THE GEOMETRY OF THREE-CANDIDATE RUNOFF ELECTIONS

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Overview

- A recent unpublished paper by Conal Duddy uses a simple geometric technique
  - to demonstrate how runoff elections are subject to monotonicity problems, and
  - to propose a compromise between runoff and plurality voting that avoids these problems.
- The purpose of this presentation is to publicize Duddy's paper and to draw connections between it and my own (non-geometric) work on monotonicity failure under Instant Runoff Voting (IRV), which in turn draws on earlier work by Lepelley et al.

Conal Duddy, "Geometry of Runoff Elections"

Nicholas R. Miller, "Monotonicity Failure in IRV Elections with Three Candidates," presented at the World Congress of the Public Choice Societies, Miami, 2012 (with subsequent revisions)

Dominique Lepelley et al., "The Likelihood of Monotonicity Paradoxes in Runoff Elections," ***Mathematical Social Sciences*****, 2016**

Overview (cont.)

- On an IRV ballot:
  - voters rank the candidates in order of preference.
  - If one candidate has a majority of first preferences, that candidate is the outright winner and is elected.
  - Otherwise, the candidate with the fewest first preferences is eliminated and his or her ballots are transferred to other candidates on the basis of second preferences.
  - This process is repeated until one candidate is supported by a majority of ballots and is elected.

- Here we consider only three-candidate contests,
  - so IRV is limited to a single 'instant runoff' in the event none of three candidates is supported by a majority of first preferences,
  - in which case IRV is formally equivalent to plurality voting with a runoff between the top two candidate.
  - Thus we can refer simply to *runoff elections.*

- A striking feature of runoff elections that they are vulnerable to monotonicity failure: getting more first preference votes may result in defeat for a candidate who would otherwise win and getting fewer votes may result in victory for a candidate who otherwise lose.

Preliminaries

- The three candidates are *X*, *Y*, and *Z*, and *n* voters cast ballots that rank them in order of preference.
  - *x* is the number of ballots that rank candidate X first;
  - x_y is the number of ballots that rank *X* first and *Y* second (and *Z* third);
  - and likewise for other candidates.

- A *plurality profile* specifies the number first-preference ballots for each candidate, i.e., (*x*, *y*, *z*).
  - A *Majority Winner* is a candidate with a majority of first-preference votes.
  - The *Plurality Winner* (PW) is the candidate with the most first- preference votes.
  - The *Plurality Runner-Up* (P2) is the candidate with the second-most first-preference votes.
  - The *Plurality Loser* (PL) is the candidate with the fewest first-preference votes.
- A profile is *competitive* if the plurality loser is supported by the first preferences of more than 25% of the ballots,
  - which implies that there is no majority winner.

Preliminaries (cont.)

- A *ballot profile* specifies the number of ballots with each ranking of the candidates, i.e., (x_y, x_z, y_x, y_z, z_x, z_y).
- Thus each plurality profile includes multiple possible ballot profiles (with different second and third preferences).
- A ballot profile determines the *majority preference relation* between pairs of candidates.
  - A *Condorcet Winner* (CW) is a candidate who is ranked higher on a majority of ballots than both other candidates – put otherwise, who *beats* both other candidates.
  - A *Condorcet Loser* (CL) is a candidate who is beaten by both other candidates.
  - But there may be neither a CW or CL but rather a *Condorcet cycle*.
  - Plurality and Condorcet status are completely independent,
    - in particular, a plurality loser may be a Condorcet winner.
- A ballot profile is *critical* if the IRV winner is not the plurality winner,
  - i.e., the plurality runner-up beats the plurality winner.

Three Types of Monotonicity Failure in Three-Candidate Runoff Elections

- A ballot profile *B* is vulnerable to *Upward Monotonicity Failure* (UMF) if *X* wins under profile *B* but there is some other profile *B*′
  - (i) that differs from *B* only in that some voters rank *X* higher in *B*′ than in *B,* and
  - (ii) under which *X* loses.

