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*Graph-Theoretical Approaches to the Theory of Voting**

In this article, language, concepts, and theorems from the theory of directed graphs are used to characterize and analyze the structure of majority preference. A number of results are then derived concerning "sincere," "sophisticated," and "cooperative" voting decisions under two common majority voting procedures. These results supplement the work of Black and Farquharson. Perhaps contrary to "common-sense" thinking, general strategic manipulation of voting processes has beneficial consequences.

It is widely recognized—and not only by political scientists—that the decisions of a voting body may be affected not only by such obviously relevant matters as the preferences of its members and their participation in or absence from particular votes, but also by such "technical" matters as the nature of the voting procedure and the order in which proposals are voted on. It is also recognized that voting may have "gamelike" characteristics offering strategic opportunities both to voters as individuals and to voters in coalitions. Finally, most political scientists—though probably few politicians or citizens—are by now aware of the "paradox of voting" and may have some sense of its connection with these questions of decision, procedure, and strategy.

Over the past decade or so a somewhat technical literature on the theory of voting has developed in the "public choice" area. The present article adds to this literature by presenting a number of new propositions concerning majority voting under two common voting procedures. These propositions pertain to the questions alluded to in the first paragraph. These new results, together with some more familiar ones, are obtained by employing language, concepts, and theorems from the mathematical theory of directed graphs. In these respects, the article will be of interest primarily to specialists in the area

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of voting theory. But, in addition, I have taken care to write this article in such a way that it can reach a wider audience,[1] and I hope that it will demonstrate more generally the utility of the theory of directed graphs in the construction of political theory. It should, therefore, serve a methodological purpose as well.

The modern theory of voting is a part of the abstract social choice theory that received its original impetus from Kenneth Arrow (1951). But the bulk of the social choice literature does not deal with the voting process per se (though some titles suggest otherwise). The portion that does derives in particular from Duncan Black (1958) and works largely within the framework that Black established. Essentially, Black was concerned, as we are here, with a voting body of any type–for example, a "committee," a "legislature," or even an "electorate"–that adopts, under some specified procedure, one discrete "proposal" (as we shall say)–or "alternative," "motion," "candidate," etc.–out of some larger finite set as its decision.

Black's work is almost entirely innocent of game-theoretical considerations and does not address strategic questions. It remained for Robin Farquharson (1969) to demonstrate that these strategic questions were amenable to resolution; but, apart from exhaustively examining the case of three voters and three proposals, Farquharson does not state any general results concerning voting decisions, nor does he clearly indicate how such results might be obtained.

Roughly speaking then, this article picks up where Black and Farquharson left off. In the first section, the necessary graph theory is introduced. In the second section, which constitutes the bulk of the article, the voting theory results are presented. In the brief concluding section, certain general patterns in these results are identified and discussed.

1. Tournaments and Majority Preference

A *directed graph,* or *digraph,* is a collection of "points," finite in number, and of "directed lines" between (some) pairs of these points. We designate by V the set of all points and use lowercase letters x, y, z, etc., to refer to

[1] No mathematical knowledge is presumed beyond familiarity with the basic language of sets. I have held notation to a minimum, and all statements are presented in verbal (though I believe quite precise) form. The reasoning is definitely of a mathematical nature but it is also elementary and–except for part of the discussion in Section 2:d–uncomplicated. Three basic graph-theoretical theorems are presented without proof; footnotes refer the reader to proofs in the standard text (Harary, et al., 1965). Otherwise the article is self-contained.

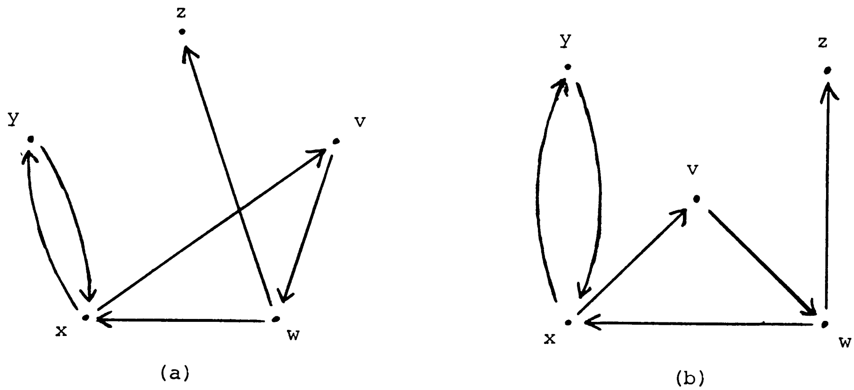

FIGURE 1

Two Diagrammatic Representations of the Same Digraph

individual points. If there is a directed line from point x to point y, we write x → y and say that x *dominates* y. Digraphs may be represented by means of diagrams such as those shown in Figure 1. The geometrical arrangement of the diagram is irrelevant to the structure of the digraph it represents; thus the two diagrams in Figure 1 represent identical digraphs.

The set of all points dominated by x is designated D(x) and the set of all points that dominate x is designated F(x). Point x is *undominated* if no other point dominates x. A *path* from x to y is a sequence of distinct points and directed lines x → z → . . . → y. A *complete path* includes all points in the digraph. Point y is *reachable* from x if there is a path from x to y. Point x is a *source* if every other point is reachable from x. A *cycle* is a path x → . . . → y together with the path y → x (that is a "path" that "repeats itself").

THEOREM 1. A digraph with no undominated point has a cycle.[2]

A *semi-path* from x to y is an "improper path" from x to y in which some of the directed lines may be pointing in the wrong direction, e.g., x → z ← v → y. A *semi-cycle* is a semi-path from x to y together with the path y → x or y ← x. A digraph is *connected* if there is a semi-path between every pair of

[2] This result is virtually self-evident; in any case, it can be easily proved. See Harary, et al. (1965), pp. 64–65, where a "point of indegree zero" is an undominated point (and a "point of outdegree zero" is a point that dominates no other point).

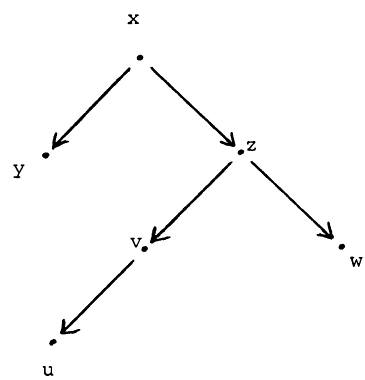

FIGURE 2

A Tree

points. A *tree,* such as the digraph represented in Figure 2, is a digraph with a source and no semi-cycles.

THEOREM 2. A connected digraph is a tree if and only if exactly one point is undominated and every other point is dominated exactly once.[3]

It follows that a tree has at least one point that dominates no others. And it is natural to designate the unique undominated point the *initial point* of the tree and to designate the points that dominate no others the *end points* of the tree.

A digraph is *complete* if, for every pair of points, at least one dominates the other. A digraph is *asymmetric* if, for every pair of points, no more than one dominates the other. A *tournament,* such as the digraph shown in Figure 3, is a digraph that is both complete and asymmetric—that is, for every pair of points exactly one dominates the other.[4] Obviously, a tournament (or any other complete digraph) has at most one undominated point, which dominates every other point.

A digraph is *transitive* if, whenever x dominates y and y dominates z, then

[3] See Harary, et al., (1965), pp. 283–284, where a "weak" digraph is a connected digraph.

[4] "The reason for the name is that such digraphs represent the structures of round-robin tournaments, in which the players (or teams) engage in game that cannot end in a tie and in which every player plays each other exactly once. Thus if the points represent players and each line represents the relationship 'defeats,' the digraph of a round-robin tournament is complete and asymmetric." (Harary, et al., 1965, p. 289.)

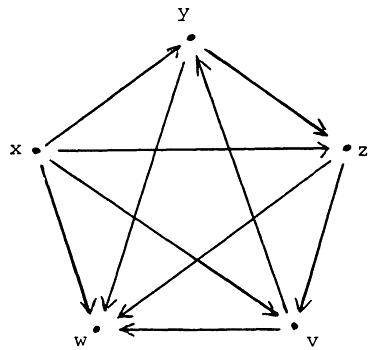

FIGURE 3

A Tournament

also x dominates z. Clearly a tournament is transitive if and only if it has no cycles, in which case it is a *strong ordering.*

The potential relevance of tournaments to the theory of majority voting should be clear. Consider a voting body that must choose one proposal out of some larger set. For any pair of proposals x and y, (1) a majority of voters prefers x to y or (2) a majority of voters prefers y to x. Clearly only one of these two statements can be true—that is, majority preference is asymmetric. Further, if we assume (as we do here) that the number of voters is odd and that they all have strong preferences (that is, none is indifferent between any two proposals), at least one of these two statements must be true—that is, majority preference is then complete. Thus we have the following:

PROPOSITION A. Given an odd number of voters with strong preferences, the system of majority preference over proposals is a tournament.[5]

It may be thought desirable for the voting body to make its decision in accordance with the principle of majority rule, for example as it has been stated by Robert Dahl (1956, pp. 37–38):

THE RULE: The principle of majority rule prescribes that in choosing among alternatives, the alternative preferred by the greater number is to be selected.

