more than their share of seats and the big parties less, because big parties face more severe strategic and coordination problems.

Monotonicity Condition 1: if party A wins more votes than party B, A wins no fewer seats than B. In the absence of *apparentement*, every apportionment formula satisfies this condition. However, as discussed in Section 7, PR makes party coalitions important in government formation, and no apportionment formula can guarantee that one coalition A of parties that collectively wins more votes than the parties in another coalition B will win no fewer seats than B.

Monotonicity Condition 2: if party A's vote share increasers from one election to the next (while the number of seats in parliament remains constant), its number of seats seat not decrease. No apportionment formula can guarantee this, because the number of seats a party wins depends not only on its own vote share but on the vote shares of all parties.[15]

Monotoncity Condition 3: if party A's vote increases relative to party B's vote from one election to the next (while the number of seats remains constant), A's seat share relative to B's should not decrease. Divisor formulas always satisfy this condition but quota formulas do not.

Monotonicity Condition 4: for fixed party vote shares, no party should lose (or gain) seats if the number of seats available is increased (or decreased). Divisor formulas always satisfy this condition (as Table 2 illustrates) but quota formulas do not.

Staying in Quota: a party should be awarded a number of seats equal to its ideal quota rounded either up or down to the next whole number. While quota formulas can satisfy this condition (and LR-H by design always does), no divisor formula can always do so. (In Table 2 with $m = 9$, party A's ideal quota is 3.94, so to stay in quota it should receive either 3 or 4 seats but it actually is awarded 5.)

Majoritarian constraints: a party that receives a majority of the vote should be awarded at least half the seats; conversely, a party than receives a minority of votes should not be awarded a majority of seats. In fact, no apportionment formula always satisfies either constraint.[16]

It is well known that FPTP can produce strange or unfortunate outcomes both at the district level (as discussed in Module [Miller]) as well as at the national level (extreme disproportionality between seats and votes, inversions, etc., as discussed below in Section 5). PR elections likewise can produce (arguably) unexpected or unfortunate outcomes. Here are a couple of examples.

PR, like Plurality, takes account of first preferences only. Imagine a society divided among three or more relatively hostile ethnic, language, religious, etc., groups, none of which constitutes a majority of the population and each of which has formed its own political party which is the most preferred party of almost all group members. List PR is commonly recommended for such a 'plural society'. However, suppose that an encompassing (multi-ethnic, multi-lingual, and/or secular)

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[15] Recall that this is also the reason that no apportionment formula entails a fixed threshold for winning seats.

[16] However, a few national PR system have such constraints written into their electoral laws, overriding the apportionment formula if necessary.

'alliance party' forms, which is the second preference of almost every voter but the first preference of very few. In this event, the 'alliance' party is the Condorcet winner among political parties but likely will win few if any seats in parliament under list PR.[17]

Threshold Requirements. A high threshold requirement under PR produces an extreme discontinuity between seats and votes in the vicinity of the threshold. For example, given the 5% threshold and parliament size of 630 in Germany, a party that gets 4.99% of the votes wins no seats, whereas a party that gets 5.01% wins about 32 seats. If a party is predicted to win less than 5% of the votes, some of its normal supporters may defect and vote 'strategically' vote for their more preferred larger party (in the manner discussed in Section . On the other hand, if a major and minor party form an tacit electoral alliance (such as the CDU+FDP or SPD+Greens in Germany), the major party may urge some of its voters to vote 'strategically' for its minor party partner (so as to help it meet the 5% threshold). In the absence of such strategic maneuvers, a PR system with a threshold can produce an extreme election inversion (in the manner of the U.S. Electoral College). Indeed, in the most recent German general election voters in effect were choosing between two prospective coalition governments: the incumbent center-right CDU+FDP coalition and the opposition center-left SPD+Green coalition. The former won a greater vote share than the latter but could not form a government because the FDP fell below the 5% threshold.

5. Strategic Effects of Electoral Systems and Duverger's Law

Module [Miller], Section 5 notes that no voting rule for electing a single candidate can be *strategyproof* if there are more than two candidates in the field. The same is true of electoral rules for electing several candidates in MMDs. Here we present a general but informal way of thinking about, and analyzing the effects of, strategic voting calculations in MMDs and in the special case of SMDs. We assume that voters want to use their votes to influence the outcome of elections in a way consistent with their preferences and not merely to sincerely express their preferences regarding candidates. More specifically, we assume that they want to influence the outcome of the *present* election and not, for example, to influence which candidates or parties may enter the field in subsequent elections (which long-term goals might justify other choices in the present election).

5.1 Strategic Voting by Individuals

We first focus on strategic voting under electoral rules using candidate-oriented ballots. Let us assume that all voters in a district initially intend to cast *sincere ballots*. If an ordinal ballot is used, a sincere ballot ranks the candidates on the ballot exactly in accord with the voter's preference ordering. If a nominal ballot is used, a sincere voter votes for candidate A only if the voter also votes for all candidates he or she prefers to A.

Consider an election in an MMD in which m candidates are to be elected. Let us suppose that, prior to the election, voters form (on the basis of the historical partisan inclinations of their district, early pre-election polls, news stories, or other information sources) broadly shared expectations

[17] Of course, such a party would probably do no better under FPTP.

concerning the relative electoral strength of the candidates, i.e., their relative popularity in the electorate and, more particularly, the order in which the candidates will finish in the election (in terms of ordinary votes, approval votes, first preference votes, Borda scores, etc., depending on the electoral rule in use). In the absence of some such information, voters cannot vote strategically.

Suppose that there are n ($> m$) candidates $C_1, C_2, \ldots, C_m, C_{m+1}, \ldots, C_n$, where the subscripts indicate the relative perceived strength of the candidates, so that C_1 is the strongest and C_n the weakest. *Leading* candidates are very likely to win, *trailing* candidates are very likely to lose, and the prospects of *competitive* are uncertain. Neither the prospective margins of victory of the leading candidates nor the prospective margins of defeat of the trailing candidates are terribly important. What is important is the relative standing of the competitive candidates, some of whom will win and others of whom will lose. Rather typically there may be just two competitive candidates C_m and C_{m+1} and what is uncertain is which one will win a seat and which will lose.[18]

To keep things simple, let us focus on this most typical case. This expectation will evidently cause voters to reconsider, in a strategic fashion, how they should vote and, in particular, will induce some voters to cast insincere ballots. Voters will be induced to change their voting intentions based on strategic calculation only if both of the following conditions hold: first, the voters must be *concerned*, i.e., have a clear preference between the competitive candidates and, second, their sincere ballots must not already (fully) reflect this preference.

Under the Single Non-Transferable Vote, concerned voters who originally intended to vote for a either a trailing candidate or for a leading candidate (especially one with a comfortable cushion of support) may be induced to vote instead for their preferred competitive candidate (since otherwise their votes cannot affect the election outcome and are 'wasted').

Under cumulative voting, concerned voters who originally intended to spread their votes over a number of candidates or to plump them on a trailing or leading candidate may be induced to plump them on their preferred competitive candidate.

Under approval voting, concerned voters who originally intended to vote for both or neither of the competitive candidates may be induced to discriminate between them, voting for the preferred one and not the other.

Under generalized plurality, concerned voters who intend to vote for fewer than m candidates or m candidates including some trailing candidates but not either competitive candidate may be induced to switch one of their votes to their preferred competitive candidate.

Under Borda rule, every voter ranks all candidates and thus already intends to express a preference between the competitive candidates, but a voter with strong preference between the two

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[18] However, three or more competitive candidates may be clustered in a near (expected) tie at the borderline between leading and trailing candidates. On the other hand, there may sometimes be a substantial gap in strength between C_m and C_{m+1}, so that the outcome of the election is pretty much a foregone conclusion, i.e., all candidates are either leading or trailing.

competitive candidates may now be induced to reinforce that expressed preference by inserting other leading and/or trailing candidates between the two competitive ones on the ordinal ballot, so as to maximize the Borda score advantage of the preferred competitive candidate

Finally, under the Single Transferable Vote, some supporters of a leading candidates who would have with surplus support if they voted sincerely may have an incentive to move another preferred candidate to the top of their ballots in order to prevent that candidate from being eliminated for lack of first preference support. However, in general strategic calculations STV appear to be so complex that voters will probably chose to vote sincerely.

In general, such strategic adjustments in voting intentions have two possible effects on the strength of non-competitive candidates. First, under all systems, leading candidates (especially those far in the lead) are likely lose some of their prior support, which migrates to competitive candidates. Second, under all systems, trailing candidates are likely to lose some or all of what little support they originally had, which also migrates to competitive candidates. Third, under Borda rule some trailing candidates may gain support and leading candidates may lose support as they are inserted between competitive candidates on ordinal ballots; under STV leading candidates may lose support to trailing candidates in an effort to make them competitive. The first two effects are more general and powerful and result from the tendency of voters to redirect their votes to 'where the action is', i.e., to competitive candidates. The third effect is the peculiar result of the few systems using ordinal ballots.

Now suppose a new round of pre-election polls (and/or other similar information), which reflect these strategic adjustments in intended votes, becomes available to voters. To continue to simplify matter, let us suppose that no new information comes to light that reflects on the merits of the candidates, no candidate has committed a major gaffe, and so forth. That is, we suppose that sincere preferences are unchanged since the first poll and that any changes in the standing of the candidates reflect only strategic adjustments in voting intentions. In light of the new polls, voters make can further strategic adjustments in their voting intentions, and so forth through several rounds of polls and adjustments.

We can now observe that the first strategic effect is *self-limiting*. For example, suppose that the effect is strong enough actually to threaten a leading (with respect to sincere preferences and original voting intentions) candidate A's status as leading (with respect to revised voting intentions after strategic adjustments). Given a new round of strategic adjustments, some voters will be induced to redirect their votes back to A, restoring A's leading status (with respect to voting intentions as well as sincere preferences). On the other hand, the second effect is *self-reinforcing*: as voters desert a trailing candidate B, B's trailing status becomes even more pronounced and evident, encouraging further desertions, so that B's trailing status becomes still more pronounced and evident, and so forth.

