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ALTERNATE DEFINITIONS OF THE COVERING RELATION: AN EXTENDED TOUR

Penn (2006) has noted that (at least) three slightly different definitions of the covering relation, and therefore of the uncovered set, have been used in the literature. This note identifies a still wider array of alternate definitions.

Miller (1980) originally examined the covering relation and the uncovered set in majority preference tournaments that result when an *odd* number *n* of voters have *strong* preferences over discrete alternatives. In such a tournament and for any pair of alternatives *x* and *y*, either

(i) $x\,P\,y$ ("*x* beats *y*") $\Longleftrightarrow |\{i : x\,P_i\,y\}| > n/2 > |\{j : y\,P_j\,x\}|$, or
(ii) $y\,P\,x$ ("*y* beats *x*") $\Longleftrightarrow |\{j : y\,P_j\,x\}| > n/2 > |\{i : x\,P_i\,y\}|$.

Once we allow for individual *indifference* (as we must in any standard spatial model and many other contexts) and/or an *even* number *n* of voters, two distinct definitions of majority rule emerge:

(a) *Absolute majority rule*: $x\,P\,y \Longleftrightarrow |\{i : x\,P_i\,y\}| > n/2$,
$x\,R\,y$ ("*x* beats or ties *y*") $\Longleftrightarrow |\{i : y\,P_i\,x\}| \le n/2$, and
$x\,T\,y$ ("*x* and *y* tie") $\Longleftrightarrow |\{i : x\,P_i\,y\}| \le n/2\ \&\ |\{j : x\,P_i\,y\}| \le n/2$.

(b) *Relative majority rule*: $x\,P\,y \Longleftrightarrow |\{i : x\,P_i\,y\}| > |\{j : y\,P_j\,x\}|$,
$x\,R\,y$ ("*x* beats or ties *y*") $\Longleftrightarrow |\{i : x\,P_i\,y\}| \ge |\{j : y\,P_j\,x\}|$, and
$x\,T\,y$ ("*x* and *y* tie") $\Longleftrightarrow |\{i : x\,P_i\,y\}| = |\{j : y\,P_j\,x\}|$.

Absolute *MR* is commonly used in voting theory formulated in game-theoretical terms, as it tends to produce slightly cleaner definitions and results by allowing *MR* to be defined in terms of "winning coalitions." On the other hand, relative *MR* is more commonly used in both social choice theory (e.g., May, 1952) and actual practice. Relative *MR* is more "resolute" than absolute *MR,* in that the latter produces "ties" that would not ordinarily be deemed as such. For example, if 3 voters prefer *x* to y, 2 voters prefer *y* to *x*, and two voters are indifferent between *x* and *y*, under absolute *MR x* and *y* are "tied," whereas under relative *MR x* beats y. In a standard spatial context with an odd number *n* of voters with "thin" individual indifference curves), the difference between the two definitions comes down mostly to the status of certain points on the boundaries of relevant sets.

We use the following additional notation. Note that each these sets may be (differently) defined with respect to either absolute or relative majority rule.

.(1) $W(x) = \{y : y\,P\,x\}$, i.e., $W(x)$ is everything that beats *x*. $W(x)$ is called the *win set* of *x*.

(2) $\overline{W}(x) = \{y : y\,R\,x\}$, i.e., $\overline{W}(x)$ is everything that beats or ties *x*. By convention, *x* ties itself, so $x \in \overline{W}(x)$. $\overline{W}(x)$ is called the *win or tie set* of *x*.

(3) $W^{-1}(x) = \{y : x\,P\,y\}$, i.e., $W^{-1}(x)$ is everything that *x* beats. $W^{-1}(x)$ is called the *inverse win set* of x.

(4) $\overline{W}^{-1}(x) = \{y : x\,R\,y\}$, i.e., $\overline{W}^{-1}(x)$ is everything that x beats or ties. Since x ties itself, $x \in \overline{W}^{-1}(x)$. $\overline{W}^{-1}(x)$ is called the *inverse win or tie set* of x.

In the special case of a tournament considered by Miller (1980), absolute and relative *MR* are equivalent; moreover, the following statements are equivalent and each implies that $x\,P\,y$.