[5] This relationship has been formally noted by Harary, et al., (1965), pp. 213–215; Harary and Moser (1966); Taylor (1968); and Bjurulf (1973). I owe a great deal to Taylor's article for stimulating my work in this area and especially first directing me towards graph theory.

That is, given two or more alternatives x, y, etc., in order for x to be government policy it is a necessary and sufficient condition that the number who prefer x to any single alternative is greater than the number who prefer any single alternative to x.

The second part, at least, of Dahl's statement clearly requires that a proposal, in order to be government policy, must be preferred by majorities to every other proposal—that is, dominate every other proposal. Thus, given our assumptions, the principle of majority rule can be translated as follows:

THE RULE: A proposal should be the voting decision—that is, should be chosen by the voting body—if and only if it is the undominated proposal in the system of majority preference.

Because majority preference is a tournament, there is at most one undominated proposal. But a tournament may have cycles, so an undominated proposal might not exist, in which case it would be logically impossible to comply with The Rule. But are we anticipating a difficulty that can never arise? While we have seen that every system of majority preference is a tournament, the converse has not yet been established. It is clear that if there are cycles in the preferences of individual voters (so that, for example, a voter prefers x to y and y to z and z to x), majority preference may be cyclical as well. But it is both conventional and plausible to assume (as we do here) that individual preferences are transitive orderings.[6] So the question is whether this restriction on individual preferences is sufficient to bring about a similar restriction on majority preference. As is by now rather well known, the answer is no; a system of majority preference may have cycles, and thus have no undominated outcome, even when generated by individual preference orderings. This "paradox of voting" is usually demonstrated by example, but here we present a more general result—the converse of Proposition A.

PROPOSITION B. Every tournament is a system of majority preference, where voters are finite in number and have strong preference orderings.[7]

[6] But in fact this assumption is necessary only for Lemma 3 and Proposition 10, which rests on the lemma.

[7] This theorem was first stated by McGarvey (1953), and I have followed his proof. This proof requires $2\binom{m}{2}$ voters, an even number, where m is the number of proposals. We can add one more voter to make the total odd; since every majority preference "carries" by exactly two "votes," this last voter, whatever his preferences, cannot alter

Consider any arbitrarily constructed tournament and any pair of points such that x dominates y. Let there be a pair of voters who both prefer x to y but who disagree with respect to every other pair of proposals. And let there be pairs of other voters with preferences of similar form with respect to each of the other pairs of proposals. Then altogether we have twice as many voters as there are pairs of proposals, and these voters have strong preference orderings such that, for each pair of proposals, the preferences of all but two of the voters cancel each other out and the common preference of the two remaining voters determines majority preference in the required manner.

Proposition B has two implications. First, the principle of majority rule must be replaced with some broader criterion. And second, we must study the structure of tournaments in their full generality, even if our interest is only in their application to majority voting.

With respect to the first implication, when no undominated proposal exists, we may wish to focus attention on the minimal set of proposals that collectively possess the property of an undominated proposal.

DEFINITION: A *minimal undominated* (or *Condorcet*) set V* of points in a tournament is a non-empty set of points such that:

Condition I: no point in V* is dominated by any point not in V*; and

Condition II: no proper subset of V* meets Condition I.[8]

It is clear that if an undominated point x exists, the minimal undominated set includes x and no other point. But a unique minimal undominated set exists in any case.

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the system of majority preference. And while $2\binom{m}{2} = m(m-1)$ voters are required for this proof, it can be further shown that no more than m+1 (m odd) or m+2 (m even) voters are ever required to generate any arbitrary system of majority preference. See Stearns (1959).

[8] There is no established term in graph theory to designate this set of points. To the best of my knowledge, this concept was first defined and employed in voting theory by Ward (1961), who used the term "majority set." The term "Condorcet set" is due to Good (1971). (The Marquis de Condorcet, an eighteenth-century French philosopher and mathematician, is generally credited with the initial discovery of the "paradox of voting" and made other significant contributions to the early theory of voting; see Black, 1958, pp. 159–180.) The term "top cycle" is sometimes used in the voting theory literature, which designation anticipates what is formally proved in Theorem 6 below. The Condorcet set is identical to the "Γ" set and equivalent, in a tournament, to the "GOCHA" set, both defined by Schwartz (1975). Also see Kadane (1966).

THEOREM 3. Every tournament has exactly one minimal undominated (or Condorcet) set V*.

We must demonstrate both existence (that every tournament has at least one such set) and uniqueness (that no tournament has more than one such set).[9]

Existence. Trivially, the set V of all points meets Condition I. If no proper subset of V does likewise, then V also meets Condition II and is a minimal undominated set. If some proper subset V′ of V meets Condition I, we repeat for V′ the argument just made for V. And so forth until we reach some subset of V that meets Condition II, as well as Condition I—which we must find, since any one-element subset of V necessarily meets Condition II.

Uniqueness. Suppose P and Q are two minimal undominated sets. Neither can be a subset of the other, for then the more inclusive set would fail to meet Condition II. Thus there must be some point x that belongs to P but not to Q and some point y that belongs to Q but not to P. Then by Condition I, and because a tournament is complete, we have x dominates y and also y dominates x. But this is impossible, because a tournament is also asymmetric. So P and Q must be identical.

Given this definition and theorem, we can replace the principle of majority rule with the following:

THE CONDORCET CRITERION. A proposal should be the voting decision only if it belongs to the minimal undominated (or Condorcet) set of proposals.

Since the Condorcet set includes only the undominated proposal whenever such a proposal exists, the Condorcet criterion incorporates the principle of majority rule whenever the latter can be followed. Otherwise—that is, in the absence of an undominated proposal—the Condorcet criterion entails no definitive prescription, since the Condorcet set then includes several (and possibly all) proposals; but it is always logically possible to comply with the criterion.

From now on we will speak of the Condorcet set, rather than the minimal undominated set. And we now turn to act on the second implication of Proposition B—that is, to investigate the structure of tournaments in more detail.

[9] Cf. Good (1971), pp. 98–99, and Schwartz (1975), pp. 22–23.

We first state what is probably the best known result concerning tournaments.

THEOREM 4. Every tournament has a complete path.[10]

We next present two "lemmas"—that is, preliminary propositions to be used in demonstrating the subsequent theorems.

LEMMA 1. In a tournament, if x is reachable from y and y is reachable from x, there is a cycle that includes both x and y.

Suppose, without loss of generality, that $x \rightarrow y$. By asymmetry, we cannot have $y \rightarrow x$, so x must be reachable from y by some path $y \rightarrow z \rightarrow \ldots \rightarrow x$, giving us the cycle $x \rightarrow y \rightarrow z \rightarrow \ldots \rightarrow x$.

LEMMA 2. In a tournament, if there is a cycle including all the points in the set V' and there is another cycle including all the points in the set V'', and if V' and V'' intersect, there is a cycle including all the points in the union of V' and V''.

Figure 4 shows two such cycles with x (at least) in common. If $y \rightarrow z$, the cycle to the right can be enlarged to include y. Otherwise, i.e., if $z \rightarrow y$, the cycle to the left can be enlarged to include z. In either case, we now have two cycles with two points (at least) in common, x and (let us say) y. Proceeding as before, the cycle to the right can now be enlarged to include v or the cycle to the left to include z. Continuing in this manner, we construct a cycle including all the points in the union of V' and V''.[11]

THEOREM 5. In a tournament, any point in the Condorcet set V^* is reachable from any other point in V^*.

Consider any two points x and y, both in V^*, and suppose—contrary to the proposition—that y is not reachable from x. Now consider the subset V^{**} of all points in V^* other than x and all points reachable from x. But then (1) V^{**} is not empty and (2) no point in V^{**} is dominated by any point not in V^{**}. Accordingly V^{**} meets Condition I and V^* fails to meet Condition II, contradicting the assumption that V^* is the Condorcet set.

[10] See Harary, et al. (1965), p. 295.
[11] Cf. Black (1958), pp. 48–49.