Thus, after several rounds of polls and strategic adjustments, we can expect (at least under the nominal ballot systems) an outcome that looks more or less like the following: candidates C_1 through C_{m+1} receive substantial support in the election and m of them are elected (most likely all but C_{m+1} or C_m), while candidates C_{m+2} through C_n receive very little support. Thus we have the following proposition: an MMD with magnitude m generally results in an election outcome with m+1 'serious' candidates who receive substantial vote support (even though additional candidates are on the ballot and may well have some support with respect to sincere preferences). Beyond this, there is some tendency for the vote support for these 'serious' candidates to be more evenly distributed than their support in sincere preferences, because some voters who most prefer very strong candidates feel free to desert them in favor of the relatively preferred competitive candidates (but not to the extent that such candidates fail to be elected).

5.2 Strategic Entry and Exit by Candidates and Parties

To this point, we have not assumed that candidates share political party affiliations — either the election is non-partisan or each candidate represents a different political party. However, given partisan elections in MMDs, one or several parties may have enough electoral support that they can expect to elect more than one candidate (depending largely on how many full quotas their electoral support contains). Recall that, under a semi-proportional electoral formula, a political party that commands the loyalty of some number of voters in an MMD can expect to elect zero, one, or several of its candidates, depending on how many (Droop) quotas its electoral support contains. On this basis we may distinguish among leading, competitive, and trailing political parties as well as candidates.

As we saw in Section 4, to realize these expectation leading parties must make strategic calculations, dependent on the electoral formula, with respect to how many candidates they should nominate and how they should urge or instruct their supporters to vote.

On the other hand, trailing parties have a clear incentive to *collaborate* (or engage in *fusion*) by nominating a common candidate so as to (try to) pool their electoral support into a bloc that approaches quota size and may allow them to elect the candidate. Likewise a competitive party can collaborate with a trailing one to gain a full quota of support and be assured of electing least one candidate. Even a leading party may have an incentive to collaborate with another party in order to be able to elect more candidates. Such collaboration could extend from a temporary expedient in which one candidate and/or party makes a strategic exit from the present election and endorses another candidate and/or party to a full and permanent merger of the parties.

Presumably, such collaboration can profitably occur only among parties that are ideologically proximate or, in any case, not entirely opposed in their policy goals. Suppose there are three parties in an MMD with 100,000 voters in which two candidates are to be elected (so the quota is 33,334) by SNTV. Suppose the Left Party is supported by 24,000 voters, the Center Party by 56,000, and the Right Party by 20,000. Only C has a full quota of support and even C can be certain to elect only one candidate. In the absence of inter-party collaboration, however, C will elect both candidates (provided C can coordinate the votes of its supports so that they are sufficiently equally divided between the two candidates) and L and R will elect none. L and R could take the second seat away from C if they pooled their support but — given that they represent opposite ideological extremes — it is unlikely that they could agree on a common candidate they would both prefer to a C candidate.

In general, we expect there to be just $m+1$ 'serious' (leading or competitive) candidates, we expect there to be no more than $m+1$ 'serious' (leading or competitive) parties under candidate-oriented systems. Indeed, we probably expect fewer than $m + 1$ parties if m is at all large, since several parties are likely to elect more than one candidate. At the same time, we would expect no fewer than two serious parties, because there should always be a competitive contest for the m^{th} seat in the district.

In contrast, given party-oriented list PR, strategic incentives for parties are greatly reduced — though not eliminated — because list PR allows very large magnitude districts (even national districts), which in turn make the quota very small, with the result that many parties — even (depending on a threshold requirement) quite small ones — can elect at least a few candidates on their own, with the result that almost all parties with significant support in the electorate are leading. Moreover, the seat-winning capacities of these major parties depends on their electoral support in an 'almost continuous' fashion, instead of in the conspicuously 'stepwise' fashion that results when a few large quota thresholds are surpassed. The principal exception to this generalization occurs when a list PR system imposes a relatively high threshold for a party to win any seats. The example of Germany has already been noted.

5.3 Strategic Voting in SMDs

Finally, we consider the special case of SMDs with Plurality Voting. With $m = 1$, the quota is a simple majority, so candidates C_1 and C_2 typically are both competitive and all other candidates are trailing. Of course, if there is something like a three-way (or more extensive) tie for first place, more than two candidates are competitive. Conversely, if C_1 has a substantial lead over C_2, C_1 has leading status and C_2 through C_n are trailing.

With respect to strategic adjustments in voting intentions, the first effect cannot occur because either there are no leading candidates or, if C_1 is leading, there are no competitive candidates, so only the self-reinforcing second strategic effect occurs. Thus we get at the district level what political scientists refer to as *Duverger's Law* (see 7.1): an election held in an SMD under Plurality Voting typically has just two 'serious' candidates, each nominated by just two 'serious' parties (though additional candidates and parties may well be on the ballot and have some support with respect to sincere preferences).

However, in this case, there is no tendency for the electoral support for these 'serious' candidates to be more evenly distributed than their support in sincere preferences, because — given that only one candidate can be elected — there is no reason for supporters of the strongest candidate migrate elsewhere. (However, as we shall observe in Section 8, competition between two candidates or parties may blunt difference between them and thereby tend to equalize their support in sincere preferences.)

Under particular circumstance, 'non-Duvergerian equilibria' involving three or more candidates with significant support in the final vote may exist. First, C_1 may be so far in the lead with respect to sincere preferences (i.e., C_1 may be a *majority winner*) that there are no competitive candidates to whom supporters of trailing candidate may be induced to migrate; for example, if C_1 is supported by 60%, C_2 by 30%, and C_3 by 10%, C_3's supporters have no incentive to switch to little incentive to switch their votes to C_1 even if they prefer C_1 to C_2 and little incentive to switch to C_2

even if they prefer C_2 to C_1. Second, C_1, C_2, and C_3 (and perhaps additional candidates) may be essentially tied with one another, so it isn't clear who should migrate where (or which candidate should make a strategic exit). Third, C_1 may have a modest lead over C_2 and C_3 (and perhaps additional candidates) who are essentially tied with one another, so even though supporters of these trailing candidates may have an incentive to coordinate their votes on one of them in order to defeat C_1, it isn't clear which should be the focal candidate (or who should make a strategic exit). Fourth, many supporters of trailing candidates may be essentially indifferent between C_1 and C_2 and so have no incentive to migrate to either of them. Fifth, if C_3 is trailing but has considerable support, its supporters may remain loyal on the off chance that C_3 may yet win, especially given a very close race between C_3 and C_3 since then a candidate can win with little more than one third of the vote; for example if C_1 is supported by 35%, C_2 by 34%, and C_3 by 31%. Finally, the ideological configuration among candidates, e.g., a strong centrist candidate (who is Condorcet winner but not a majority winner) bracketed by somewhat weaker leftist and rightist candidates) may induce equilibrium in the manner described earlier.

5. Duverger's Law

6. Votes and Seats at the National Level

Parliamentary elections have the direct effect (if voters are voting for parties) or the indirect effect (if voters are voting for candidates affiliated with parties) of allocating seats to political parties. Thus, British general elections (indirectly) allocate seats in the House of Commons among parties and U.S. Congressional elections (indirectly) allocate seats in the House of Representatives and the Senate to the Democratic and Republican (and possibly other) parties. Even U.S. Presidential elections can be interpreted as (indirectly) allocating electoral votes ('seats' in the Electoral College) to parties (and thus determining which party nominee becomes President). List-PR elections directly allocate seats to parties (and indirectly allocate party seats to individual candidates).

As we have seen, most electoral systems are *districted*, with the result that this allocation takes place in (at least) two steps: first, an electoral rule translates votes into seats within each district and, second, seats allocated within each district are aggregated across districts into an overall allocation of seats in the national parliament or other assembly. Let us recall the terminology that classifies parties by their competitive position within a district. Trailing parties have little electoral support and are unlikely to win even one seat in the district. Leading parties have sufficient support that they can confidently expect to win one or more seats. Competitive parties stand between the two other types with respect to electoral strength and may or may not win a single seat. Given a (semi-) proportional electoral rule, a party with the support of distinctly less than about one quota of the electorate is likely to be trailing, a party with the support of more than one quota is likely to be leading, and a party with the support of about one quota or a bit is likely to be marginal. Recall that the actual level of voter support necessary to put a party into one or other category depends on the district magnitude as well as its level of support.

6.1 Votes and Seats with Multi-Member Districts

Given a large MMD, any list-PR electoral formula translates the vote for parties into seat allocations among the parties in a manner that can be (and usually is) highly proportional. For example, a party that receives 27.139% of the vote in an MMD with $m = 38$ would, under an ideally proportional system, be entitled to its *precise quota* of $.27139 \times 38 = 10.313$ seats. Since seats must be awarded in whole numbers, it cannot receive its precise quota of 10.313 seats but, regardless of whether it actually receives 10 seats (26.3% of 38 seats) or 11 seats (28.9%), the result is quite close to proportional.[19] And if results within districts are close to proportional, the national results aggregated across districts must be close to proportional also. Indeed, a leading or competitive party that is slightly penalized by virtue of being 'rounded down' in one district is likely to be slightly rewarded in a compensating fashion by virtue of being 'rounded up' in another district, so national results with respect to such parties are likely to be more closely proportional than most district

[19] As noted in Section 3, the LR-H apportionment formulas used in list-PR systems by construction 'stays in quota', i.e., always gives each party its quota rounded up or down to the nearest integer. Divisor formulas may not 'stay in quota' but usually do so, even if the formula gives the party as few as 9 seats (23.7%) or as many as 12 seats (31.6%), it produces a result that is not very far from proportional. We will in any event speak informally of parties being 'rounded up' or 'rounded down' in their seat allocations within districts.

results. This tendency for the ‘rounding errors’ to roughly balance out in the national seat allocation may be called the *compensation effect*.[20] However, a party that is trailing in (almost) all districts cannot expect to be held even approximately harmless by the compensation effect, as such a party is consistently ‘rounded down’ to zero seats in (almost) all districts, even though national proportionality would entitle it to several seats in parliament.[21] For example, a party that wins about 1% of the vote in each of 10 MMDs of magnitude 38 is ideally entitled to about 0.38 seats in each district but would probably be “rounded down” to zero seats in all districts and thus would be allocated zero seats nationally, even though national proportionality and a parliament size of 300-400 would entitle it to about 3 or 4 seats.[22]

Within a small (say magnitude 3-6) MMD, any list-PR or semi-proportional rule can translate votes for parties into seats in a manner that is only roughly proportional. For example, a party that receives 27.139% of the vote in an MMD with 5 seats would, under an ideally proportional system, be entitled to $.27139 \times 5 = 1.357$ seats. In practice, it must receive either one (20%) or two (40%) seats, and in either event the result not very proportional. Again leading and competitive parties are more or less equally likely to be ‘rounded up’ or ‘rounded down’, so the compensation effect again implies that the national seat allocation among such parties is likely to be more proportional than most district allocations. But trailing parties winning no more than about 10-15% of the vote in any district will probably fail to win seats in any district, though such a party would win about its proportional share if large MMDs had been used.