(1a)	$W(x) \subset W(y)$	(1b)	$W(x) \subseteq W(y)$
(2a)	$\overline{W}(x) \subset \overline{W}(y)$	(2b)	$\overline{W}(x) \subseteq \overline{W}(y)$
(3a)	$W^{-1}(y) \subset W^{-1}(x)$	(3b)	$W^{-1}(y) \subseteq W^{-1}(x)$
(4a)	$\overline{W}^{-1}(y) \subset \overline{W}^{-1}(x)$	(4b)	$\overline{W}^{-1}(y) \subseteq \overline{W}^{-1}(x)$

In Miller (1980) the covering relation C happened to be defined in terms of (3b), though different notation was used:

$$x\,C\,y \iff W^{-1}(y) \subseteq W^{-1}(x).$$

In words, x covers y if and only if every point beaten by y is also beaten by x. By the equivalences noted above, covering could just as well have been defined in terms of any of the expressions (1a) through (4b), and $x\,C\,y$ implies $x\,P\,y$.

Regardless of the definition of the covering relation C, and regardless of whether it is defined with respect to absolute or relative *MR*, the *uncovered set* is defined as

$$UC(X) = \{x \in X \mid \sim y\,C\,x \text{ for all } y \in X\}.$$

Provided that C is defined in way that makes it both *asymmetric* and *acyclic* (as in the tournament case), the covering relation has at least one maximal element, so $UC(X)$ is never empty.

In more general contexts than tournaments ties may occur as a result of

(i) individual indifference, and/or
(ii) an infinite alternative space, and/or
(iii) an even number of voters n.

Given (ii) and standard assumptions about spatial preferences, we must allow for (i) to produce the indifference curves that describe individual preference, and with (i) ties may occur even with n odd. However, given (i) and (ii) and provided that (iii) does not hold (i.e., that n remains odd), tie sets are "thin" and merely define the boundaries between win sets and inverse win sets (in the same way that individual indifference curves define the boundaries of individual preference sets). However, given both (ii) and (iii) tie sets are typically "thick" even while individual indifference remains "thin."

In any context that allows ties, the equivalences among (1a) through (4b) above break down, as does the implication that $x\,P\,y$. Even apart from the distinction between absolute and relative *MR*, a number of different covering relations C can now be identified, many of which have been proposed in the literature and which have the following properties.

First, every covering relation requires at a minimum that either (i) $W(x) \subseteq W(y)$ or (ii) $W^{-1}(y) \subseteq W^{-1}(x)$. If the set inclusion defining covering is *improper* (e.g., if we may have $W(x) = W(y)$), an *weak covering relation* C' results, with the property that we may have both $x\,C'\,y$ and $y\,C'\,x$. But if the set inclusion is required to be proper (e.g., always $W(x) \subset W(y)$), a (*strict*) *covering relation* C results, which precludes having both $x\ C\ y$ and $y\ C\ x$.

Second, as it is based on proper set inclusion, any kind of (strict) covering is (i) *transitive*, i.e., $x\ C\ y$ and $y\ C\ z$ imply $x\ C\ z$, and also (ii) *asymmetric*, i.e., $x\ C\ y$ precludes $y\ C\ x$. These two properties together assure that the uncovered set $UC(X)$ is never empty.

Finally, $y\ P\ x$ implies $W(x) \not\subseteq W(y)$ and $W^{-1}(y) \not\subseteq W^{-1}(x)$, so any kind of (weak or strict) covering $x\ C\ y$ implies that $x\ R\ y$.

The first two (weak and strict, respectively) covering relations take account of "upstream dominance" (in the terminology of Bordes, 1983) only.

(1) $x\ C'_1\ y \iff W(x) \subseteq W(y)$.

In words, everything that beats x also beats y. If $x\ T\ y$ and $W(x) = W(y)$, we have both $x\ C_1\ y$ and $y\ C_1\ x$.

(2) $x\ C_2\ y \iff W(x) \subset W(y) \iff x\ C'_1\ y\ \&\ \sim y\ C'_1\ x$.

In words, everything that beats x also beats y, and something (perhaps x itself) beats y but not x. Put otherwise, C_2 is the asymmetric component of the weak relation C'_1. Note that (with ties permitted) $x\ C_2\ y$ does not imply that $W^{-1}(y) \subseteq W^{-1}(x)$; in Example 1, $W(x) = \phi$ and $W(y) = \{v\}$ so $x\ C_2\ y$; nevertheless, $z \in W^{-1}(y)$ but $z \notin W^{-1}(x)$. C_2 is sometimes called the "Fishburn function" and $UC_2(X)$ the "Fishburn set" (Fishburn, 1977; also see Richelson, 1980 and 1981, and Bordes, 1983).

C_1 and C_2 have obvious "downstream" counterparts defined in terms of inverse win sets.

(3) $x\ C'_3\ y \iff W^{-1}(y) \subseteq W^{-1}(x)$.