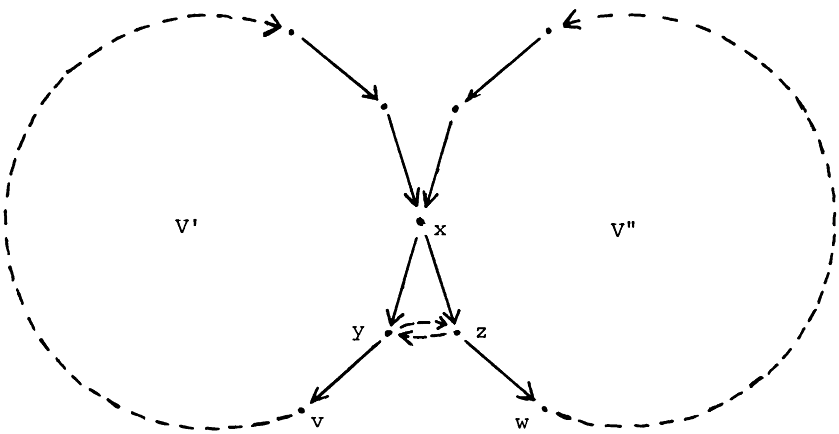


FIGURE 4

Intersecting Cycles

THEOREM 6. In a tournament without an undominated point, there is a cycle including all the points in the Condorcet set V*.

This follows directly from the previous theorem and the two lemmas. Clearly no such cycle can also include points not in V*, for them some point in V* would be dominated by some point not in V*, contradicting Condition I. Thus it further follows that if there is no undominated point, the Condorcet set includes at least three points.

THEOREM 7. In a tournament, the following statements are equivalent:
(1) there is a complete path beginning with point x;
(2) point x is a source; and
(3) point x belongs to the Condorcet set V*.

First, it is obvious and immediate that (1) implies (2).

Next, we show that (2) implies (3). Suppose that x does not belong to V* and consider some point y that does belong to V*. By Condition I, we cannot have $x \to y$; nor can we have $x \to z \to y$, for then $x \to z$ implies that z is not in V* while $z \to y$ implies the reverse; nor can we have $x \to z \to v \to y$; and so forth. Thus x cannot be a source. So if x is a source, it must belong to the Condorcet set.

Next we show that (3) implies (1). We know by Theorem 4 that the tournament has a complete path, and if x is undominated—that is, if x is the only point in V*—clearly any complete path must begin with x. Otherwise—

that is, if there is no undominated point and V* includes three or more points—we know from Theorem 6 that there is a cycle including all the points in V*; thus there is a path including all the points in V* that begins with x, say $x \to \ldots \to y$. And by Theorem 4, the tournament including only the points not in V* has a complete path, say $v \to \ldots \to z$. Finally, by Condition I, we have $y \to v$, giving us the complete path in the full tournament $x \to \ldots \to y \to v \to \ldots \to z$.

We now have a cycle of implications: (1) implies (2) implies (3) implies (1). Since the relation of logical implication is transitive, the three reverse implications must hold as well, so the three statements are equivalent.

Clearly it is convenient to represent majority preference by means of a tournament. We now turn to consider the operation of two specific voting procedures, and we will find that the graph-theoretic concepts that have been presented, and the results stated in Theorem 7 especially, are extremely useful in proving a number of propositions concerning voting decisions.

2. The Theory of Voting

In the first section, we considered voters' preferences and analyzed the structure of the tournament generated by the preferences of majorities of voters. We now turn to consider actual voting decisions under two specific "binary" procedures. We consider the general case in which a voting decision must be reached out of a set of m proposals.

A voting procedure is *binary* if, at every vote, a voter has exactly two choices (e.g., "yea" or "nay," "vote for x" or "vote for y") other than abstain.

Under *amendment* procedure, two proposals are paired for a majority vote, the defeated proposal being eliminated as a possible decision; the surviving proposal is then paired with a third at the second vote; and so forth. The proposal that survives the final or $(m-1)$th vote is the decision.

Under *successive* procedure, the first proposal is voted up or down on a majority basis; if voted up, it is the decision and voting terminates; otherwise, the second proposal is voted up or down; and so forth. If the first $m-1$ proposals are voted down, the one remaining proposal (probably an "implicit" proposal to preserve the status quo or to "do nothing") is the decision.[12]

[12] In naming these procedures, I have followed the usage of Farquharson (1969), pp. 10–11, 61. Black (1958) calls the former "procedure (α)" on p. 21 and "ordinary committee procedure" thereafter.

Amendment procedure approximates parliamentary voting in Anglo-American legislative bodies in its logical structure, though typical parliamentary practice differs in a consequential way when separate amendments to the main motion are before the voting body. According to Niemi and Bjurulf (1976, p. 12), successive procedure is used in a number of continental European legislatures. Thus all the results presented below have practical relevance.

It is important to note that under both procedures, the proposals must be voted on in some definite order. One critical question is whether—or under what conditions—and how the voting decision is affected by the order in which proposals are voted on, as well as by the procedure that is used.

The voting decision depends, of course, not only on the preferences of the voters (and possibly the procedure and voting order) but more directly on their voting behavior—that is, on the choices they actually make at each vote. We assume in the first place that no voter ever abstains. And initially we assume that each voter votes "sincerely," so that his voting choices simply express his preferences and are not influenced by any strategic calculations. Then in the latter part of this section, we consider two types of "strategic" voting. A second critical question of course is whether—or under what conditions—and how the voting decision depends on whether voting is sincere or strategic.

a. Sincere Voting Decisions under Amendment Procedure

Under amendment procedure, according to which proposals are explicitly paired for votes, a "sincere" voter simply votes for his more preferred proposal in each pair. The outcome of a vote thus "reveals" the direction of majority preference between the pair of proposals in question.

It is plausible to suppose that the voting decision may depend on the order in which the proposals are voted on. So the first question is to identify the set of proposals that can possibly be the voting decision, given an appropriate voting order. We have the following necessary condition:

PROPOSITION 1. Under amendment procedure, proposal x is the sincere voting decision only if x is a source.

Consider the portion of the majority preference tournament "revealed" by sincere voting under amendment procedure. The digraph representing "revealed" majority preference clearly is incomplete (it includes only m-1 directed lines but all $\frac{m}{2}(m-1)$ pairs of proposals). But by the systematic nature of the procedure, it is connected, and every proposal but one is dominated (that is, has been defeated) exactly once and the one remaining proposal (which is the decision x) is undominated. Thus by Theorem 2 the digraph representing "revealed" majority preference is a tree and the winning proposal x is its source.[13] It follows that x must also be a source in the complete majority preference tournament.

According to Theorem 7, a proposal is a source if and only if it belongs to the Condorcet set. Thus we can restate Proposition 1 in this manner:

PROPOSITION 1′. Under amendment procedure, the sincere voting decision belongs to the Condorcet set.

Thus sincere voting under amendment procedure complies with the Condorcet criterion. And it follows that, if there is a unique source (that is, an undominated proposal), it is the sincere voting decision regardless of the order in which the proposals are voted on.[14]

We have seen that the set of possible sincere voting decisions under amendment procedure includes no proposals not in the Condorcet set. But does it include all proposals in the Condorcet set? That is, can any proposal in the Condorcet set be the decision, given an appropriate voting order? This question can be answered affirmatively.

PROPOSITION 2. Under amendment procedure, for any proposal x that belongs to the Condorcet set, there is some order of voting such that x is the sincere voting decision.

By Theorem 7, there is a complete path in the majority preference tournament that begins with x. Let the proposals be voted on in the order that is the reverse of that entailed by this path; at each vote after the first, the newly introduced proposal defeats the proposal surviving from the previous vote, and x–introduced in the final vote–is the voting decision.[15] Thus there is an order of voting–specifically one that puts x last–such that x is the decision.

[13] Since the digraph that represents "revealed" majority preference under amendment procedure is *always* a tree, it follows that sincere voting under amendment procedure can *never* reveal the extent which majority preference is cyclical or acyclical. (But *any* additional vote would reveal some semi-cycle.) Cf. Marz, et al. (1973).

[14] For this result, together with the demonstration that such a proposal exists if preferences are "single-peaked," Black (1958) is best known; see pp. 18, 43.

[15] Cf. Taylor (1968), p. 40. On p. 42 Taylor comes very close to stating Proposition 2. Also see Black (1958), pp. 40–41.

Note that the proof of Proposition 2 implies only that there is *some* order of voting that puts x last such that x is the decision. It does not imply that x is the decision under *every* order that puts x last nor that x *must* be last (or even last in the Condorcet set) to be the decision. However, it is clear that every proposal that dominates x must be introduced *and defeated* before x is introduced, if x is to be the decision. Therefore, in the absence of an undominated proposal, we have the following necessary condition for proposal x to be the sincere voting decision under amendment procedure:

Condition III. Every proposal in F(x) and one additional proposal in V* must precede x in the order of voting.