6.2 Votes and Seats with Single-Member Districts

A common criticism of FPTP systems is that they often generate highly disproportional seat allocations at the national level. While this is often blamed on plurality rule, it is worth noting that all list-PR formulas, as well as most semi-proportional formulas, are logically equivalent to plurality rule when applied to SMDs. Plurality is in fact as proportional as possible given that there is only one seat to be allocated. The lack of proportionality of such systems derives from aggregation across SMDs, so the criticism should really be directed at the SMD system, not on the electoral rule applied within each district.

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[20] Of course, if the apportionment formula used within districts somewhat biased in favor of larger parties, this bias will tend to accumulate when seats are aggregated nationwide.

[21] The situation is different if different parties have trailing status in different districts. We subsequently consider the case of a highly sectionalized party system in an SMD context.

[22] To the extent that all such trailing parties together win a significant proportion of the national vote but no seats, all other (leading and marginal) parties must be allocated a greater than proportional number of seats. And as noted earlier, most national list-PR systems impose a vote threshold that parties must meet before being allocated any seats, with the result that parties that do win seats often end up with a greater than proportional share of seats as the apportionment formula is applied, not to the total vote cast, but to the total vote for all parties that meet the threshold.

Given plurality voting within each district, at most one party can command a full quota (50%) of support and, whether or not it commands a full quota, the leading party is always ‘rounded up’ to one seat while all other parties are ‘rounded down’ to zero seats. But with SMDs, the scope of the compensation effect is quite restricted. If the same party is leading in all (or almost all) districts, it is always (or almost always) ‘rounded up’ while other parties are ‘rounded down’, so the leading party wins all (or almost all) seats nationally.

The discussion in 5.3 suggested that SMD systems are likely to produce just two competitive candidates in each district. If candidates of the same two parties are competitive across all districts, the compensation effect operates (at least approximately) between these two parties but not with respect to any trailing parties, which are (as always) consistently ‘rounded down’. Thus minor parties (other than those with a regional basis of support) are severely unrepresented nationally, both with respect to their actual electoral support and even more so with respect to their ‘sincere support’ in the electorate (already attenuated by the strategic incentives discussed in 5.3).

We now consider the FPTP case in more detail, assuming that district-level strategic effects are operating so powerfully that only two parties, which we will call *L* and *R* (for Left and Right), contest elections, winning V_L and V_R percent of the votes respectively and S_L and S_R percent of the seats. We shall see that, while ‘rounding errors’ may balance out between the two parties over the long haul, they are unlikely to do so in any particular election; rather there is typically an imbalance that favors whichever party wins a majority of the national vote, the more so the greater the magnitude of its victory; thus the votes-seats is supraprortional as the leading party wins more than its proportionate share of seats (and the trailing party less). Moreover, we shall see that ‘election inversions’ may occur — that is, one party may win more votes while the other party wins more seats (a phenomenon noted in Module [Miller]).

Electoral districts almost certainly vary with respect to the characteristic level of support they give each party — for example, heavily working-class districts may regularly give very disproportionate support to the *L* party, while rural or suburban districts may equally regularly give disproportionate support to the *R* party. Let us specify the partisan composition of a district by its *district partisanship* (DP) score (here defined with respect to party *R*, though it could just as well be *L*), which is the (typical or average) difference between support for the *R* party in the district and its support nationally. Thus, a ‘marginal’ (or ‘bellwether’) district that typically mimics the nation as a whole in its support for the parties has a DP score of (just about) 0%, while a heavily pro-*R* district may have a score of +20% and a very heavily pro-*L* district a score of −30%. However, factors that pertain to particular elections rather than to longstanding partisan loyalties — such as economic conditions, the quality of party leaders, the governing performance of the party that won the last election, a prevalent ‘time for a change’ sentiment, etc. — produce ‘swings of the pendulum’ that cause vote support for the parties to move up or down nationally; moreover, these swings typically move in a (more or less) uniform (percentage point) amount from one election to the next. Thus in an election in which the *R* party wins 60% of the national vote, it would be expected carry the marginal district with the same 60% of the vote and the pro-*R* district with 80% of the vote, but lose the pro-*L* district 30% of the vote.[23] Note that, if the same number of votes are cast in each district, the average (mean) DP score is 0% and that average district vote percent for a party is equal to its national vote percent.

The nature of the votes-seats relationship under FPTP depends critically on the *distribution* of districts over the possible range of DP scores. The schematic distributions displayed in Figures 1-7 illustrate this dependence. Each figure shows the distribution to the left and the resulting votes-seats relationship to the right. The top horizontal scale in each left panel shows DP scores ranging from −50% to +50%; thus districts are more supportive of the *L* party the further left they are and more supportive of the *R* party the further right they are. The lower horizontal scale in each panel shows national vote shares for the *L* party as the 'pendulum' swings in its favor or against. (The vote for *R* of course is always 100% − *L*%.) The shaded area in each panel (a) shows the distribution of districts, where the whole shaded area represents 100% of the districts and the proportion of this area that lies to the left (or right) of a given DP score (or *L* vote) represents the proportion of districts with lower (or higher) DP scores (or *L* votes). The vertical axis in each panel on the right shows the share of seats won by party *L*. Each votes-seats graph is the *cumulative* distribution of districts as shown in the corresponding panel with reference to the national vote scale, and it shows how *L*'s seat share increases as its national vote increases from 0% to 100%.

Since the distributions shown in Figures 1-6 are all *symmetric* (that is, they are identical to their mirror images), they all imply that when the national vote is split equally the two parties win equal numbers of seats (thus each votes-seats curve passes through the center of the graph). But they have very different implications when the parties split the national unequally.

The distribution in Figure 1 is *rectangular* — that is, districts are uniformly spread over the full range of DP scores. Thus if party L wins X% of the vote, it likewise wins X% of the seats, as shown the votes-seats diagram to the right — that is, seats shares are proportional to vote shares. Iin this special case, FPTP is equivalent to proportional representation.

The distribution in Figure 2 is also rectangular but now the districts are uniformly spread over only the restricted ranged of DP scores from −25% to +25%. Thus no districts overwhelmingly favor either party and many districts are relatively marginal. In this event, a party that wins less than 25% of the national vote wins no seats, and a party than win more 75% of the national vote wins all the seats, as shown in the votes-seats diagrams. In closer elections, the losing party wins some seats but less than its proportional share and, while the winning party fails to win all the seats, it wins more than its proportionate share. Thus seat shares follow the supraproportional pattern typical of FPTP.

The distribution in Figure 3 is *triangular* (composed of two adjacent right triangles) with peak in the center. Only a few districts overwhelmingly favor either party, while relatively many districts are marginal or close to being so. Though the distribution itself is defined by straight line segments,

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[23] An approximately 'uniform swing' over districts is typically observed from one election to the next, though purely local factors can produce deviations from uniformity. And of course the factors that shape the partisan of districts (and or the demographic composition of districts) evolves over the long run. The alert reader will notice that the 'uniform swing' assumption leads to logical inconsistency in extreme cases, e.g., given a 60% national victory the R party would win 105% of the vote in district with a DP score of +45%. But few if any districts have such extreme DP scores, and national electoral victories rarely exceed 60%.

the cumulative distribution that defines the votes-seats relationship in panel (b) is curvilinear. As a party's national vote increases, its seat share increases increasingly rapidly until its national vote reaches 50%, after which its seat share continues to increase but decreasingly rapidly. Seat shares again follow the supraproportional pattern typical of FPTP.

In Figure 4 the two right triangles are flipped around to create a *bimodal* distribution with two peaks, one at either extreme of the DP scale; thus most districts overwhelmingly support one or other party and very few are marginal or close to being so. The seats-votes relationship is again curvilinear but exhibits the reverse pattern from the previous one. As a party's national vote increases, its seat share initially increases very rapidly at first but the rate of increases diminishes until the vote reach 50%, after which its seat share continues to increase and does so increasingly rapidly. Thus seat shares follow a *subproportional* pattern.

Figures 5 and 6 exhibit versions of Figures 2 and 4 pushed to their logical limits. In Figure 5, the range of DP scores over which districts are uniformly spread has become restricted to the point that it has (essentially) zero width, so that all districts have DP scores of (essentially) 0% — that is, in terms of party support, every district becomes a replica of the nation as a whole. Thus, whichever party wins the national vote also carries all the districts and wins all the seats. In this extreme case, FPTP operates in the 'winner-take-all' manner of MMD plurality rule but with the nation as a whole constituting a single MMD.