In words, x beats everything that y beats. If $x\ T\ y$ and $W^{-1}(y) = W^{-1}(x)$, we have both $x\ C_3\ y$ and $y\ C_3\ x$.

(4) $x\ C_4\ y \iff W^{-1}(y) \subset W^{-1}(x) \iff x\ C'_3\ y\ \&\ \sim y\ C'_3\ x$.

In words, x beats everything that y beats, and x beats something (perhaps y itself) that y fails to beat. Put otherwise, C_4 is the asymmetric component of the weak relation C'_3. Note that $x\ C_4\ y$ does not imply that $W(x) \subseteq W(y)$; in Example 1, $W^{-1}(y) = \{z\}$ and $W^{-1}(x) = \phi$ so $x\ C_4\ y$; nevertheless, $v \in W(y)$ but $v \notin W(x)$. Bordes (1983) calls $UC_4(X)$ the F_D set.

We can combine C_1 and C_3, and also C_2 and C_4, into (weak and strict, respectively) covering relations that take equal account of "upstream" and "downstream" dominance (that are "symmetric," in Bordes' terminology).

(5) $x\ C'_5\ y \iff x\ C'_1\ y\ \&\ x\ C'_3\ y$.

In words, everything that beats x also beats y and x beats everything that y beats. McKelvey (1986, p. 287-288) calls C'_5 the *weak dominance relation*. If x and y are thought of as candidate strategies (platforms) in a symmetric zero-sum (win, draw, lose) two-player game of electoral competition associated with the n-voter system of social preference, C'_5 may be characterized as the relation of "dominance or equivalence" over strategies.

(6) $x\, C_6\, y \iff x\, C_2\, y \;\&\; x\, C_4\, y.$

In words, everything that beats *x* also beats *y*, and something (perhaps *x* itself) beats *y* but not *x*, and also *x* beats everything that *y* beats, and *x* beats something (perhaps *y* itself) that *y* fails to beat. Miller (1979) proposed $UC_6(X)$ as the appropriate generalization of the uncovered set once we allow for ties. Thus $UC_6(X)$ has sometimes been associated with Miller's name (Richelson, 1980 and 1981; Bordes, 1983).

In discussing $UC_6(X)$, Bordes actually cites Miller (1980), though no such definition was proposed there; instead, as a result of a manuscript typographical error (in a footnote on p. 94), covering was defined as C'_5. The definition I intended to give in Miller (1980), and did give in a correcting update (Miller, 1983, p. 385, except that, as a result of another typographical error, the set inclusion symbols were missing entirely), stands between C'_5 and C_6.

(7) $x\, C_7\, y \iff x\, C'_5\, y \;\&\; \sim y\, C'_5\, x \iff W(x) \subset W(y) \;\&\; W^{-1}(y) \subseteq W^{-1}(x)$ *or* $W(x) \subseteq W(y) \;\&\; W^{-1}(y) \subset W^{-1}(x).$

In words, everything that beats *x* also beats *y*, *x* beats everything that *y* beats, and either something (perhaps *x* itself) beats *y* but not *x* or something (perhaps *y* itself) is beaten by *x* but not by *y*. As McKelvey (1986) notes, C_7 is the (weak) dominance relation over strategies in the associated electoral game. Thus Richelson (1980 and 1981), Bordes (1983), and McKelvey (1986) call $UC_7(X)$ the *undominated set*.

Under definitions (1)-(7), $x\, C\, y$ implies that $x\, R\, y$ but (if ties may occur) does not require that $x\, P\, y$. McKelvey (1986) proposed the following more stringent definition of covering.

(6) $x\, C_8\, y \iff x\, P\, y \;\&\; x\, C_7\, y.$

In words, *x* beats *y*, everything that beats *x* also beats *y*, and *x* beats everything that *y* beats. However, given $x\, P\, y$, $W(x) \subseteq W(y)$ implies $W(x) \subset W(y)$ and $W^{-1}(y) \subseteq W^{-1}(x)$ implies $W^{-1}(y) \subset W^{-1}(x)$, so we can also define C_8 as follows.

(6a) $x\, C_8\, y \iff x\, P\, y \;\&\; x\, C_5\, y \iff x\, P\, y \;\&\; x\, C_6\, y.$

On the whole, C_8 appears to be the most authoritative definition of covering, in that researchers who invoke the covering relation and the uncovered set in the spatial context normally cite McKelvey (1986). In particular, Duggan et al. (2000, p. 4?) and Duggan and Jackson (2004, pp. 7-8) both cite McKelvey and define the covering relation as C_8. This definition is also used by Austen-Smith and Banks (2005, p. 269) for covering in the spatial model. C_8 is Definition 3 of covering in Penn (2006). But as Penn points out, other researchers — even while citing McKelvey — sometimes actually give one of the two following definitions.