From this it follows that neither of the first two proposals in the Condorcet set to be introduced into the voting can be the voting decision; and therefore, if the Condorcet set includes exactly three proposals, the last of the three to be introduced is the decision.

b. Sincere Voting Decisions under Successive Procedure

Under successive procedure, according to which proposals are voted up or down in sequence, a "sincere" voter votes for a proposal if it is the proposal he most prefers of those that remain undefeated; otherwise he votes against. Therefore sincere voting under successive procedure does not involve an explicit *pairing* of proposals (except at the (m−1)th vote, if taken), and the majority preference tournament is not sufficient to determine which proposal is the sincere voting decision in any given case. However, our first proposition is a negative one that can be demonstrated simply by example.

PROPOSITION 3. Under successive procedure, the sincere voting decision may not belong to the Condorcet set.

Accordingly, sincere voting under successive procedure may fail to comply with the Condorcet criterion. Suppose that there are three proposals x, y, and z, and three voters. Voter 1 prefers x to y to z, voter 2 prefers y to x to z, and voter 3 prefers z to y to x. Proposal y is undominated and should be the decision, according to the Condorcet criterion. But if successive procedure is used and y is voted on first, y will be rejected if voting is sincere, since voters 1 and 3, who most prefer x and z respectively, will both vote against it. (Proposal x will then win the second vote and be the decision.)

Therefore, the set of possible decisions under successive procedure, in contrast to amendment procedure, includes proposals not in the Condorcet set. But it is also clear that, in the example that demonstrated this point, proposal y would be the decision if it were not voted on first. We may ask whether this is generally true. That is, can any proposal in the Condorcet set be the decision, given an appropriate voting order? As in the case of amendment procedure, this question can be answered affirmatively.

PROPOSITION 4. Under successive procedure, for any proposal x that belongs to the Condorcet set, there is some order of voting such that x is the sincere voting decision.

First we note that the sincere voting decision is the first proposal x in the voting order such that a majority of voters have x as their highest preference in the set of proposals that remain undefeated–that is, in the set including x and all proposals that follow x in the voting order. Therefore we have the following necessary condition for proposal x to be the sincere voting decision under successive procedure:

Condition IV. Every proposal that follows x in the order of voting must belong to D(x).

Now we refer again to Theorem 7 and proceed as in the proof of Proposition 2. Let the proposals be voted on in the order that is the reverse of that entailed by a complete path beginning with x. Since each proposal is dominated by the immediately subsequent proposal, no proposal preceding x meets Condition IV; every proposal is defeated until only x–last in the voting order–remains, making x the decision. (Proposal x is also the decision if it is next to last in the voting order.) Thus there is an order of voting–specifically one that puts x last (or next to last)–such that x is the decision.

Condition IV has a number of other implications: (1) the first proposal in the voting order is the decision only if it is undominated; (2) the second proposal in the voting order is the decision only if it belongs to the Condorcet set or the first proposal is undominated; (3) the third proposal in the voting order is the decision only if it belongs to the Condorcet set or the first or second proposal is undominated; and (4) any proposal later in the voting order, if not in the Condorcet set, is the decision only if all proposals in the Condorcet set precede it in the order of voting. It is also clear that a proposal dominated by every other proposal cannot be the decision; and that if an undominated proposal exists, and if it is last or next to last in the voting order, it is the decision.

c. Sophisticated Voting Decisions under Successive Procedure

It is widely acknowledged that, given three or more proposals, sincere voting may be inexpedient in some circumstances. This point is probably most widely recognized in multi-candidate plurality elections (in which a vote for a preferred "minor" candidate may be "wasted"). But it may also be true under the binary procedures we are considering. Indeed, the inexpediency of sincere voting is quite clear in the example used to demonstrate Proposition 3. Voting sincerely, voter 3 opposes y at the first vote and thereby brings about its defeat and the ultimate adoption of x–his lowest preference. Provided that he is informed about the preferences of his fellow voters, voter 3 can anticipate the unfortunate consequence of his sincere voting and can calculate that he would do better by joining voter 2 (for whom sincere voting clearly is expedient) in voting for y and thereby achieve the adoption of y–his second, rather than lowest, preference. Thus in this case at least the "sophisticated" voting decision under successive procedure complies with the Condorcet criterion, even though the sincere voting decision does not.

We can generalize this conclusion by generalizing this analysis. We start out with the set of all proposals as possible decisions. As a result of the first vote under successive procedure, one proposal either becomes the decision or is eliminated as a possible decision. Thus at the second vote (if taken) the set of possible decisions has been reduced by one and, as a result of the second vote, one of these either becomes the decision or is eliminated. And so forth, until the (m–1)th vote, after which only one proposal remains as a possible decision. This process of narrowing down the set of possible decisions may be represented by a *division tree,* such as that shown in Figure 5(a) for the case of five proposals, x, y, z, v, and w. Note that the division tree is indeed a "tree," as the term has been formally defined. The initial point of the tree corresponds to the first vote (at which all proposals are possible decisions), and each end point corresponds to a termination of voting (at which every proposal but one has been eliminated as a possible decision–in other words, a decision has been reached). Each path from the initial point to an end point represents a particular sequence of votes. In the case of successive procedure (but not generally), there is exactly one end point for each proposal–that is, there is only one way (one sequence of votes) to reach any given decision.

Now suppose that the majority preference tournament is that shown in Figure 5(b) and that all voters are "sophisticated." Each "sophisticated" voter knows that if it comes down to a choice between v and w at the fourth and final vote, v becomes the decision. We may call v the *anticipated decision* at this vote and accordingly v has been underlined in Figure 5(a). Now each

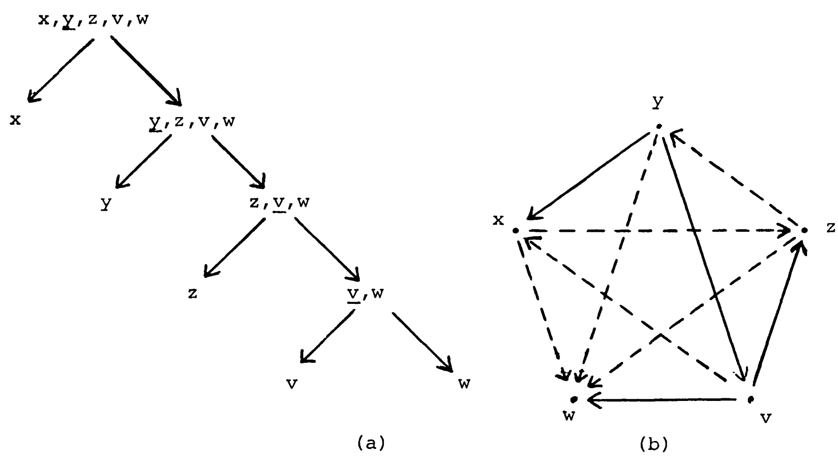


FIGURE 5

Division Tree for Successive Procedure and an Example of Sophisticated Voting

"sophisticated" voter knows further that the choice at the third vote, though manifestly a vote on whether to accept or reject z, is in effect a pairwise choice between z and v. Since v dominates z, if all voters are "sophisticated," v would win this vote also, and v is also the anticipated decision at the third vote, so the second vote is in effect a pairwise choice between y and v. And so forth. "Pushing" the anticipated decision (underlined) to the initial point in the division tree, we see that y is the anticipated decision at the very first vote. In other words, y is the sophisticated voting decision in this case.[16]

More generally, we see that sophisticated voting under successive procedure works *as if* the last two proposals in the voting order were paired for a sincere majority vote, the defeated proposal being eliminated and the sur-

[16] This "tree method" for determining the sophisticated voting decision is different from—and certainly simpler than—the "reduction method" presented by Farquharson (1969), pp. 38–49. For further discussion of this method and its relationship to Farquharson's method, see Miller (1973), pp. 345ff, Niemi, et al. (1974), pp. 1–10, and McKelvey & Niemi (1976). Niemi, et al., exhaustively examine sincere and sophisticated voting decisions under amendment and successive procedure in the case of three proposals. In a more recent paper, Niemi and Bjurulf (1976) extend this examination to four and (in part) five proposals. Their orientation is one of providing strategic advice: how to make a preferred proposal the decision. Most or all of their strategic advice can be derived from the more general descriptive propositions presented here.

viving proposal being paired with the third to last proposal in the voting order for a sincere majority vote, and so forth. In other words, the logical structure of sophisticated voting under successive procedure is identical to that of sincere voting under amendment procedure *when the order of voting is reversed.* By virtue of this identity, any valid statement concerning sincere voting decisions under amendment procedure is valid also for sophisticated voting decisions under successive procedure, provided only that any reference to the order of voting is reversed.