- An example:

Original Profile B

35	10	25	30
X	Y	Y	Z
Y	X	Z	X
Z	Z	X	Y

Revised (Companion) Profile B'

35	10	25	30
X	X	Y	Z
Y	Y	Z	X
Z	Z	X	Y

Three Types of Monotonicity Failure (cont.)

- A ballot profile *B* is vulnerable to *Downward Monotonicity Failure* (DMF) if *X* loses under *B* but there is some other profile *B*′
  - (i) that differs from *B* only in that some voters rank *X* lower in *B*′ than in *B,* and
  - (ii) under which *X* wins.

- An example:

Original Profile B

32	17	30	20
X	*X*	*Y*	*Z*
Y	*Z*	*X*	*Y*
Z	*Y*	*Z*	*X*

Revised (Companion) Profile B'

32	17	30	20
X	*Z*	*Y*	*Z*
Y	*X*	*X*	*Y*
Z	*Y*	*Z*	*X*

Three Types of Monotonicity Failure (cont.)

- *A ballot profile B is vulnerable to Double Monotonicity Failure* (2MF) if B is vulnerable to both UMF and DMF Failure.

- An example:

Original Profile

38	32	30
Y	*X*	*Z*
Z	*Y*	*X*
X	*Z*	*Y*

Companion Profile 1 (UMF)

31	9	32	30
Y	*X*	*X*	*Z*
Z	*Y*	*Y*	*X*
X	*Z*	*Z*	*Y*

Companion Profile 2 (DMF)

38	3	32	30
Y	*Z*	*X*	*Z*
Z	*Y*	*Y*	*X*
X	*X*	*Z*	*Y*

Conditions for Monotonicity Failure

Given that *Z* is the plurality loser under profile *B*, two conditions are necessary and jointly sufficient to make *B* vulnerable to (upward or downward) monotonicity failure in the event that *X* is moved up or down in some ballot orderings, creating profile *B*'.

Condition 1 pertains to the *runoff pair* and requires that the ballot changes that produce *B*′ from *B* must deprive *Y* of enough first preferences (for UMF), or give *Z* enough additional first preferences (for DMF), that the runoff that had been between *X* and *Y* is now between *X* and *Z*.

Condition 2 condition pertains to the *runoff outcome* and requires that *X*, which won (for UMF) or lost (for DMF) the runoff against *Y* under *B* must lose (for UMF) or win (for DMF) the runoff against *Z* under *B*′.

Three-Candidate Plurality Profiles and the Election Triangle

Every point in the triangle (including the vertices and edges) corresponds to a plurality profile.

Of course the reverse is not true (unless the number of voters is infinite)

Donald Saari, many works, but earlier:

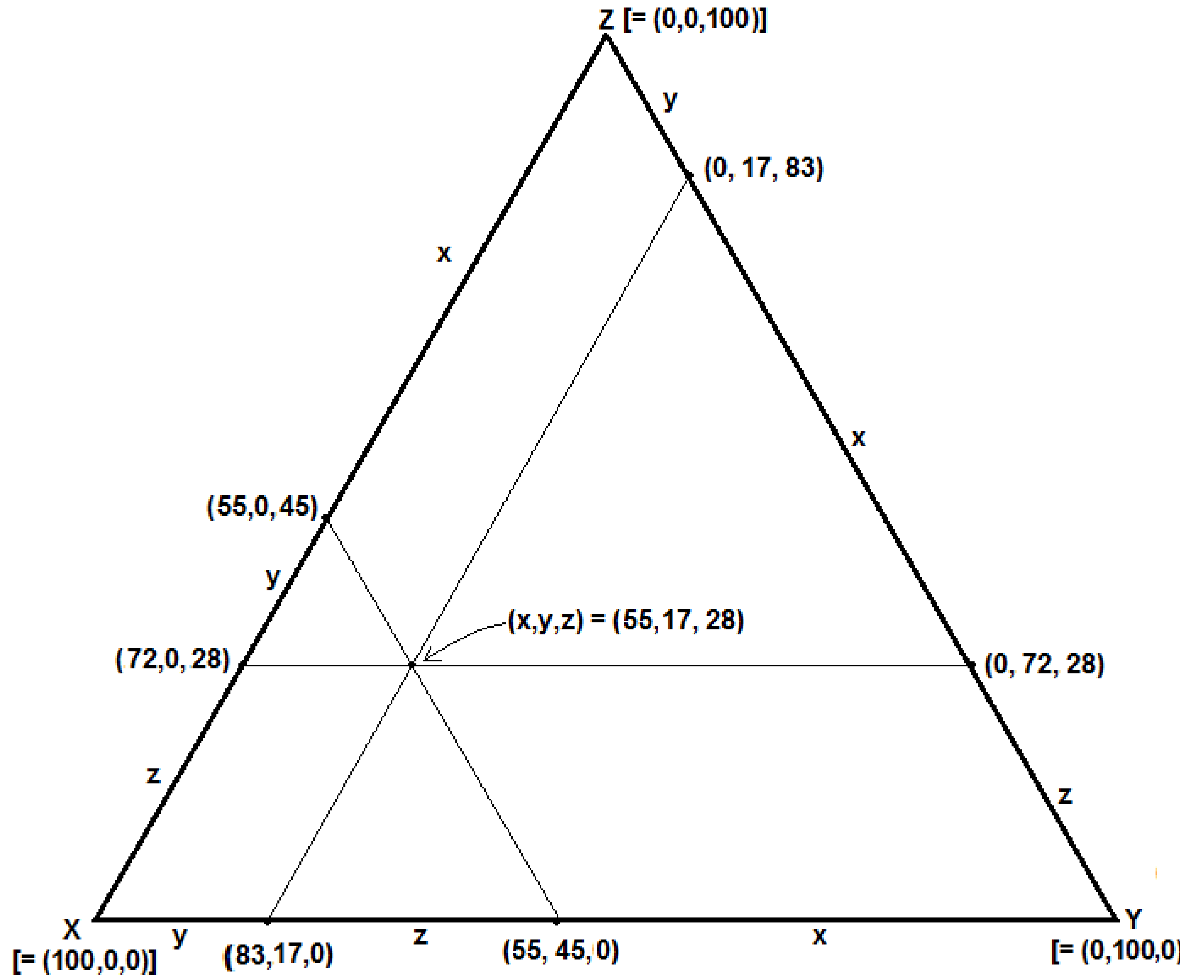
W. Shelton, "Majorities and Pluralities in Elections," ***The American Statistician*****, Dec. 1972**

G. Upton, "Diagrammatic Representation of Three-Party Contests," ***Political Studies*****, 1976**

G. Gudgin and P. Taylor, ***Seats Votes, and Spatial Organization of Elections*** **(1979), Chapter 5**

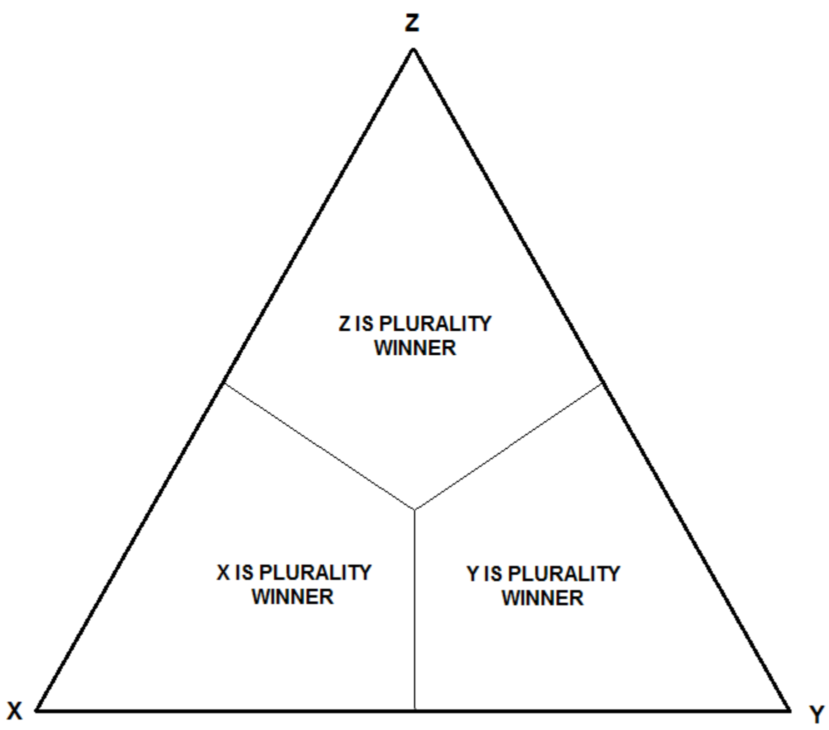
B. Grofman et al., "Comparing and Contrasting Two Graphical Tools for Displaying Patterns of Multiparty Competition," ***Party Politics*****, 2004**