Figure 6 shows the bimodalism of Figure 4 is pushed to its logical limit so that half the districts are at one extreme of the DP scale and the other half at the other — that is, every district typically is maximally supportive of one or other party. While a party that wins 50% of the national votes wins half the seats, so does a party that wins virtually any other percent of the vote.[24]

Of particular interest in each seat-votes diagrams is *slope* of the line specifying the relationship. This slope is commonly referred to as the *swing ratio*, as it indicates the rate at which a party gains or loses seat shares as it national vote 'swings' up or down. Thus in Figure 1(b), the swing ratio is precisely 1 throughout its range, so that a party gains or loses 1% of the seats for every 1% swing in its national vote. In Figure 2(b), the swing ratio is precisely 2 in the range of 25% to 75% , so that a party gains or loses 2% of the seats for every 1% swing in its national vote, and is zero outside of this range (since there are no more seats to change party hands). In Figure 3(b), the swing ratio is approximately 3 in the vicinity of the 50% but diminishes in either direction. In Figure 4(b), the swing ratio is about 1/3 in the vicinity of the 50% mark but increases in either direction. In Figure 5(b), the swing ratio is (essentially) infinite exactly at the 50% mark but is zero everywhere else while in the last diagram the swing ratio is zero throughout almost the entire range but is (essentially) infinite at either extreme. It should be clear that, since the votes-seats curve must start at the lower left corner of the graph and end up at the upper right corner and since it is *monotonic*, i.e., seat share can never go down as vote share goes up, the swing ratio can be constant over the whole range of vote shares only if it is 1 (as in Figure 1). However the swing ratio is typically close to constant in the vicinity of equal vote shares, which is where most two-party elections end up.

[24] However, this extreme possibility presents the problem noted in the last sentence of the previous footnote.

Being symmetric, the distributions in Figure 1-6 all imply that, in an election in which the national vote is split equally, the two parties win equal numbers of seats. In contrast Figure 7 displays a (radically) *asymmetric* (or *skewed*) distribution composed of two distinct rectangles which have equal areas and therefore each represents half the districts. The *median* DP score (the score such that half the districts have a lower score and half higher) thus is located at boundary between the two rectangles, i.e., at +10%. This implies that the *L* party must win (slightly) more than 60% of the vote in order to win a majority of seats; with 50% of the vote, the *L* party wins only 40% of the seats.[25] Thus, this distribution of districts by DP scores produces a severe *bias* again the *L* party, which is clearly reflected in the corresponding seats-votes diagram. This bias produces an interval in the national vote for *L* from 50% to 60% in which an *election inversion* occurs —that is, the *R* party wins a majority seats even though the *L* party wins a majority of votes. It is true that the *L* party wins all the seats with 70% of the vote, whereas the *R* party must win 90% of the vote to win all the seats, but this is small compensation for *L* since all either party needs is a majority of seats to control the government (and extreme splits in the national vote are very unlikely).

All the distributions shown in Figures1-7 are schematic, designed to show a range of logical possibilities in the votes-seats relationship. The next question is what kind of distributions are most likely actually to arise. The plausible general answer is that DP scores are likely to be distributed in an (approximately) 'normal' manner — that is, in bell-shaped patterns such as those displayed in Figure 8. The so-called *normal distribution* results when many small independent effects are aggregated to produce an overall effect, e.g., when many social factors combine to determine the partisan composition of districts. The further question is how spread out such a normal distribution of DP score is likely to be and what effect this has on the seats-votes relationship. The normal distribution shown in Figure 8 by the dark curve has a *standard deviation* (SD) of 10%, while the less and more spread out ones shown by lighter curves lines have SDs of 5% and 15% respectively.[26] The corresponding cumulative normal distributions, i.e., the votes-seats curves, are shown in the right panel of Figure 8.

Figure 9 shows a normal distribution with a large SD that has been skewed to the right in somewhat the same manner as Figure 7, with the corresponding (biased) votes-seats curve. We have seen that in such a skewed distribution, the median and mean do not coincide; rather the mean is pulled in the direction of the longer 'tail' (to the left in Figure 7). As previously noted, such a

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[25] Note that the average or *mean* DP of all districts in the 'horizontal' rectangle is the midpoint between −40% and +10% (since districts are uniformly spread over this range), i.e., −15%, and by like reasoning the mean DP of the districts in the 'horizontal' rectangle is +15%, so the overall mean is 0%. Thus, despite its asymmetry, this distribution is centered on the 0% DP score like the previous ones. Median and mean scores differ only if the distribution is asymmetric; indeed the difference between the mean and median is often used as a measure of skewness.

[26] The *variance* of a distribution of scores is the average of the squared deviation from the mean score; since the mean DP score is 0%, the variance of DP scores is simply the average of the squared DP scores. The *standard deviation* of a distribution is the square root of its variance and thus is in the same units as the scores. Given a symmetric DP distributions, its SD may be thought of as the average amount by which above average scores exceed 0% and likewise the average amount by which below average scores fall below 0%.

distribution can produce an election inversion, in which the L party wins the most votes but fails to win the most seats.[27] This can come about because it wins a minority of districts by (on average) large margins while it loses a majority of districts by (on average) small margins.[28]

It turns out votes-seats curves almost identical to those produced by (unskewed) normal distributions can be generated 'power laws' of the following form:

$$\frac{S_L}{S_R} = \left(\frac{V_L}{V_R}\right)^h ,$$

where the exponent h (the 'power' to which the vote ratio is raised) plays a role analogous to the SD in the normal distribution in shaping the votes-seats curve, except that a larger h corresponds to a smaller SD. Moreover, the exponent h is essentially equivalent to the swing ratio given by the corresponding votes-seats curve at the point where votes are equally split between the two parties. The curves displayed in Figure 8 correspond to powers of approximately 1.5, 2.4 and 3.3. As h becomes very large, the corresponding SD approaches zero, giving the seats-votes relationship in Figure 5. As h becomes smaller, the corresponding SD becomes too large for a normal distribution to fit within the 0-100% range. When $h = 1$, the distribution become rectangular and seat shares are proportional to votes shares, as in Figure 1.[29] An exponent smaller than one corresponds to a bimodal DP distribution, as exemplified by the votes-seats curve in Figure 4 (or the one in Figure 8 labelled '$h \approx .5$'). As the exponent approaches zero, the corresponding distribution becomes as maximally bimodal, so that the parties split the seats equally regardless of their vote shares as in Figure 6.

One specific votes-seats power law has attracted special attention — namely the one given by $h = 3$. The so-called *cube law* states that in two-party British-style elections the ratio of seats won by the two major parties is approximately proportional to their vote ratio cubed. The exponent $h = 3$ corresponds to a DP distribution SD of approximately 13.5%, suggesting that the social factors that combine to determine the partisan composition of districts do so in a way that not only forms a normal distribution but also typically one with (approximately) this particular degree of spread over districts. However, while this may be true typically, it certainly is not true invariably. Much depends the nature of these social factors, as well as the size of the electorate in each district.

In this respect, the diagrams in Figure 10 are suggestive. Each left hand panel shows a large square with about half of its area shaded and the other half unshaded and a grid system defining

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[27] The 2000 U.S. Presidential election, the 1951 U.K. general election, and a fair number of other FPTP elections provide examples.

[28] As practical matter, district distributions are likely to be sufficiently symmetrical that the distance between the mean and median is unlikely to be more than one or two percentage points, which means that in practice inversions occur only in very close elections. The logical bounds on inversions are very broad, however. In principle, a party can win a majority of seats with barely more than 25% of the national vote (by winning a bare majority of votes in a bare majority of districts and no votes at all in the remaining districts) or fail to win a majority seats with barely less than 75% of the national vote.

[29] Any number raised to the power of zero is equal to one.

smaller squares laid over it. In the top panel, the shaded areas are coarse and much larger than the small squares, in the middle panel they are finer, and in the bottom panel they are finer still. Each small square can be classified in terms of the proportion of its area that is shaded. The coarse clusters of shaded areas produce the bimodal distribution shown, the intermediate clusters produce a more or less rectangular distribution, and the fine clusters produce of an approximately normal distribution with a moderate SD. It is evident that if the clusters became finer still, the distribution would remain approximately normal but it its SD would shrink further.

Now that each large square represents a geographical area over which voters are spread evenly. The shaded areas represent clusters of working class voters who preponderantly support the *L* party, the surrounding unshaded areas represent middle class suburban or rural areas that preponderantly support the *R* party, and the grid defines electoral districts. One implication of the diagrams in Figure 10 is that, given a fixed system of districts, the nature of the DP distribution (and the seats-votes relationship) depends on the nature of these partisan clusters. If the clusters are coarse, the DP distribution is bimodal and the votes seats curve resembles Figure 4; if they are fine, the DP distribution is normal with a small SD, and the votes-seats curve resembles Figure 3 or the relevant parts of Figure 8. In the intermediate case, the distribution is approximately rectangular and the votes-seats curve resembles Figure 1.

Note that what matters is not how social diversity influences partisan loyalties per se but how this diversity relates to districts. We can illustrate this point in a way that is both more concrete and more extreme, if rather implausible. If almost all women support the *L* party and almost all men support the *R* party (or vice versa), partisan clusters are minuscule and districts are all marginal, the the DP distribution has almost no spread, and the swing ratio is extremely high at the 50% mark. On the other hand, if almost all northerners support the *L* party and almost all southerners support the *R* party (or vice versa, and supposing the two regions have equal populations), there are only two huge partisan clusters almost all districts support one or other party overwhelmingly, the DP distribution is extremely bimodal, and the swing ratio is essentially zero.

To this point, we have interpreted the panels in Figure 10 in terms of constant districts and the varying coarseness of partisan clusters. Alternatively, we can interpret them in term constant cluster and districts that vary in size, so that the top panel shows a small geographic area and a few very small districts (much smaller than the typical cluster) while the bottom panel shows a much larger area with many much larger districts (much larger than the typical cluster). This implies that the workings of a FPTP electoral system depend on the size of districts and thus on assembly size in relation to the size of the electorate. This interpretation reinforces the general expectation that small districts are likely to be less socially diverse internally than large districts and thus more likely to be more firmly supportive or one party or the other producing a low swing ratio. But this also means there will more diversity across small districts than large ones, i.e., the smaller the districts, the greater their spread, and the lower the swing ratio. This implies, for example, that the winning party will usually control the U.S. Senate (larger districts, higher swing ratio) by larger margins than the U.S. House of Representatives (smaller districts, lower swing ratio). It also implies that if presidential electors were selected from SMDs (as has been proposed), rather than statewide on a general ticket, the electoral vote in Presidential elections would be much less lopsidedly in favor of the winning candidate.[30]

It is well known that district boundaries are often drawn with electoral considerations in mind. (In U.S. politics, this practice has long been referred to as *gerrymandering*, and the term has spread into academic and international political science; see Module [Bickerstaff].) If both parties have veto power over the districting process, the result is likely that they draw the boundaries to create a lot of districts that are safe for one or other party, with very few close districts, producing a low swing ratio. Non-partisan districting may create more close districts and a higher swing ratio. If the party *R* unilaterally controls the districting process, it may try to 'pack' as many *L* voters as possible into a relatively few districts that are extremely (and 'wastefully') safe for the L party, while creating a large majority of districts that are close to marginal but lean in the R direction. This will probably produce moderate swing ratio but it also will produce a skewed distribution that may allow the R party to win a majority of seats with a minority of votes, in the manner of Figures 7 and 9.