Thus we can move directly to the two following counterparts of Propositions 1′ and 2:

PROPOSITION 5. Under successive procedure, the sophisticated voting decision belongs to the Condorcet set.

Thus sophisticated voting (unlike sincere voting) under successive procedure complies with the Condorcet criterion. And it follows that, if there is an undominated proposal, it is the sophisticated voting decision regardless of the order in which the proposals are voted on (and the example that introduced this section was not atypical).

PROPOSITION 6. Under successive procedure, for any proposal x that belongs to the Condorcet set, there is some order of voting such that x is the sophisticated voting decision.

Specifically, there is some order of voting that puts x first.

Reversing the reference to the order of voting, we have the following counterpart of Condition III as a necessary condition, in the absence of an undominated proposal, for proposal x to be the sophisticated voting decision under successive procedure:

Condition V. Every proposal in F(x) and one additional proposal in V* must follow x in the order of voting.

From this it follows that neither of the last two proposals in the Condorcet set to be introduced into the voting can be the voting decision; and therefore, if the Condorcet set includes exactly three proposals, the first of the three to be introduced is the decision.

It is probably intuitively clear that, when voting is sincere, successive procedure favors proposals later in the voting order whereas, when voting is sophisticated, the reverse is true.[17] We can, in fact demonstrate the following:

PROPOSITION 7. Under successive procedure and a given order of voting, if the sincere and sophisticated voting decisions differ, the proposal that is the sophisticated decision precedes the proposal that is the sincere decision in the order of voting.

Let x be the sincere decision and y the sophisticated decision, and suppose, contrary to the proposition, that y follows x in the order of voting. But then Condition IV implies that x dominates y and Condition V implies that y dominates x, which is impossible since majority preference is a tournament. So x must precede y in the order of voting.

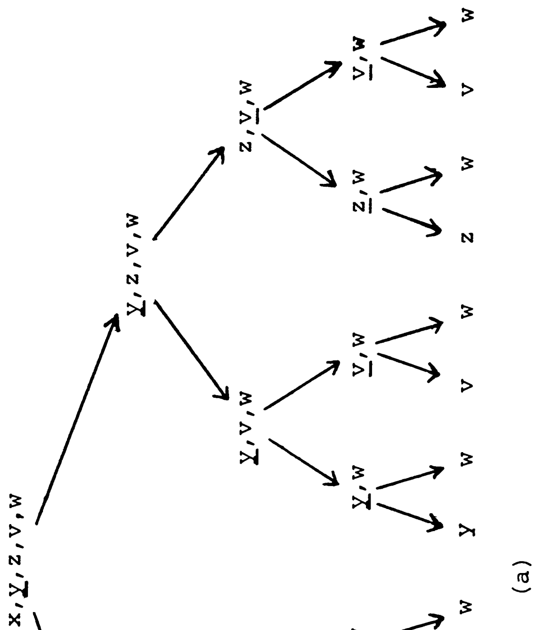
d. Sophisticated Voting Decisions under Amendment Procedure

As should be clear, the mode of analysis used in the previous section can also be used to determine sophisticated voting decisions under other binary procedures, including amendment procedure. Matters do become slightly more complicated, but the analysis remains quite straightforward.

First, we must construct the appropriate division tree. Under amendment procedure, two proposals are paired at the first vote and one or the other is eliminated as a possible decision. Consider again the case of five proposals, x, y, z, v, and w voted on in that order. As a result of the first vote, either x, z, v, and w (if x wins) or y, z, v, and w (if y wins) remain as possible decisions. As a result of subsequent votes, the sets of possible decisions are narrowed down in similar fashion until each is a one-element set—that is, until a decision is reached. Figure 6(a) shows the complete division tree for this case. Because under amendment procedure, m-1 votes are taken regardless of the results of previous votes, all paths from the initial point to an end point are of equal length, some decisions can be reached by several different paths, and the tree as a whole is considerably larger than that for successive procedure.

We can now analyze sophisticated voting by the same logic used before.

[17] According to Fraquharson (1969), p. 62: "If voting is sincere, the later a proposal is voted on, the better its chance," while "if voting is sophisticated, the earlier a proposal is voted on, the better its chance." Proposition 7 is one way of rendering precise these somewhat imprecise statements. For other ways, see Miller (1975b), Niemi, et al. (1974), and Niemi and Bjurulf (1976).



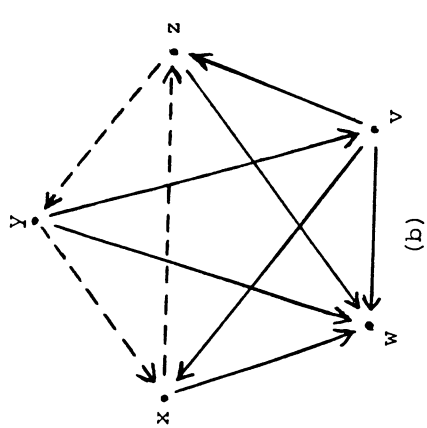


FIGURE 6
Division Tree for Amendment Procedure and an Example of Sophisticated Voting

This time, as we "push" the anticipated decision up the division tree, we compare one anticipated decision with another (rather than, as in the case of successive procedure, one anticipated with one actual decision). And, precisely because there may be several different paths to the same decision, some votes, though manifestly between two distinct proposals, become in effect inconsequential choices between identical anticipated decisions. But neither of these matters causes difficulty and, as in the case of successive procedure, we can quickly identify the anticipated decision at the first vote, which is the sophisticated voting decision. Figure 6(b) shows a majority preference tournament, and the corresponding anticipated decisions are underline in the division tree in Figure 6(a).

PROPOSITION 8. Under amendment procedure, the sophisticated voting decision belongs to the Condorcet set.

Consider any proposal x that belongs to the Condorcet set and any path in the division tree that leads to x as the decision. In this path, if x is not the anticipated decision at the final vote, then the anticipated decision at that vote must be some y that dominates x, in which case y belongs to the Condorcet set. Likewise, at the next to last vote in this path, the anticipated decision must be x, or y, or some proposal z that belongs to the Condorcet set. And so forth. Thus the anticipated decision at the initial vote, which is the sophisticated decision, belongs to the Condorcet set.

While this argument was made with amendment procedure particularly in mind, it is clear that it does not rest on any special characteristics of the amendment procedure division tree and that it is equally applicable to other binary procedures. We can therefore generalize Proposition 8:

PROPOSITION 8′. Under any binary procedure, the sophisticated voting decision belongs to the Condorcet set.

Thus sophisticated voting under any binary procedure complies with the Condorcet criterion. And it follows that, if there is an undominated proposal, it is the sophisticated voting decision regardless of the order in which the proposals are voted on.[18] (Proposition 8′, of course, subsumes Propositions 5 and 8.)

If we focus specifically on the division tree for amendment procedure, we

[18] This point, for amendment procedure, was conjectured by Grofman (1969), p. 73, and has also been demonstrated by Niemi, et al. (1974), p. 9.

can analyze sophisticated voting more closely. Under amendment procedure, the final or (m-1)th vote always involves the last proposal in the voting order, say proposal z. Thus the anticipated decision at any final vote is either z or some proposal that dominates z. Accordingly, the anticipated decision at any earlier vote, and also the sophisticated decision, belongs to $F(z) \cup \{z\}$.

Now consider some proposal x that dominates z and any path in the division tree that leads to x. Since x dominates z, x is the anticipated decision at the final or (m-1)th vote in this path. If x is not also the anticipated decision at the (m-2)th vote in this path, that anticipated decision must be some y such that (1) y belongs to F(z) and (2) y dominates x. And then if x (or y) is not the anticipated decision at the (m-3)th vote in this path, that anticipated decision must be some v such that (1) v belongs to F(z) and (2) v dominates x (or y). And so forth, to the initial point in the division tree. It follows that z, the last proposal in the voting order, is the sophisticated decision only if z is undominated. And otherwise—that is, if z is dominated—we have the following necessary condition for proposal x, not last in the voting order, to be the sophisticated voting decision under amendment procedure:

Condition VI. Proposal x must belong to F(z), where z is the last proposal in the voting order.