By definition, $x\, C_8$ y only if $x\, P\, y$, and C_8 results from adding the prerequisite $x\, P\, y$ to C_5, C_5, or C_6. The same prerequisite for covering can be added to C'_1 and C'_3, which — given the stipulation that $x\, P\, y$ — are equivalent to C_2 and C_4 respectively.

(9) $x\, C_9\, y \iff x\, P\, y \;\&\; x\, C'_1\, \mathrm{y} \iff x\, P\, y \;\&\; x\, C_2\, y.$

In words, x beats y and everything that beats x also beats y. This is Penn's (2006) Definition 2 of covering. Bordes et al. (1992, p. 511) call C_9 the "Gillies' subrelation." Bordes (1983) calls $UC_9(X)$ the F_u set, while Banks and Bordes (1988) call it $UC_u(X)$. As Penn (2006) notes, Shepsle and Weingast (1984, p. 58), Cox (1987, p. 412), Epstein (1998, p. 84), and Bianco, Jeliazkov, and Sened (2004, p. 260), all define the covering relation as C_9, even while citing McKelvey (1986). Austen-Smith and Banks (2005, p.134) use C_9 as the definition of covering in finite binary agenda voting games (but ties are ruled out, so the stipulation that $x\ P\ y$ is superfluous).

(10) $x\ C_{10}\ y \iff x\ P\ y\ \ \&\ \ x\ C'_3\ y \iff x\ P\ y\ \ \&\ \ x\ C_4\ y.$

In words, x beats y and x beats everything that y beats. This is Penn's (2006) Definition 1 of covering. Bordes et al. (1992, p. 511) call C_{10} the "Miller's subrelation." Bordes (1983) calls $UC_{10}(X)$ the F_d set, while Banks and Bordes (1988) call it $UC_d(X)$. As Penn notes, Banks (1985, p. 297), Penn (2005, p. 8), Dutta, Jackson, and Le Breton (2004, p. 10) all define the covering relation as C_{10}, as do Laslier and Picard (2002, p. 111) and Fey (2002, p. 4).

Penn (2006) further shows that, in the context of a divide-the-dollar distributive voting game, $UC_8(X) = UC_9(X)$ and are equal to the entire Pareto set, while $UC_{10}(X)$ includes only those Pareto allocations that give over half the players a strictly positive payoff.

It is evident the definitions of these covering relations are logically interconnected in a way that is summarized in Figure 1, where an arrow from C_h to C_k means $x\ C_h\ y$ implies $x\ C_k\ y$.

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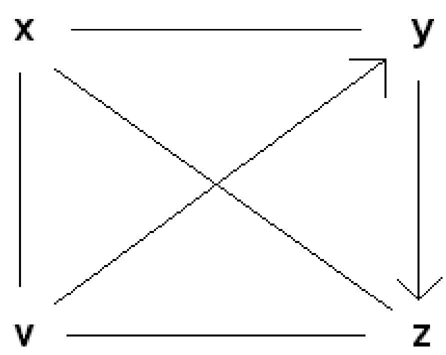

EXAMPLE 1

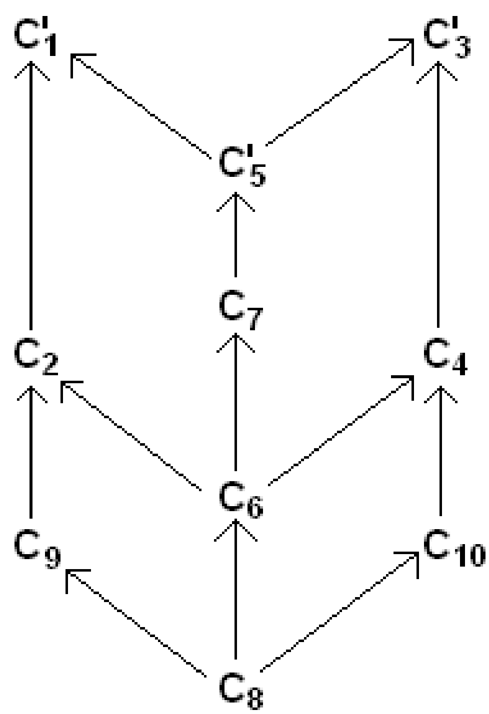

FIGURE 1