The Covering Relation in Tournaments: Two Corrections

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An earlier article on the "uncovered set" contained errors in two proofs, which are corrected here. Most important, it is V^{**u}, not the uncovered set V^{**} itself, which always contains a complete cycle if the core is empty. Other more substantive results presented in the earlier article are not affected by these corrections.

In my article on the "uncovered set" in tournaments and majority voting that appeared in this journal (Miller, 1980), Lemma 4 and Theorem 3 are incorrect as they stand.[1] Correction of these errors does not disturb in any way the other, more substantive results, but it is appropriate to bring these errors to the attention of readers of the article.

With respect to Lemma 4, what is true is that statements (*a*), (*c*), (*d*), and (*e*) are equivalent and that each of these implies (*b*). However, (*b*) does not imply the others unless it is given that x dominates y.

Examination of the alleged proof that (*b*) implies (*c*) shows that the argument is actually that (*a*) and (*b*) together—or in any case, $x \rightarrow y$ and (*b*) together—imply (*c*)[2]. Finally, it is "obvious" that (*b*) and (*e*) are *not* equivalent, for (*e*) implies $x \rightarrow y$, but (*b*) does not.

Thus Lemma 4 and its proof should be replaced by the following:

LEMMA 4A: In a tournament, if x covers y, then $F(x) \cap D(y) = \emptyset$.

This is proved in the manner of (*a*) implying (*b*) in the original Lemma 4.

LEMMA 4B: In a tournment, if x dominates y and $F(x) \cap D(y) = \emptyset$, then x covers y.

Suppose x dominates y and $F(x) \cap D(y) = \emptyset$. Then $D(y) \subseteq V - F(x) = D(x) \cup \{x\}$; also $x \notin D(y)$, so $D(y) \subseteq D(x)$, which in a tournament means $D(y) \subset D(x)$. Thus x covers y.

LEMMA 4C: In a tournament, the following statements are equivalent:

(*a*) x covers y, that is, $D(y) \subset D(x)$;
(*c*) $F(x) \subset F(y)$;
(*d*) $D(x) \cup F(y) =$ *V*hr ; and
(*e*) $x \notin R_2(y)$.

Suppose $D(y) \subset D(x)$. Then the complement in V of $D(x)$ is properly contained in the complement in V of $D(y)$, that is, $F(x) \cup \{x\} \subset F(y) \cup \{y\}$. Since x covers y, x dominates y and $y \notin F(x)$. Thus, $F(x) \cup \{x\} \subset F(y)$ and $F(x) \subset F(y)$. So (*a*) implies (*c*). And (*c*) implies (*d*) implies (*a*) as in the original Lemma 4. Thus (*a*), (*c*), and (*d*) are equivalent. Finally, (*a*) implies $x \rightarrow y$ (Lemma 3); (*a*) also implies $F(x) \cap D(y) = \emptyset$ (Lemma 4A), so there is no $z \epsilon V$ such that $y \rightarrow z \rightarrow x$. These two implications together are equivalent to saying $x \notin R_2(y)$.

Lemma 4 was used three times in the subsequent development of my article. In the first two instances (in demonstrating Lemma 6 on p. 74 and Lemma 9 on p. 88), the equivalence of (*a*) and (*e*) is invoked—properly according to the revised Lemma 4C. In the final instance (in demonstrating Theorem 7), the alleged equivalence of (*a*) and (*b*) is invoked (on p. 91). The proof of Theorem 7 should be corrected by replacing the sentence referring to Lemma 4 with the following: And by Lemmas 4A and 4B, the requirement that $F(x) \cap D(y)$ be empty is equivalent—given the initial supposition that x is majority preferred to y—to the requirement that x cover y.

Theorem 3 is wrong, as demonstrated by the counter example in Figure 1, in which $V^* = V$ and $V^{**} = \{v_1, v_2, v_3, v_4\}$ but there is no cycle including precisely these points. The argument on pp. 76–77 is a valid proof for a slightly different theorem,[3] namely the following:

THEOREM: 3′: In a tournament, if V^{***} is empty, there is a cycle including precisely the points in V^{**u}.