A THREE-CANDIDATE ELECTION TRIANGLE AND A PLURALITY PROFILE

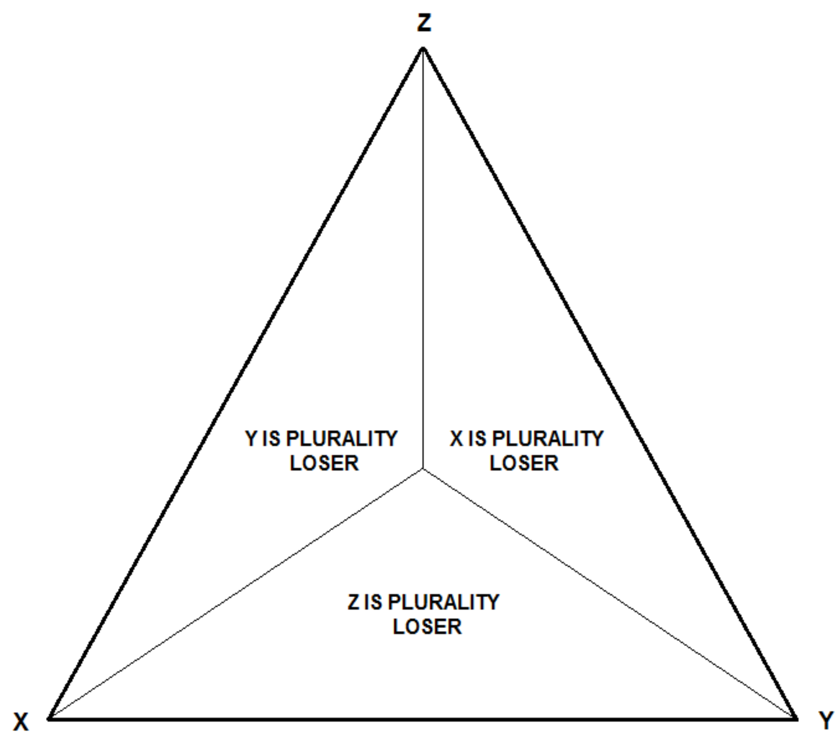


Plurality Winner and Loser Regions

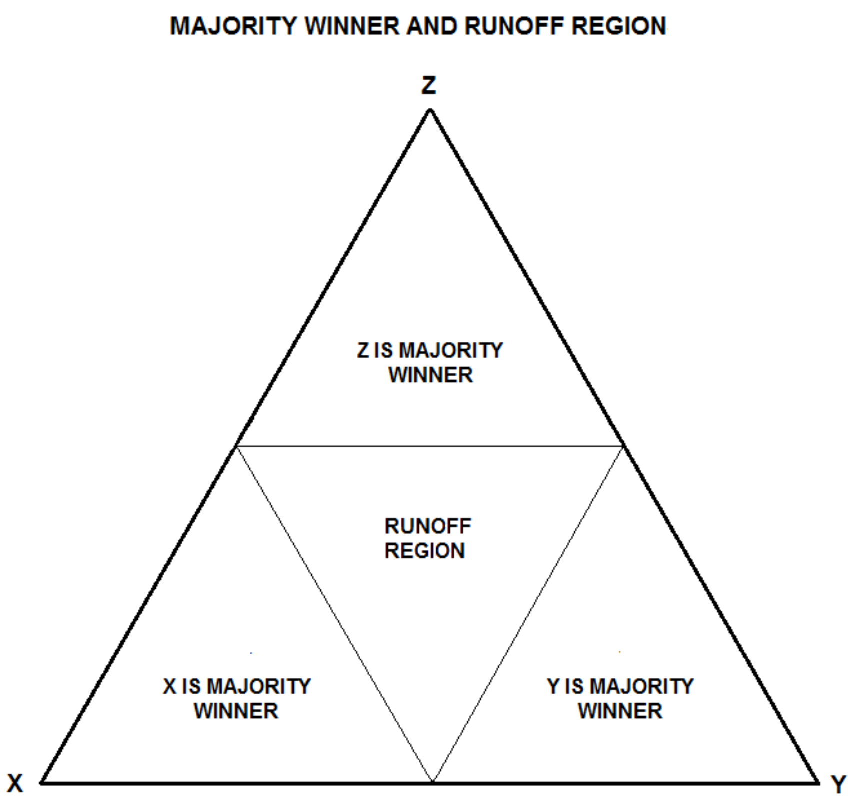