6.3 Three-Party Elections under FPTP

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30 The swing ratio would probably be about 3, in contrast with a swing ratio of about 8 under the current Electoral College system.

7. Electoral Systems and Party Systems

7.1 Generalized Duverger's Law and the Effective Number Parties

7.2 Party Entry and Exit, Composition and Competition

8. Electoral Systems and Government Selection

8.1 Electoral Decisiveness

I. Electoral Decisiveness versus Coalition Formation

1. Electoral systems serve two functions
    - i. elect a representative assembly
    - ii. (at least indirectly) elect a government/executive
  - b. in a presidential system (e.g., U.S., Latin America, a elsewhere), these are done through separate election (and electoral systems)
    - i. so the executive and assembly may not be politically aligned
      - (1) divided government/'cohabitation'
  - c. in a parliamentary systems (e.g., UK, most of Europe, Japan, Israel), the government/executive is selected by and accountable to the assembly
    - i. so the executive must be supported by an assembly/parliamentary majority
  - d. our present concern is with parliamentary systems

2. An electoral system is *decisive* to the extent it directly produces an executive/government
  - a. given Duverger's Law, a FPTP electoral system is typically decisive
    - i. it typically produces a (more or less) two party system
    - ii. following each election one or other of the two major parties typically commands a majority of seats in parliament and immediately forms a government which serves until the next election
    - iii. the governing party can thus be held accountable (by parliament and the electorate) for its performance and given credit or blame at the next election, at which time it will either be maintained in office or replaced by a government formed by the other major party
    - iv. in so far as Duverger's Law does not operate fully, a 'hung parliament' (in which no party commands a majority of seats) is possible but is considered unlikely and exceptional
      - (1) in this event either a coalition government or a minority government (with the tacit support of one or small oparties and/or independent MPs) must be formed

3. In contrast, a relative pure list-PR electoral system accommodating a multi-party system, so typically produces 'hung parliament' that the term is not applied
  - a. there are exceptions that may gives voters something like the decisive choice they typically have under FPTP
    - i. one large party that may achieve majority status after some elections
      - (1) CD in pre-1992 Italy

ii. two more or less enduring alignments of parties, one of which typically wins a majority of seats
(1) CDU+FDP vs. SPD+Greens in Germany
iii. "enhanced PR" giving largest party a seat bonus, that may give it majority or 'dominant' status (and perhaps produce a two party system in the longer run)

b. typically produces an indecisive result, so election sets up a 'bargaining game' among the parties represented in parliament to form a governing coalition (probably with majority support)
i. put otherwise, the elections distributes the resources (seats) available to parties in this bargaining process, rather that directly selecting the government

II. The Basic Arithmetic of Coalition Formation

4. While PR election distributes resources/seats among parties, the number of consequentially different ways it which it can do this is quite limited, if the number of seat-winning parties is fairly small (e.g., no more than about 6-7)
a. put otherwise, different seat distributions may produce the same bargaining power for all parties
b. thus bargaining power of parties is not proportional to to their resources/seats or what we will call their *weight*
c. what we can say is that, with respect to basic arithmetic of coalition formation, 'bargaining power is weakly monotonic to weight'
i. i.e., if party A has more seats than party B, it never has less power than B and it may (but need not) have more power than B
ii. cf. module [F&M] on a priori voting power.

8.2. Strong Simple Games

A *simple game* pertains to a set *N* of *players* (parties in the present context) and is defined in terms of the 'winning' status each possible subset or *coalition* of players. Let's designate players as A, B, C, etc. For example, a four-player simple game with has this set of 16 possible coalitions:

{A,B,C,D}	(the 'grand coalition' of all players);
{A,B,C},{A,B,D},{A,C,D},{B,C,D}	(all three-player coalitions);
{A,B},{A,C},{A,D},{B,C},{B,D},{C,D}	(all two-player coalitions);
{A},{B},{C},{D}	(all one-player 'coalitions'); and
{ϕ}	(the 'null coalition').

A simple game is defined by specifying its *winning coalitions*, where this specification must satisfy several conditions. The first condition formalizes one aspect of our intuitive sense of what may constitute a winning coalition.

(1) If a given coalition *S* is winning, every more inclusive coalition, i.e., every superset of *S*, is

also winning.

For example, in a four-player simple game, if the coalition {A,B} is winning, so are coalitions {A,B,C}, {A,B,D}, and {A,B,C,D}.

The second condition stipulates that a simple game must have both winning and non-winning coalitions. Given Condition (1), this can be guaranteed by the following condition.

(2) The (grand) coalition of all players is winning and the (null) coalition of no players is non-winning.

Given (1), if the first part of (2) were not true, there would be no winning coalitions and, if the second part were not true, there would be no losing coalitions.

The third condition formalizes another aspect of our intuitive sense of what may constitute a winning coalition.

(3) The complement of every winning coalition is non-winning.[31]

Thus, a four-player simple game in which the coalitions {A,B} and {C,D} are both winning is ruled out. Rather complementary pairs of coalitions are of two types: (i) one is winning and the other is non-winning, or (ii) both are non-winning. In the event of (i), the coalitions are called *winning* and *losing*, respectively; in the event of (ii), both coalitions are called *blocking*.

However, in the present context we will restrict our attention to *strong* simple games without blocking coalitions, by stipulating this fourth condition.[32]

(4) The complement of every non-winning coalition is winning.

A winning coalition *S* is *minimal* if every proper subset of *S* is non-winning, i.e., if the loss of *any* member converts *S* into a losing coalition. Given the Condition (1), a simple game can be defined by specifying its minimal winning coalitions. All supersets thereof are winning and all proper subsets thereof are non-winning. It is possible that one or more players does not belong to any minimal winning coalition; such a player a *dummy*.

A *weighted majority game* is strong simple game in which each player can be (or actually is) assigned some numerical *weight* (e.g., number of votes) and a coalition is winning if and only if its total weight is greater than half the weight of all the players. Clearly a parliament in which seats are allocated among several parties and a coalition of parties with majority support must be formed constitutes a weighted majority game.

Given these conditions that have been stipulated and a fixed small number of players (e.g., parties), the number of distinct strong simple games is (perhaps) surprisingly constrained. Consider

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[31] Simple games that do not meet this condition are said to be *improper*. Improper simple games may arise in some contexts but not in the present one.

[32] In effect, we are assuming that the parliament in which coalition formation takes has an odd number of seats. For convenience, we will also assume that no two parties control the same of seats.

the four-party example, let A, B, C, and D not only label the parties but also designate the number of seats the parties holds out of a total of *m* seats. Let's label the parties so that let A > B > C > D and then consider all possible coalitions as listed above.

The null coalition cannot be winning by Condition 2. If a one-party coalition is winning, it must be {A}, since A control the most seats; indeed, if {A} is winning, party A controls a majority of seats and need not find a coalition partner to win. Thus B, C, and D are dummies. So this is the first possible strong simple game with four players. Whatever the actual weights (e.g., seats) of the four players, this game may be represented by these minimal or *von Neumann-Morgenstern* (vN-M) *weights* and *quota*: (1:1,0,0,0)

If no one-party coalition is winning, it must be that the two-party coalition {A,B} is (minimal) winning (since A > C and B > D) and likewise {A,C} (since A > B and C > D). The remaining question is whether {B,C} is winning. If so, D is a dummy and we have the second possible game for which the vN-M weights are (2:1,1,1,0). Otherwise, {B,C,D} is winning, and we have the third possible game four-player game — and the only one in which none of the players is a dummy — for which the weights are (3:2,1,1,1)

A *homogenous* weighted majority game is a weighted majority game in which all minimal winning coalitions have the same total weight.

An (unweighted) *majority game* is a simple game in which a coalition is winning if and only if its total membership exceeds that of its complement. It is the special case of a weighted majority game in which all players have the same weight.

Let us consider different possible strong simple games with N = {A,B,C,D} as above. Apart from the labelling of the players, there are just three possibilities.

(a) There is a single one-player minimal winning coalition, say {A} (In this case, A is a dictator and B, C, and D are dummies.)

(b) There are three two-player minimal winning coalitions, say {A,B}, {B,C}, and {A,C}. (In this case, A, B, and C are equally powerful and D is a dummy.)

(c) There is one "privileged" player (say A) who constitutes a minimal winning coalition when combined with any one of the other three players, and the coalition of these three other players likewise constitutes a minimal winning coalition (so {A,B}, {A,C}, {A,D}, and {B,C,D} are the minimal winning coalitions). (In this case, there is no dictator and no dummy, but A is evidently "more powerful" than B, C, and D, the latter having equal, lesser amounts of power.)