We have seen that all anticipated decisions at the several final votes belong to $F(z) \cup \{z\}$. By the same argument that supports Proposition 8, the anticipated decision at the first vote, and thus the sophisticated decision, belongs to the minimal undominated set of proposals in the majority preference tournament that includes only the proposals in $F(z) \cup \{z\}$, which is obviously identical to the minimal undominated set of proposals in the tournament that includes only the proposals in F(z). Designate this minimal undominated set F*(z).[19] We can therefore refine Condition VI as follows:

Condition VI′. Proposal x must belong to F*(z), where z is the last proposal in the voting order.

Thus if F*(z) includes precisely one proposal, that proposal is the sophisticated decision. Otherwise—that is, if F*(z) includes three or more proposals—

[19] That is, F*(z) is the "Condorcet set" of this smaller tournament; as such, F*(z) exists and is unique and includes either precisely one proposal or at least three proposals in a cycle. If z does not belong to V*, then clearly F*(z) = V*; otherwise F*(z) is contained in V*.

the sophisticated decision depends on which proposal in $F^*(z)$ is last in the voting order. Suppose that this proposal is v. By an argument similar to that supporting Condition VI, the sophisticated decision cannot be v and belongs to $F(v)$. In conjunction with Condition VI′, this means that the sophisticated decision belongs to the intersection $F^*(z) \cap F(v)$.[20] And by the same argument that supports Proposition 8 and Condition VI′, the sophisticated decision belongs to the minimal dominated set of proposals in the majority preference tournament that includes only the proposals in $F^*(z) \cap F(v)$. Designate this minimal undominated set $[F^*(z) \cap F(v)]^*$. Thus, if $F^*(z)$ is not a one-element set, we can further refine Condition VI as follows:

Condition VI″. Proposal x must belong to $[F^*(z) \cap F(v)]^*$, where z is the last proposal in the voting order and v is the proposal in $F^*(z)$ last in the voting order.

Thus, if $[F^*(z) \cap F(v)]^*$ includes precisely one proposal, that proposal is the sophisticated decision. Otherwise, the sophisticated decision depends on which proposal in $[F^*(z) \cap F(v)]^*$ is last in the voting order. And so forth. But at some point we reach a one-element set,[21] which identifies the sophisticated decision.

We have seen that under amendment procedure, the set of possible sophisticated voting decisions includes no proposal not in the Condorcet set. But again we can ask whether this set includes all such proposals. That is, can any proposal in the Condorcet set be the sophisticated voting decision, given an appropriate order of voting? Here for the first time we reach a negative conclusion, which can be demonstrated by example.

PROPOSITION 9. Under amendment procedure, a proposal may belong to the Condorcet set yet not be the sophisticated voting decision under any order of voting.

Every proposal belonging to the Condorcet set meets Condition VI for some z. But such a proposal may fail to meet Condition VI′ or some further refinement of Condition VI. Consider, for example, the majority preference

[20] Since the proposals in $F^*(z)$ are in a cycle, at least one proposal in $F^*(z)$ belongs to $F(v)$ and at least one, in addition to v, does not; thus the intersection $F^*(z) \cap F(v)$ is a non-empty proper subset of $F^*(z)$.

[21] This must be true because (1) the number of proposals is finite and (2) the number of possible sophisticated decisions is reduced in each "cycle" of the argument (by the point made in the previous footnote).

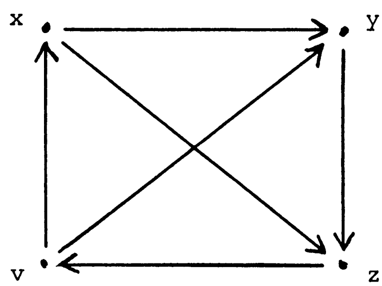


FIGURE 7

tournament shown in Figure 7. All proposals, including y, belong to the Condorcet set. Proposal y dominates only z; thus by Condition VI, y can be the sophisticated decision only if z is voted on last. But, in this case, the decision belongs to F*(z)—that is, the decision is x. So y is not the sophisticated decision under any voting order.

The finding that the set of possible sophisticated decisions under amendment procedure may exclude certain proposals in the Condorcet set raises the question of whether this exclusion works in some "reasonable" manner. Obviously this exclusion cannot be justified on the basis of the Condorcet criterion. But is it consistent with some other normative principle? Consider the following. A proposal x is *Pareto-optimal* if no other proposal y is unanimously preferred to x. Then we have:

THE PARETO CRITERION: A proposal should be the voting decision only if it is Pareto-optimal.

At first glance, it may appear that the Pareto criterion is redundant in conjunction with the Condorcet criterion—that is, it might seem that any proposal meeting the latter condition meets the former as well. But in fact the two criteria are independent, and the Condorcet set may include proposals that fail to be Pareto-optimal. Suppose there are four proposals, x, y, z, and v, and three voters: voter 1 prefers v to x to y to z, voter 2 prefers z to v to x to y, and voter 3 prefers x to y to z to v. These preferences generate the majority preference tournament considered just above and depicted in Figure 7. Proposal x is unanimously preferred to y so, while all four proposals belong to the Condorcet set, only x, z, and v are Pareto-optimal. From this and Propositions 2, 4, and 6, it follows that sincere voting under amendment procedure and both sincere and sophisticated voting under successive procedure may fail to comply with the Pareto criterion—that is, in these cases a voting body may adopt a proposal y even when *every* voter prefers some other proposal x to y.[22] But we have just seen in the present example that proposal y, which fails to be Pareto-optimal, cannot be the sophisticated voting decision under amendment procedure. Does sophisticated voting under amendment procedure generally exclude non-Pareto-optimal proposals? We can answer this question affirmatively.

PROPOSITION 10. Under amendment procedure, the sophisticated voting decision is Pareto-optimal.

Thus sophisticated voting under amendment procedure complies with the Pareto criterion. This can be demonstrated readily once we establish the following simple lemma:

LEMMA 3. If a proposal x is unanimously preferred to y, then x dominates every proposal dominated by y.

Consider a proposal x that is unanimously preferred to (and thus dominates) y and any z dominated by y. All voters prefer x to y, and a majority prefer y to z; by the transitivity of individual preference, at least the same majority prefers x to z; thus x dominates z.

The proposition now follows directly from Condition VI and its refinements. Suppose x is unanimously preferred to y. If y meets Condition VI, so does x (by Lemma 3); if y meets Condition VI′, so does x (because x dominates y); if y meets Condition VI″, so does x (by Lemma 3 and because x dominates y); and so forth. So when we reach the one-element set that is the sophisticated decision, it cannot be y.[23]

Only a bit less clearly than in the case of successive procedure, sincere voting under amendment procedure appears to favor proposals later in the voting order.[24] We can in fact prove the following:

[22] This point has been noted by Plott (1971), p. 27n, and Fishburn (1973), pp. 96–97. Also see Schwartz (1975), pp. 29–30.

[23] But sophisticated voting under amendment procedure does not exclude *only* proposals in the Condorcet set that fail to be Pareto-optimal as possible decisions, for the sophisticated decision depends only on the majority preference tournament, and not on the exact distribution of preferences that underlies the tournament. And the discussion in footnote 7 tells us that, in the tournament shown in Figure 7, majority preference between x and y may "carry" by just one "vote" and certainly need not reflect unanimity.

[24] Cf. Black (1958), p. 40, and the references cited in footnote 17.

PROPOSITION 11. Under amendment procedure and a given order of voting, if the sincere and sophisticated voting decisions differ, the proposal that is the sophisticated decision precedes the proposal that is the sincere decision in the order of voting.

In demonstrating this proposition and the next, it is convenient to designate the proposals $v_1, v_2, \ldots, v_m$, so that the subscripts indicate the order of voting. Suppose that v_h is the sincere decision, which by supposition is different from the sophisticated decision. This implies that there is no undominated proposal, so $F(v_m)$, and also $F^*(v_m)$, is not empty and includes the sophisticated decision. By Condition III, $F(v_h)$ is contained in $\{v_1, \ldots, v_{h-1}\}$ or, equivalently, $\{v_{h+1}, \ldots, v_m\}$ is contained in $D(v_h)$, which implies that $F^*(v_m)$ is not contained in $\{v_{h+1}, \ldots, v_{m-1}\}$ and accordingly intersects $\{v_1, \ldots, v_h\}$. It follows that if $F^*(v_m)$ is a one-element set, the proposition is confirmed. Otherwise, one or several (but not all) of the proposals in $F^*(v_m)$ may belong to $\{v_{h+1}, \ldots, v_{m-1}\}$. Suppose that v_k is the proposal in $F^*(v_m)$ last in the voting order and that $h < k$, i.e., that $F^*(v_m)$ intersects $\{v_{h+1}, \ldots, v_{m-1}\}$. It remains true that $\{v_{h+1}, \ldots, v_{m-1}\}$ is contained in $D(v_h)$, which implies that $[F^*(v_m) \cap F(v_k)]^*$ is not contained in $\{v_{h+1}, \ldots, v_{m-1}\}$ and accordingly intersects $\{v_1, \ldots, v_h\}$. And so forth. So when we reach the one element set that is the sophisticated decision, it must belong to $\{v_1, \ldots, v_h\}$; and, if the sophisticated decision is not v_h, it must precede v_h in the order of voting.