PLURALITY WINNER REGIONS

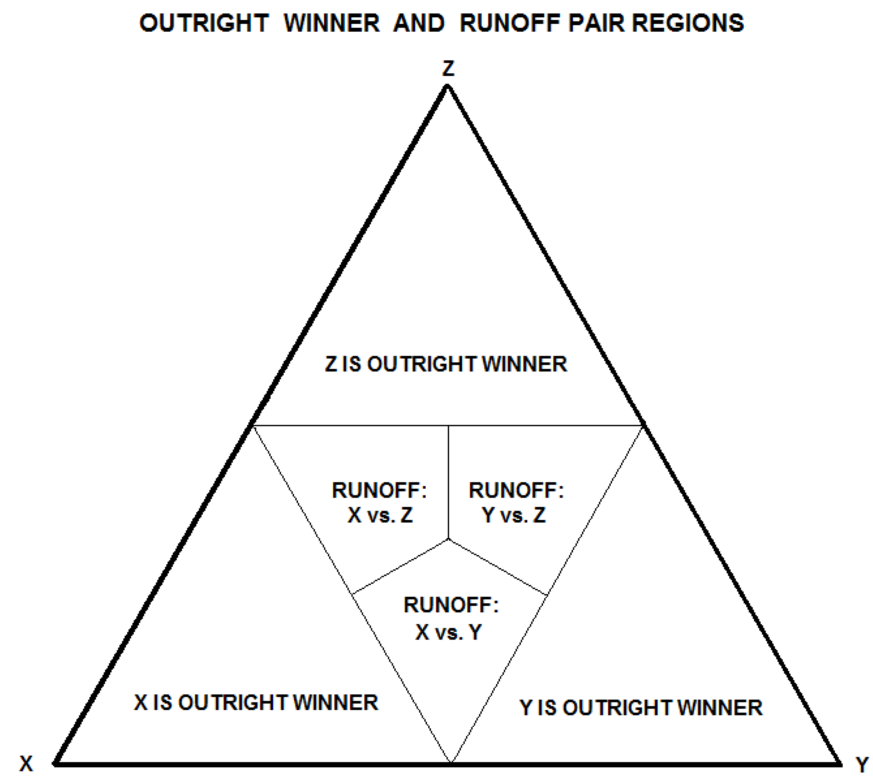


PLURALITY LOSER REGIONS



IRV and the Election Triangle

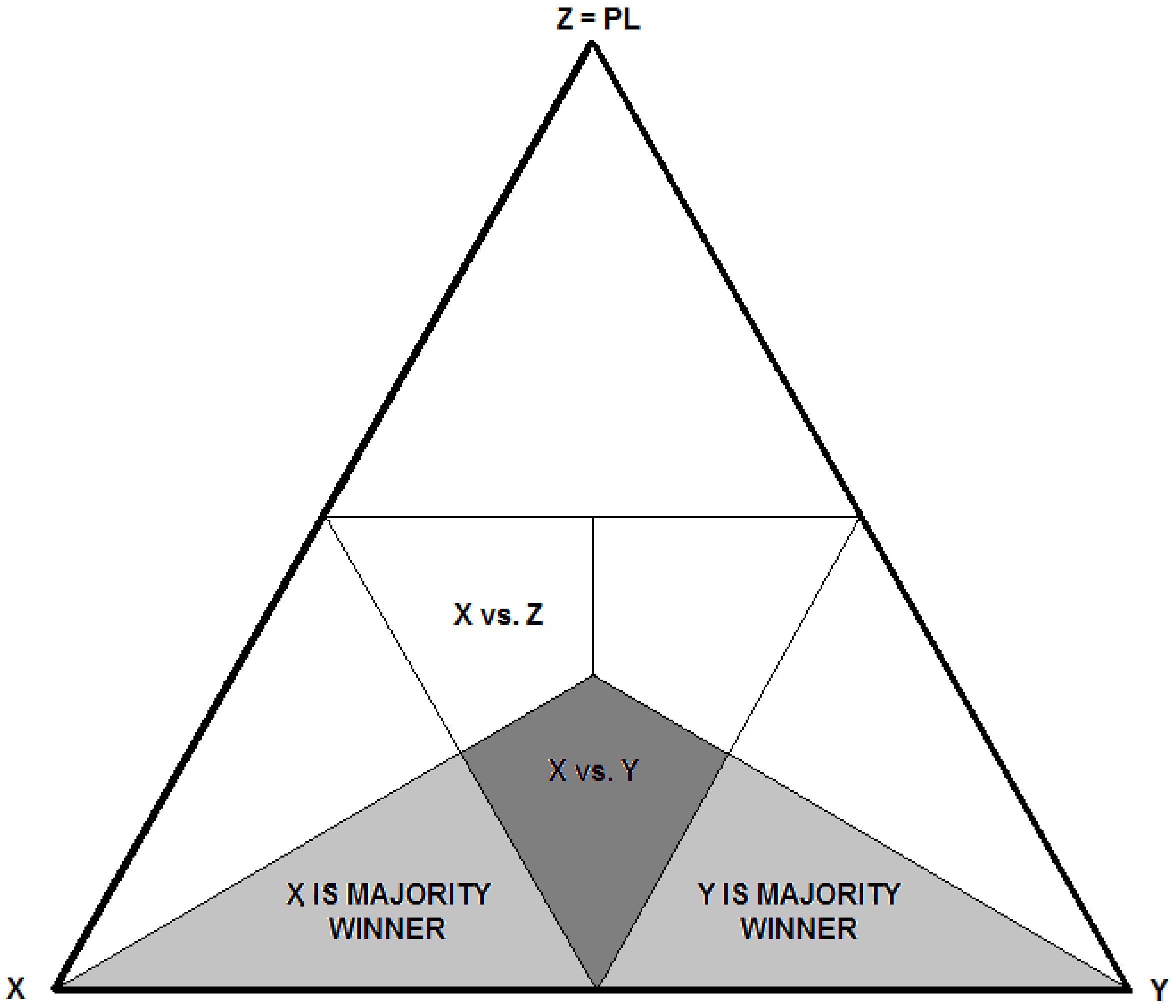