The set V^{**u} is defined on p. 93 of the article. The argument on pp. 76–77 applies to V^{**u}, not V^{**}, because—when the argument is repeatedly applied in the manner called for on the bottom of p. 77—it is applied to nested subtournaments each considered anew and, *within these subtournaments,* points may be covered that were originally uncovered. Therefore, the argument terminates only when we reach a subtournament X such that, for any pair of points x and y in X, either $x \rightarrow y$ or there is some z *belonging to* X such that $x \rightarrow z \rightarrow y$; that is, it terminates only when we reach the subtournament including precisely the points in V^{**u}. (Thus, for the example in Figure 1, the argument might first exclude the covered point v_5 and would then show that there is a cycle including precisely the remaining points. But in the subtournament including precisely the remaining points, previously uncovered v_4 is now covered by v_3, and the argument will not terminate until we eliminate v_4, as well as v_6 and v_7, leaving us with a cycle including precisely the points in $\{v_1, v_2, v_3\} = V^{**u}$.)

Theorem 3′ does not require such a cumbersome proof, however,

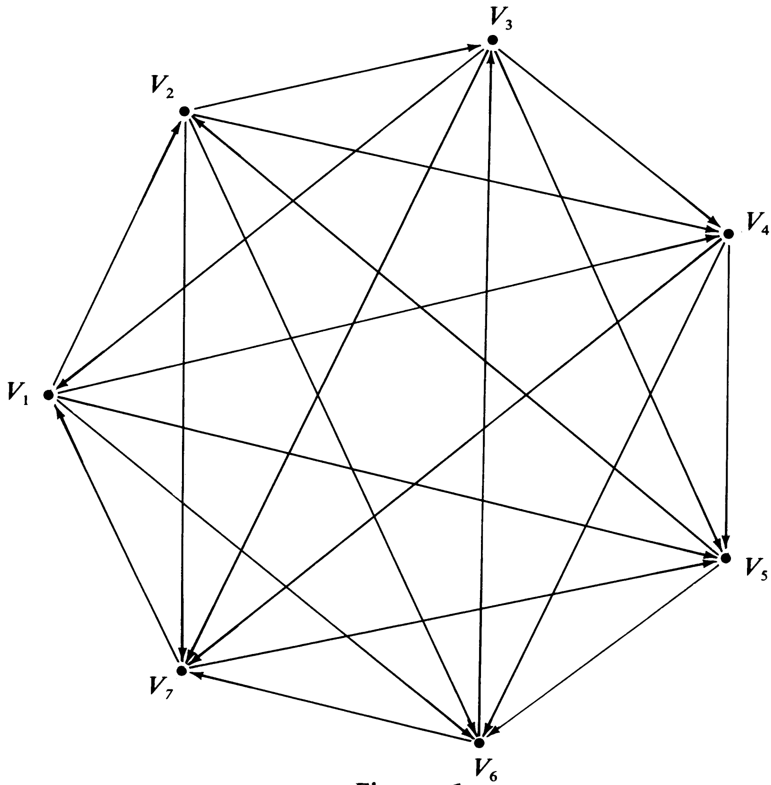

Figure 1

A Tournament in Which the Uncovered Set Does Not Have a Complete Cycle

We need consider only the tournament consisting of the points in V^{**u} and the arrows between them. In this tournament, every point is reachable from every other point (by a path of domination of no more than two steps). Thus, this tournament is strong and, by established graph-theoretical results (Harary, Norman, and Cartwright, 1965, pp. 305–306), has a complete cycle.

Theorem 3 is invoked on p. 93 to support the contention that parties never reach a (pure strategy) electoral equilibrium if V^{***} is empty, But it is invoked specifically with reference to V^{***}, so Theorem 3' does just as well.

Thus, all subsequent results in the article stand undisturbed by these two corrections.[4] Moreover, correction of the theorem strongly supports the conjecture in footnote 11.

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REFERENCES

Harary, Frank, Robert Z. Norman, and Dorwin Cartwright. 1965. *Structural models: An introduction to the theory of directed graphs.* New York: Wiley

Miller, Nicholas R. 1980. A new solution set for tournaments and majority voting: Further graph-theoretical approaches to the theory of voting. *American Journal of Political Science,* 24 (February): 68–96.

[1] I am indebted to Arthur Pittenger of the Department of Mathematics at the University of Maryland Baltimore County for bringing these errors to my attention.

[2] There was also a typographical error in the manuscript, which should have read ". . . x dominates y, i.e., $y \notin F(x)$. . . ."

[3] There was another typographical error, however, introduced in production after page proof stage. The ninth line on p. 77 appears also as the second line, which should instead read ". . . . (in modulo m^*) such that $v_{h-1} \rightarrow v_g$; then there is a cycle $v_{h-1} \rightarrow v_g \rightarrow$ . . . ."

[4] There was, however, one additional manuscript typographical error. Footnote 16 (p. 94) should generalize that "x covers y" to $D(y)$ $D(x)$ and $F(x)$ $F(y)$ *or* $D(y)$ $D(x)$ and $F(x)$ $F(y)$.