One other proposition relating sincere and sophisticated voting decisions under amendment procedure is of interest.

PROPOSITION 12. Under amendment procedure and a given order of voting, if the sincere and sophisticated voting decisions differ, the proposal that is the sophisticated decision dominates the proposal that is the sincere decision.[25]

Let v_h be the sophisticated decision and v_k be the sincere decision. Of course, if the two decisions differ, (1) there is no undominated outcome (by Propositions 1′ and 8′) and (2) $h < k$ (by Proposition 11). If $h < k = m$, the proposition follows immediately, since v_h belongs to $F(v_m)$ (by Condition VI).

If $h < k < m$, the proposition follows a bit less directly. By Condition III,

[25] This proposition does not hold for successive procedure; for an example, see Miller (1975b).

v_k belongs to $F(v_m)$ and, by Condition VI′, v_h belongs to $F^*(v_m)$. Thus if, contrary to the proposition, v_k dominates v_h, v_k also must belong to $F^*(v_m)$; accordingly $F^*(v_m)$ includes at least three proposals, v_h, v_k, and at least one other, and the sophisticated decision depends on which proposal in $F^*(v_m)$ is last in the voting order. This last proposal cannot be v_h, since $h < k$. If this last proposal is v_k, then by Condition VI″ v_h belongs to $[F^*(v_m) \cap F(v_k)]^*$, which implies that v_h dominates v_k. So if v_h does not dominate v_k, this last proposal must be some v_j, such that $k < j$ and v_h belongs to $[F^*(v_m) \cap F(v_j)]^*$. Since $k < j$, Condition III implies that v_k belongs to $F(v_j)$ and of course, if v_k dominates v_h, v_k also belongs to $F^*(v_m)$. Thus once again if, contrary to the proposition, v_k dominates v_h, v_k also must belong to $[F^*(v_m) \cap F(v_j)]^*$; accordingly $[F^*(v_m) \cap F(v_j)]^*$ includes at least three proposals, v_h, v_k, and at least one other that falls between v_k and v_j in the voting order. And so forth. In summary, the supposition that v_k dominates v_h implies that v_k belongs to $F^*(v_m)$, and to $[F^*(v_m) \cap F(v_j)]^*$, etc. But since the number of proposals is finite, we must reach a one-element set that includes only the sophisticated decision v_h. Thus v_h must dominate v_k.

e. Cooperative Voting

Opportunities for general strategic manipulation of voting decisions have not been exhausted by consideration of sophisticated voting, since sophisticated (like sincere) voters make their voting choices in isolation from one another. But if voters can cooperate—that is, if they can communicate and make binding agreements (or "contracts")—further strategic opportunities are opened up.

For example, suppose there are three proposals, x, y, and z, and three voters: voter 1 perfers x to y to z, voter 2 prefers y to z to x, and voter 3 prefers z to x to y (thus all three proposals belong to the Condorcet set). Suppose further that the proposals are voted on in alphabetical order under amendment procedure. If voting is sincere, voters 1 and 2 vote together for x at the first vote, and voters 2 and 3 vote together for z at the second vote, so z is the sincere decision. But if voter 1 is sophisticated, he can anticipate that his sincere vote for x at the first vote will bring about the ultimate adoption of z, his lowest preference; and he can calculate that he would do better to join voter 2 in voting for y at the first vote and thereby achieve the adoption of y, his second preference. Neither of the other two voters has reason to vote nonsincerely, so y is the sophisticated decision.

But if the voters can cooperate to the extent of making binding agreements, further strategic manipulation is surely possible, for voters 1 and 3 both prefer x to y and—since they constitute a majority—they have the power, acting in concert, to make x (or any other proposal) the decision (by voting as a bloc for x at both votes).[26] But the matter does not end here, since—if voters 1 and 3 are about to make such an agreement—voter 2 (who risks getting his lowest preference) can offer voter 3 better terms: an agreement to make z the decision. And then voters 1 and 2 can do better by agreeing to make y the decision.

A *decisive coalition* is a set of voters who, acting in concert, have the power to impose any decision on the voting body. A *majoritarian* procedure makes any majority coalition decisive. Clearly both procedures we have considered are majoritarian. Proposal x is *effectively preferred* to y if there is a decisive coalition all of whose members prefer x to y. Clearly in a majoritarian voting body majority preference and effective preference are equivalent. Thus in the example just considered, in which all three proposals belong to the Condorcet set, we find a cycle of effective preference: x is effectively preferred to y, y to z, and z to x. In general, under fully cooperative conditions, any tentative agreement or "contract" to make any proposal y the decision may be "upset" or "preempted" by another tentative agreement or "contract" to make some proposal x the decision, where x is effectively preferred to y. And this process of "recontracting" continues until an agreement is reached to make an undominated proposal the decision (which agreement cannot be "upset") or until the process is broken off essentially arbitrarily.

This discussion supports several conclusions. First, the cooperative voting decision depends on the outcome of "pre-play" negotiations; once some agreement is struck, voting itself is only a formality to be "played out."

Second (but related to the first point), the cooperative voting decision depends only on the preferences of the voters and on the fact that the procedure is majoritarian[27] —the procedure itself (e.g., amendment versus successive) and the order in which proposals are voted on are irrelevant under fully cooperative conditions.

Third, the tournament we have previously analyzed in some detail represents not only majority preference but also effective preference in the voting body and thus determines the course of bargaining in "pre-play" negotiations.

[26] Note that a "contract" between voters 1 and 3 is required, since otherwise 3 would always vote for z at the second vote and, anticipating such a "defection,"1 would vote for y at the first vote.

[27] Or, more generally, on the power relations entailed by the procedure, majoritarian or otherwise.

By Theorem 7, there is a path from any proposal in the Condorcet set to any other proposal; thus "recontracting" will in due course lead into the Condorcet set (if it does not begin there) and, once in the Condorcet set, "recontracting" can never lead out of it. This argument supports the following proposition.

PROPOSITION 13. Under any majoritarian voting procedure, the cooperative voting decision belongs to the Condorcet set.

There is, however, no basis for specifying one such proposal rather than another as the cooperative decision, since the particularities of procedure and voting order, which determine which of the several proposals in the Condorcet set will be the sincere or sophisticated decision, are now irrelevant. Thus, unless there is an undominated proposal, the cooperative voting decision is indeterminate.[28]

Before concluding this section, it may be worthwhile to discuss one other point, since—though it has by now been adequately treated elsewhere—the underlying graph-theoretical considerations have not always been made explicit. Implicitly to this point, we have been considering a voting body confronted with a single "issue" that can be resolved in any one of several ways or with a single "bill" that can be passed, perhaps with amendments, or defeated. We now consider a voting body confronted with several such issues or bills when some general agreement may be made concerning the resolution of several or all such matters simultaneously—that is, when "logrolling" of some sort is feasible.[29]

We consider a voting body that must resolve n *issues,* X, Y, . . . , Z, where each issue (for example, X) is made up of several *positions* (for example, x_1, $x_2, \ldots, x_m$). If we retain the word "proposal" to designate the basic objects of decision, a proposal is now a bundle of n positions, one from each issue (for example, $[x_h, y_k, \ldots, z_j]$).

As before, we assume that voters have strong preference orderings over these proposals. But we now consider one natural restriction on voters' preferences over such proposals. A *complement* $\overline{x}_r$ of issue X is a bundle of n-1 positions, one from each other issue. Thus, $(x_h, \overline{x}_r)$ is a proposal. A voter's

[28] There are ways, however, in which the set of possible cooperative decisions may be further pared down; for some suggestions, see Schwartz (1975), Section 8.

[29] Note that, as used here, "logrolling" implies full cooperation among voters and is to be distinguished from other forms of "vote trading" that involve partial but less than full cooperation. For a formalization of this distinction, see Miller (1977).

preferences are *separable* if, whenever he prefers $(x_h,\overline{x}_r)$ to $(x_k,\overline{x}_r)$ for *some* complement $\overline{x}_r$, he also prefers $(x_h,\overline{x}_t)$ to $(x_k,\overline{x}_t)$ for *any other* complement $\overline{x}_t$. In other words, he prefers x_h to x_k regardless of how the other issues are resolved. (Otherwise, his preferences on issue X cannot be "separated" from the resolution of other issues.) Thus if all voters have separable preferences (as we now assume), it is possible to specify each voter's preferences, and also majority preference, over the positions in a single issue, as well as over the proposals that are now bundles of positions.