8.3 Coalition Formation with Office-Seeking Parties

8.4 Coalition Formation with Policy-Seeking Parties

9. Conclusion

Quota type	Party	Party A	Party B	Party C	Party D	Party E	Total
	Votes	48,537	29,084	14,029	7,475	1,112	100,237
(m = 5)	Vote share	0.4842	0.2902	0.1400	0.0746	0.0111	1.0000
	Ideal quota	2.4211	1.4507	0.6998	0.3729	0.0555	5.0000
Q_H =	Quotas contained	2	1	0	0	0	3
	Remainders	8,441	9,036	14,029	7,475	1,112	40,093
20,048	Seats	2	2	1	0	0	5
Q_D =	Quotas contained	2	1	0	0	0	3
	Remainders	15,123	12,377	14,029	7,475	1,112	50,116
16,707	Seats	3	1	1	0	0	5
Q_I =	Quotas contained	3	1	0	0	0	4
	Remainders	5,577	14,764	14,029	7,475	1,112	42,957
14,320	Seats	3	2	0	0	0	5
Q_{I+} = 12,530	Quotas and Seats	3	2	1	0	0	6

Table 1

Formula	Divisor / Seats	Party A	Party B	Party C	Party C	Party D
D'Hondt	1	44,382 (1)	21,462 (3)	17,682 (4)	9,475 (8)	8,382
	2	22,191 (2)	10,731 (7)	8,841	4,738	
	3	14,749 (5)	7,151			
	4	11,096 (6)				
	5	8,876 (9)				
	Seats for $m = 5$	3	1	1	0	0
	Seats for $m = 6$	4	1	1	0	0
	Seats for $m = 7$	4	2	1	0	0
	Seats for $m = 8$	4	2	1	1	0
	Seats for $m = 9$	5	2	1	1	0
Sainte-Laguë	1	44,382 (1)	21,462 (2)	17,682 (3)	9,475 (5)	8,382 (7)
	3	14,749 (5)	7,151(8)	5,894	3,158	2,794
	5	8,876 (6)	4,292			
	7	6,340 (9)				
	Seats for $m = 5$	2	1	1	1	0
	Seats for $m = 6$	3	1	1	1	0
	Seats for $m = 7$	3	1	1	1	1
	Seats for $m = 8$	3	2	1	1	0
	Seats for $m = 9$	4	2	1	1	1
Modified Sainte-Laguë	1.00	44,382 (1)	21,462 (2)	17,682 (4)	9,475 (7)	8,382 (9)
	2.14	20,739 (3)	10,029 (6)	8,263	4,428	3,917
	3.57	12,432 (5)	6,012	4,953		
	5.00	8,876 (8)				
	Seats for $m = 5$	3	1	1	0	0
	Seats for $m = 6$	3	2	1	0	0
	Seats for $m = 7$	3	2	1	1	0
	Seats for $m = 8$	4	2	1	1	0
	Seats for $m = 9$	4	2	1	1	1