Let us subscript the positions on issue X so the x_1 is majority preferred to x_2, x_2 is majority preferred to x_3, and so forth; and likewise for the other issues. Theorem 4 assures us that such a subscripting convention is possible. And Theorem 7 tells us that a position in X can be labeled x_1 if and only if it belongs to the Condorcet set of positions in X; and likewise for the other issues. Thus we can label the positions in a way consistent with the subscripting convention and also so that $(x_1,y_1,\ldots,z_1)$ is the noncooperative voting decision whenever voting complies, on an issue-by-issue basis, with the Condorcet criterion (e.g., in every case we have considered except sincere voting under successive procedure). Now we can state the following:

PROPOSITION 14. The proposal $(x_1,y_1,\ldots,z_1)$ belongs to the Condorcet set of proposals.

By Theorem 7, it is sufficient to show that $(x_1,y_1,\ldots,z_1)$ is a source. Consider any other proposal, say $(x_h,y_k,\ldots,z_j)$. The latter is dominated by any other proposal that differs from it with respect to a single issue and, on this issue, includes the position with the next lowest subscript. For example, $(x_h,y_k,\ldots,z_j)$ is dominated by $(x_h,y_{k-1},\ldots,z_j)$, for the attention of all voters is focused on the single issue Y with respect to which the two proposals differ, and in this respect a majority of voters prefers the former (by the labeling convention). Likewise, $(x_h,y_{k-1},\ldots,z_j)$ is dominated by $(x_h,y_{k-2},\ldots,z_j)$, and so forth. So there is a path from $(x_h,y_1,\ldots,z_j)$ to $(x_h,y_k,\ldots,z_j)$. Repeat this argument for issue X (for example) and we have a path from $(x_1,y_1,\ldots,z_j)$ to $(x_h,y_1,\ldots,z_j)$ and hence to $(x_h,y_k,\ldots,z_j)$. Repeat again for each remaining issue and we have a path from $(x_1,y_1,\ldots,z_1)$ to $(x_h,y_k,\ldots,z_j)$. Thus there is a path from $(x_1,y_1,\ldots,z_1)$ to any other proposal, and $(x_1,y_1,\ldots,z_1)$ is a source.[30]

It follows from the proposition that, if any proposal is undominated, it is

[30] For generally similar statements and proofs, see Kadane (1972), Miller (1975a), and Schwartz (1975), pp. 12–13.

$(x_1, y_1, \ldots, z_1)$. Thus, provided that noncooperative voting complies with the Condorcet criterion, "logrolling" in the sense of full cooperation among voters across issues, can change the voting decision—that is, $(x_1, y_1, \ldots, z_1)$ is dominated—if and only if majority preference over proposals is cyclical, the Condorcet set includes several proposals, and the cooperative "logrolling" decision is indeterminate.[31]

3. Summary and Conclusion

The results presented in the previous section may be somewhat difficult to comprehend and digest. Perhaps their significance can best be highlighted by identifying certain patterns that they exhibit.

At the risk of oversimplification, we may identify several different views of the voting process, characterized by varying levels of sophistication. According to the most "naive" view, voting serves to realize the "will of the majority" and any "reasonable" voting procedure accomplishes this goal—or at least does so if all members of the voting body cast "honest" (that is, sincere) votes. According to a more "practical" view of the voting process, likely to be held by parliamentary practitioners, the decision of a voting body may be affected by which one of several "reasonable" procedures is in use, and especially by the order in which proposals are voted on. This dependence probably comes about because preferences change and different strategic opportunities present themselves, as voting proceeds. But by now most political scientists are aware of the "paradox of voting" and of its destructive implications for any notion of the "will of the majority"; thus according to a more "sophisticated" view of the voting process, the paradox is likely to be intimately connected with the implications of procedure, voting order, and strategy for voting decisions.[32] The remaining problem is, of course, to specify the precise nature of this connection. Such specification has been the principal purpose of this article.

To summarize, we first recognize that majority preference may be cyclical and that an undominated proposal may not exist—that is, there may be no "will of the majority." We therefore focus on the Condorcet set: the minimal set of proposals that collectively possess the property of an undominated proposal. As an initial conjecture, we might guess that the set of possible majority voting decisions includes only proposals in the Condorcet set, and

[31] Cf. Park (1967), Oppenheimer (1972), Bernholz (1973) and (1974), Koehler (1975), Schwartz (1975), and Miller (1975a) and (1976).

[32] For one of the earliest investigations along these lines, see Riker (1958).

that the details of procedure, voting order, and strategy determine only which proposal in the Condorcet set is the decision (and thus are irrelevant if an undominated proposal, i.e., a one-element Condorcet set, exists). This conjecture is confirmed in four of the five "cases" studied (Propositions 1′, 5, 8, and 13)—the exception being sincere voting under successive procedure (Proposition 3), the one case in which the voting decision may depend on the distribution of preferences that underlies the majority preference tournament. A second plausible conjecture is that the set of possible decisions under either procedure includes all proposals in the Condorcet set—that is, that any proposal in the Condorcet set can be the voting decision, given an appropriate order of voting. This conjecture is confirmed in the first three cases studied (Propositions 2, 4, and 6) but fails in the case of sophisticated voting under amendment procedure (Proposition 9).[33]

Common sense suggests that voting processes may be undermined if voters do not vote "honestly" but rather engage in "devious" strategic manipulations; thus common sense suggests that such machinations should be condemned. But condemnation is unlikely to be effective, and it is apparently impossible to devise any proper voting procedure that is immune to strategic manipulation[34] ; thus it is significant that our analysis shows that the common-sense belief must be modified—at least when all voters are equally "devious" or "sophisticated." When some voters engage in strategic manipulations and others do not, no doubt perverse decisions can result—beneficial to the "devious" voters, harmful to some others, and perhaps harmful in some sense (for example, according to the Condorcet criterion) to "social welfare" as well. But we have seen that if all voters are equally "devious" or "sophisticated," the consequences of strategic manipulation are if anything beneficial.

Specifically, under successive procedure, sophisticated voting complies with the Condorcet criterion (Proposition 5), while sincere voting does not (Proposition 3). Put otherwise, the set of possible sophisticated decisions is less inclusive than the set of possible sincere decisions, and this paring down is accomplished in accordance with the Condorcet criterion.

Under amendment procedure, sincere voting already complies with the Condorcet criterion (Proposition 1′), but sophisticated voting pares down the set of possible decisions still further and accomplishes this in accordance with the Pareto criterion (Proposition 10). Moreover, under amendment procedure

[33] The matter remains somewhat unclear in the case of cooperative voting; recall footnote 28.

[34] See in particular Zeckhauser (1973), Gibbard (1973), and Satterthwaite (1975).

and any given voting order, if the sincere and sophisticated decisions differ, a majority of voters prefers the latter to the former (Proposition 12).[35]

Finally, if full strategic cooperation is possible, the set of possible voting decisions is pared down at least to the Condorcet set under any majoritarian voting procedure whatsoever (Proposition 13)—while noncooperative voting under many procedures may be far more arbitrary.

It also appears that, under amendment and successive procedure, the sophisticated voting decision is less order-dependent than the sincere decision. And of course under any procedure the cooperative voting decision is entirely independent of order.

In several significant respects, therefore, this analysis has modified the common-sense condemnation of strategic voting. And the summary just presented also indicates that, whether voting is sincere or sophisticated, amendment procedure is superior to successive procedure. Finally, the two remaining results (Propositions 7 and 11) are consistent with previous claims regarding the effects of voting order under sincere and sophisticated voting.

Having summarized the principal substantive results in this article, I will conclude on the methodological point. Possibly the graph-theoretical framework of this article will strike some readers as so much window-dressing—elegant but unnecessary. Several of the results can be (and have been) obtained more easily and directly by other methods. But I can assure the skeptical reader that other results presented here cannot be so readily obtained—nor so coherently integrated—without the aid of the logic provided by graph theory. I hope, therefore, that this article, as well as adding to the stock of voting theory results, may serve also to demonstrate more generally the utility of graph-theoretical reasoning in political inquiry.

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[35] Of course, this is not as clearly beneficial—even from a majoritarian point of view—as it may appear to be at first blush, for both proposals belong to a common cycle of majority preference (by Propositions 1 and 8 and Theorem 6).

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