Table 2

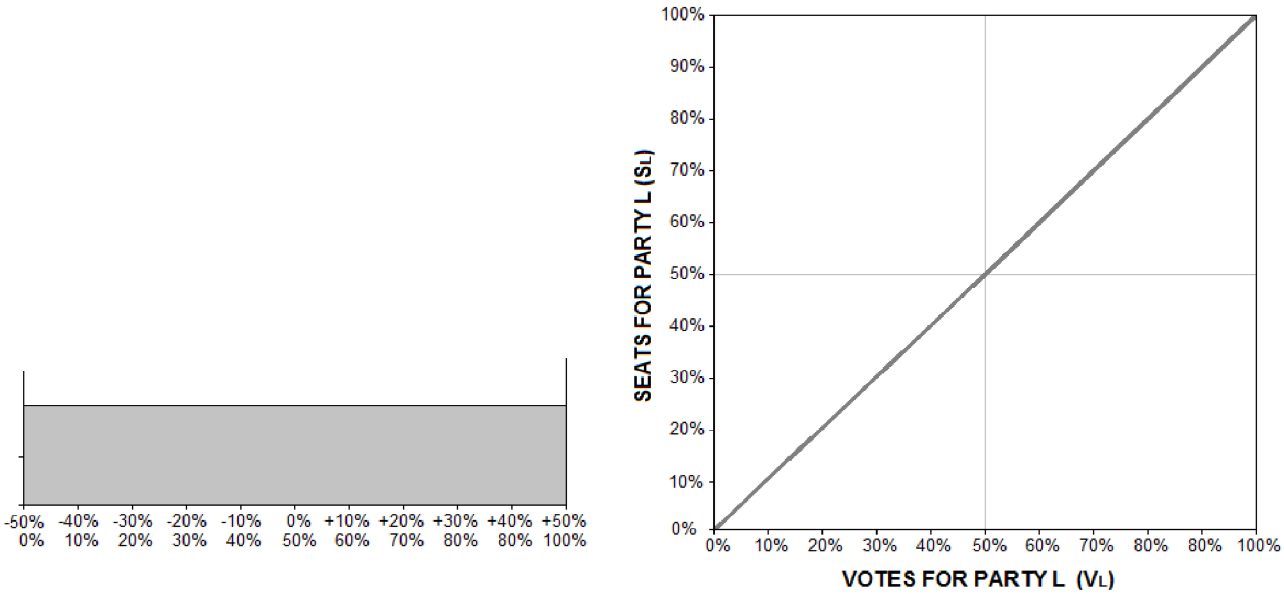

Figure 1

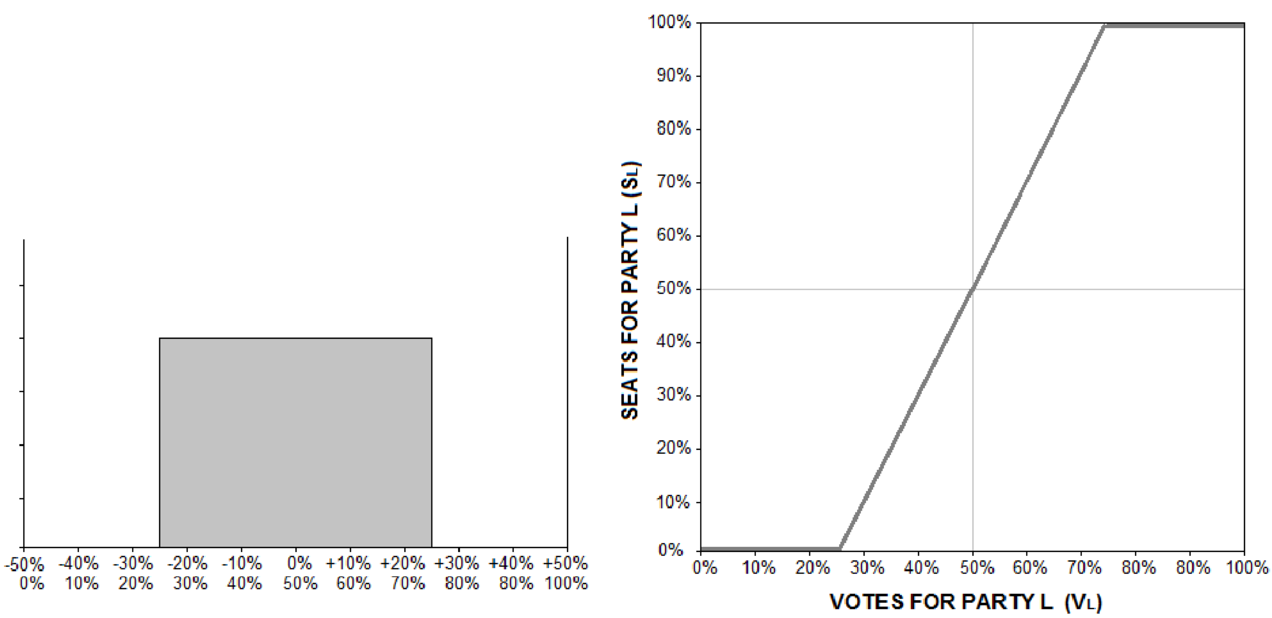

Figure 2

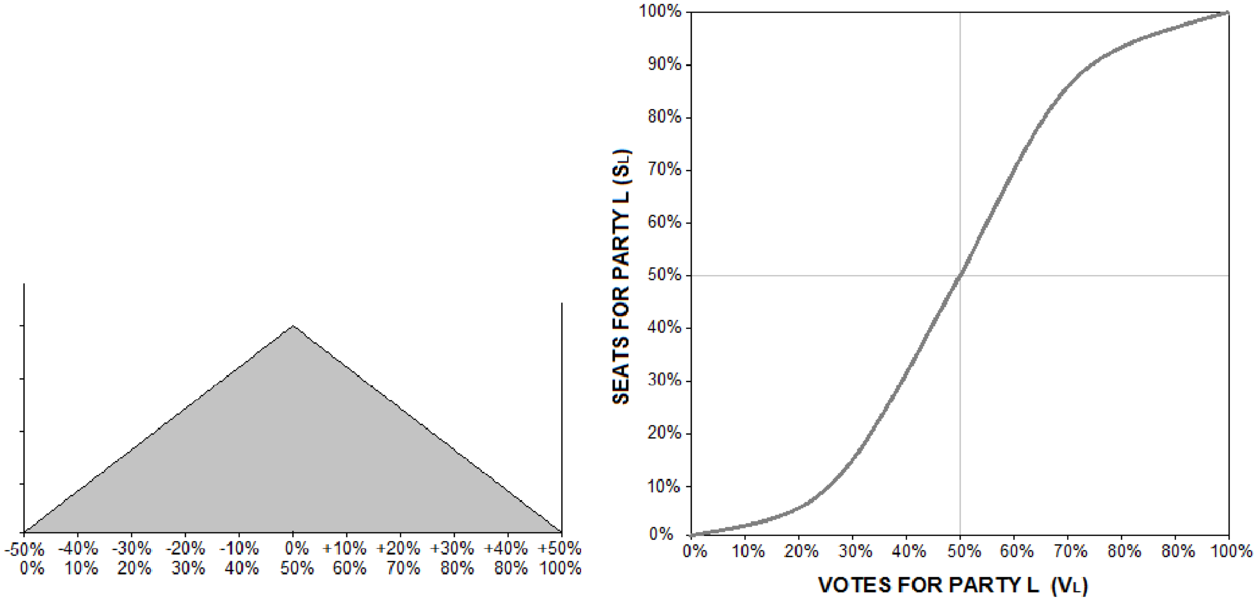

Figure 3

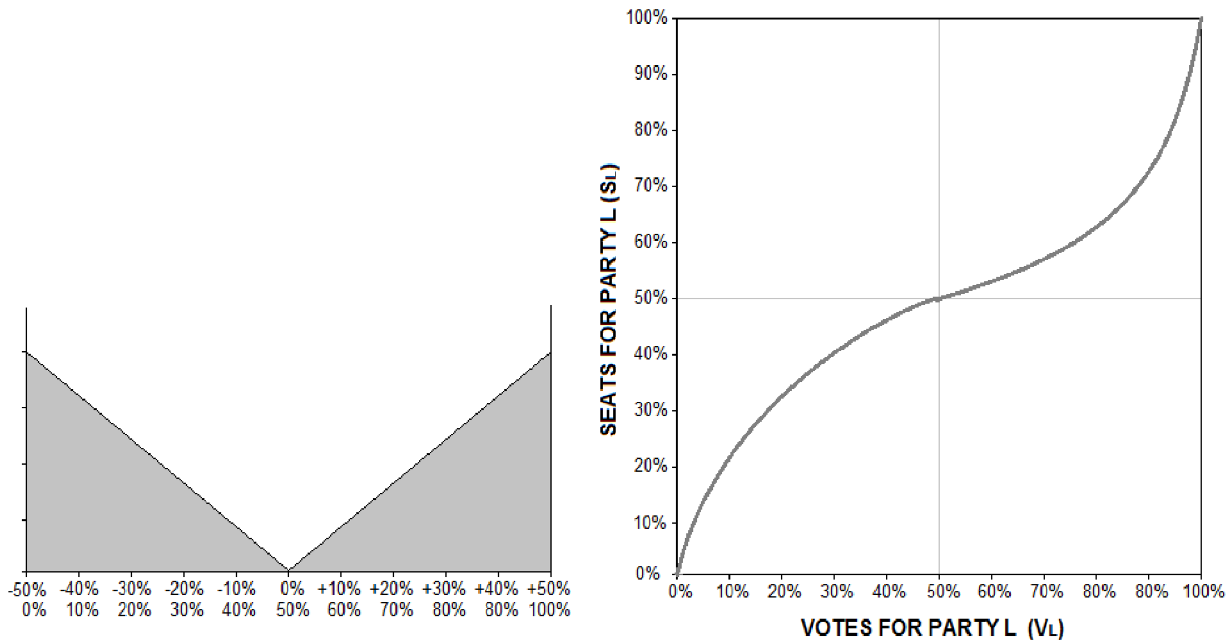

Figure 4

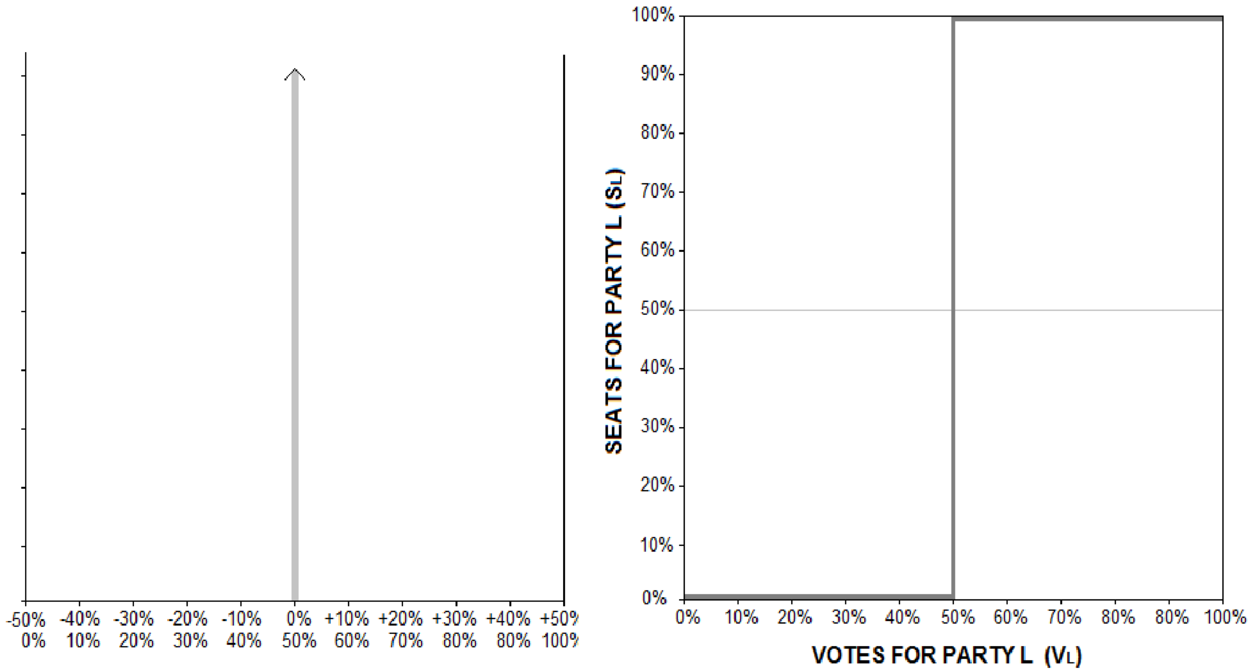

Figure 5

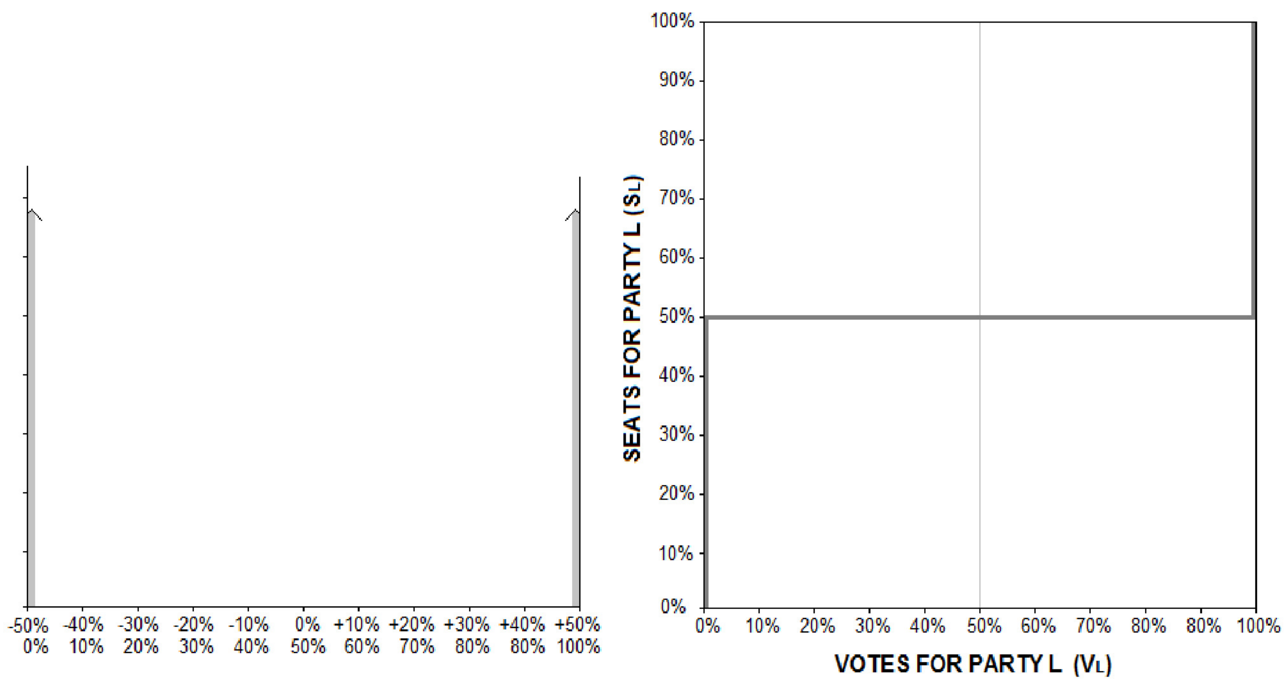

Figure 6

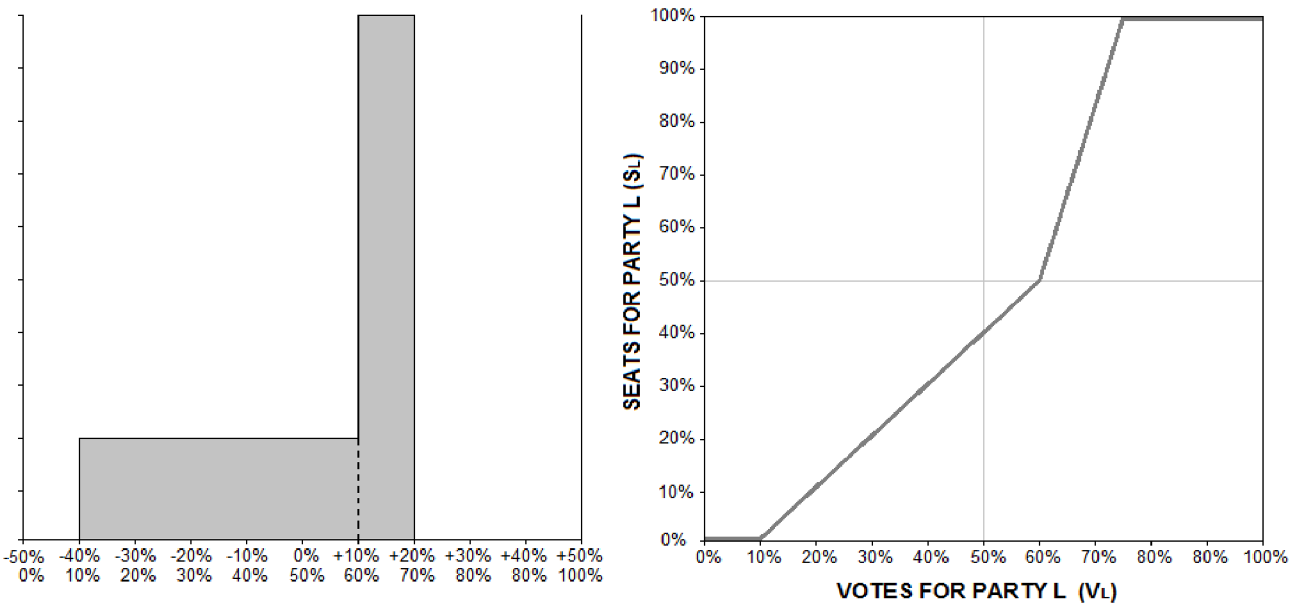

Figure 7

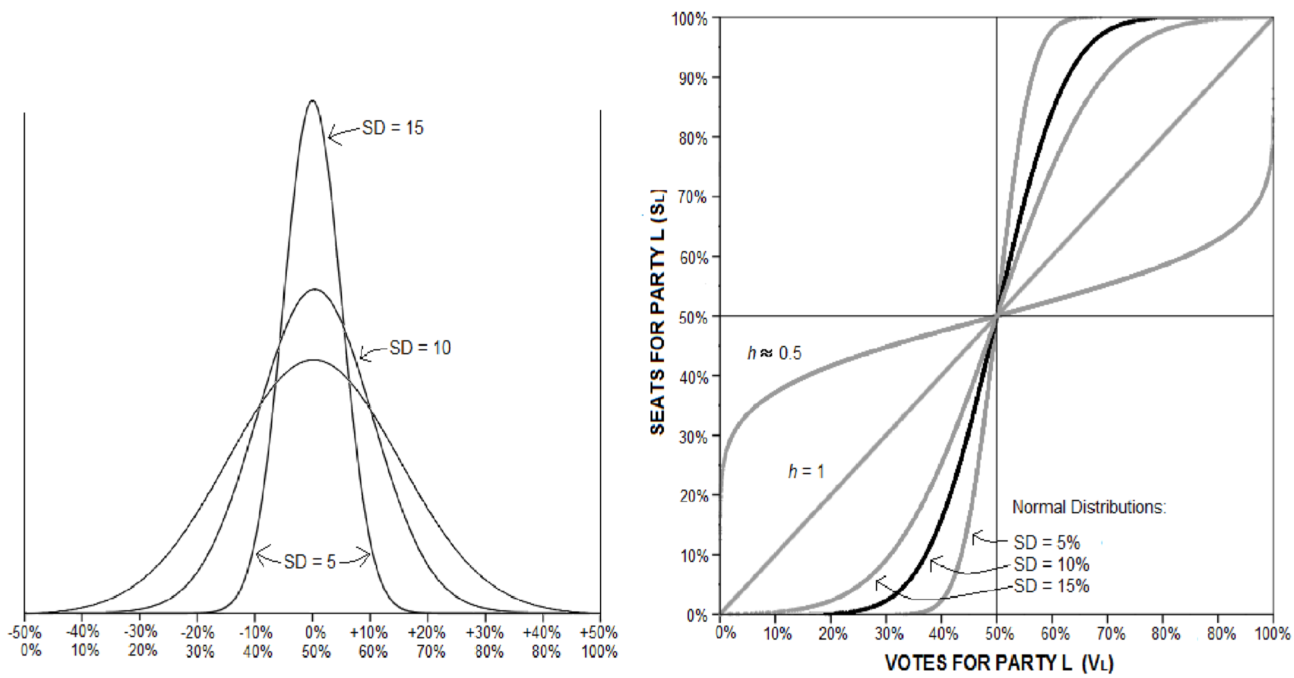

Figure 8

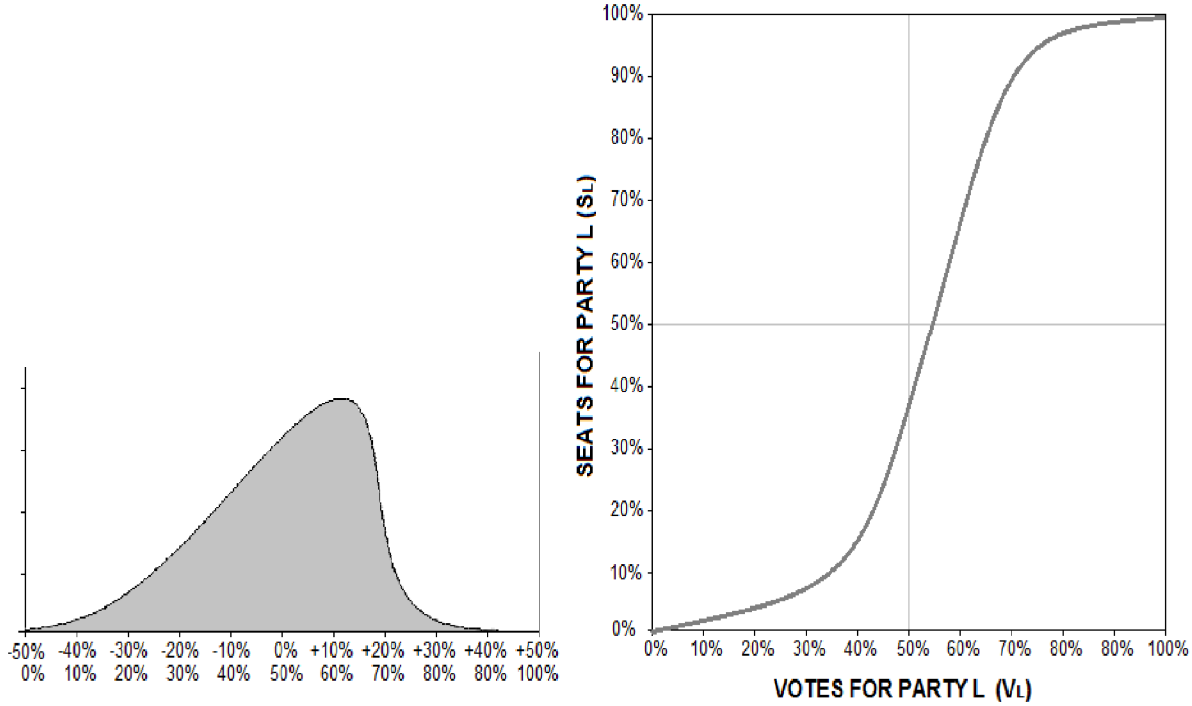
-50% -40% -30% -20% -10% 0% +10% +20% +30% +40% +50%
0% 10% 20% 30% 40% 50% 60% 70% 80% 80% 100%
SEATS FOR PARTY L (SL)
100% 90% 80% 70% 60% 50% 40% 30% 20% 10% 0%
0% 10% 20% 30% 40% 50% 60% 70% 80% 80% 100%
VOTES FOR PARTY L (VL)

Figure 9

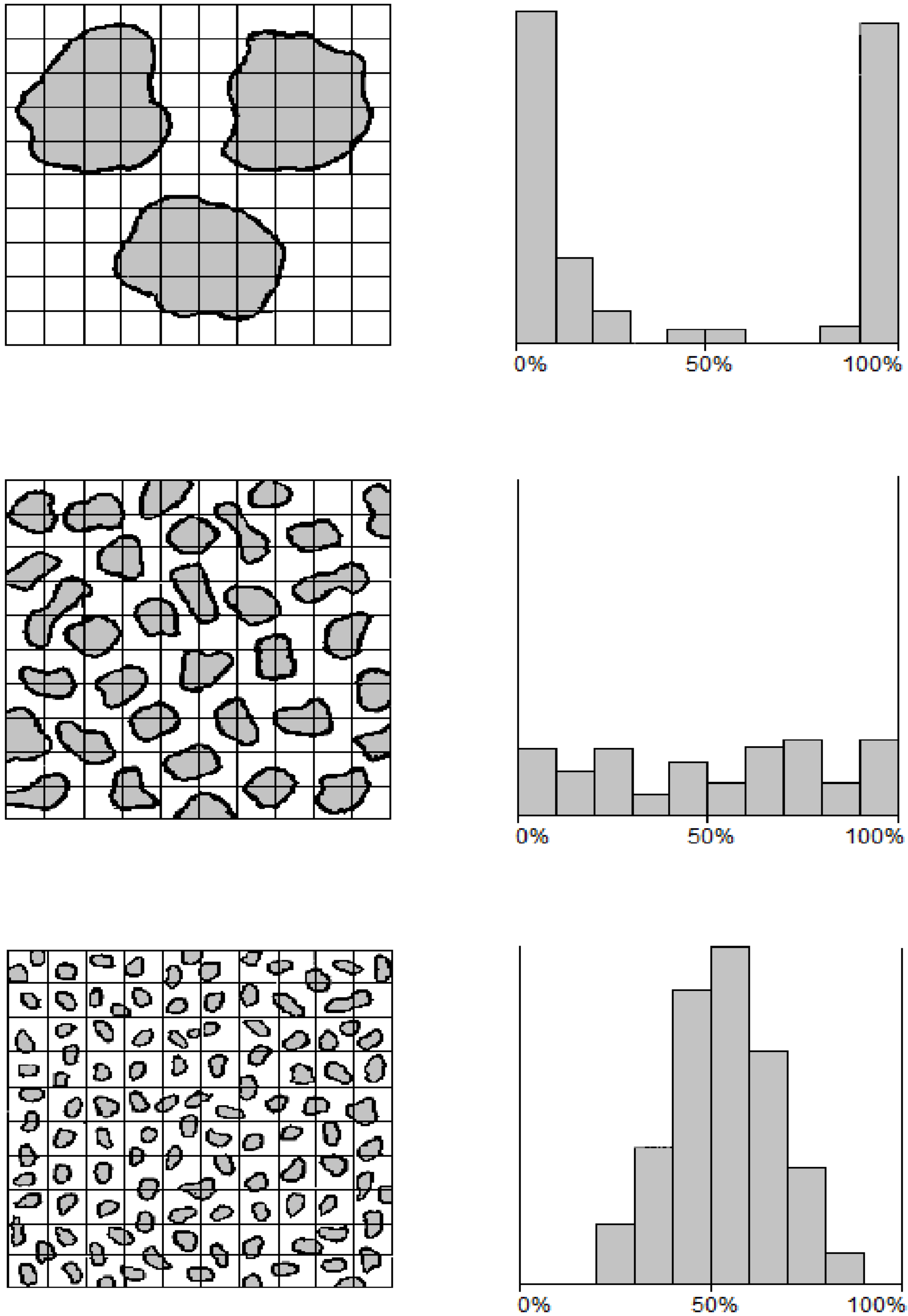

Figure 10

Source: Gudgin and Taylor (1979)