Conditions for Monotonicity Failure

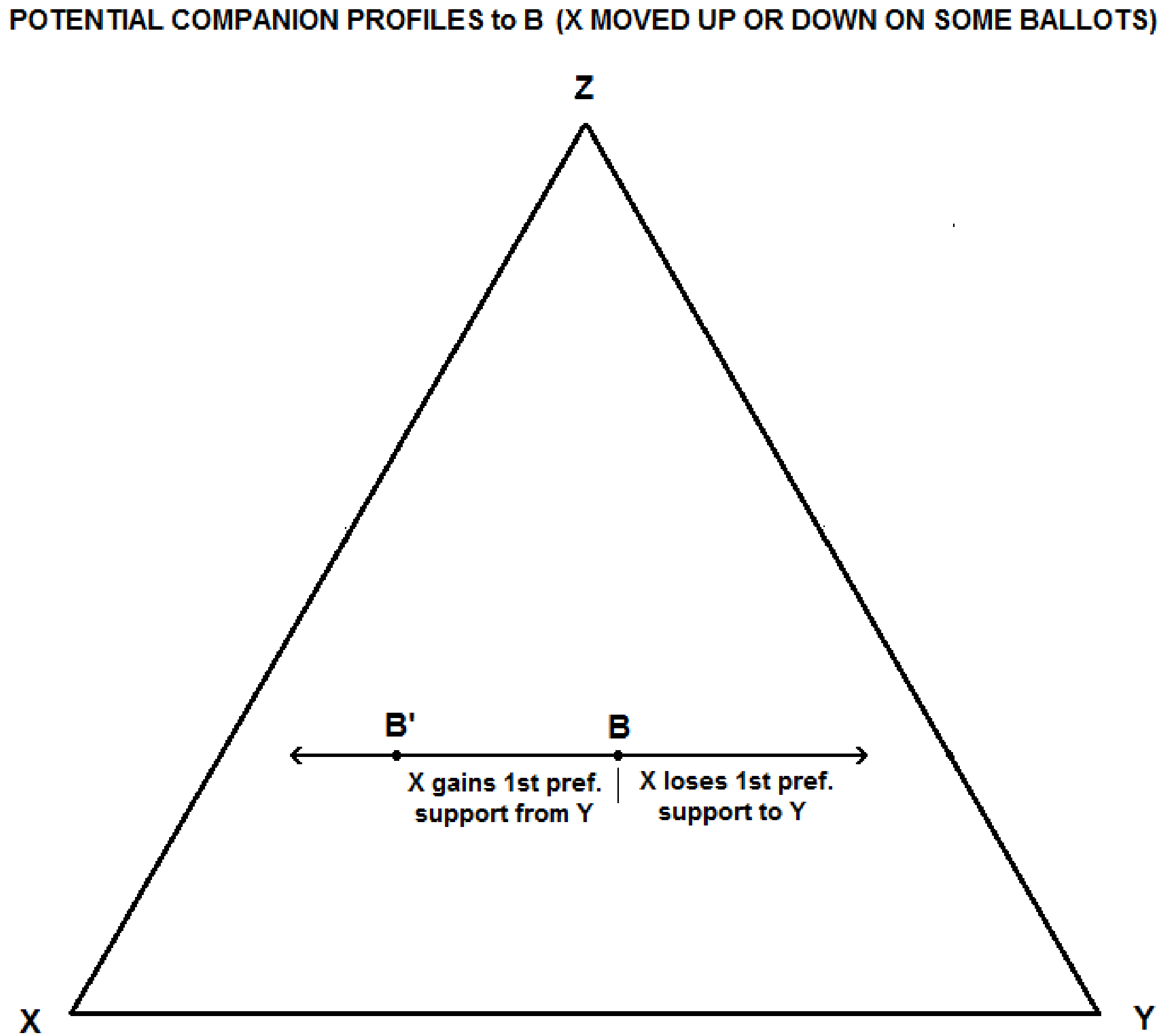
In considering UMF and DMF, in a vulnerable profile *B*:

- *Z* is the plurality loser;
- *X* gains first preferences from *Y* or loses them to *Z*
- so that runoff pair switches from *X* vs. *Y* to *X* vs. *Z.*



Companion Profiles and the Election Triangle

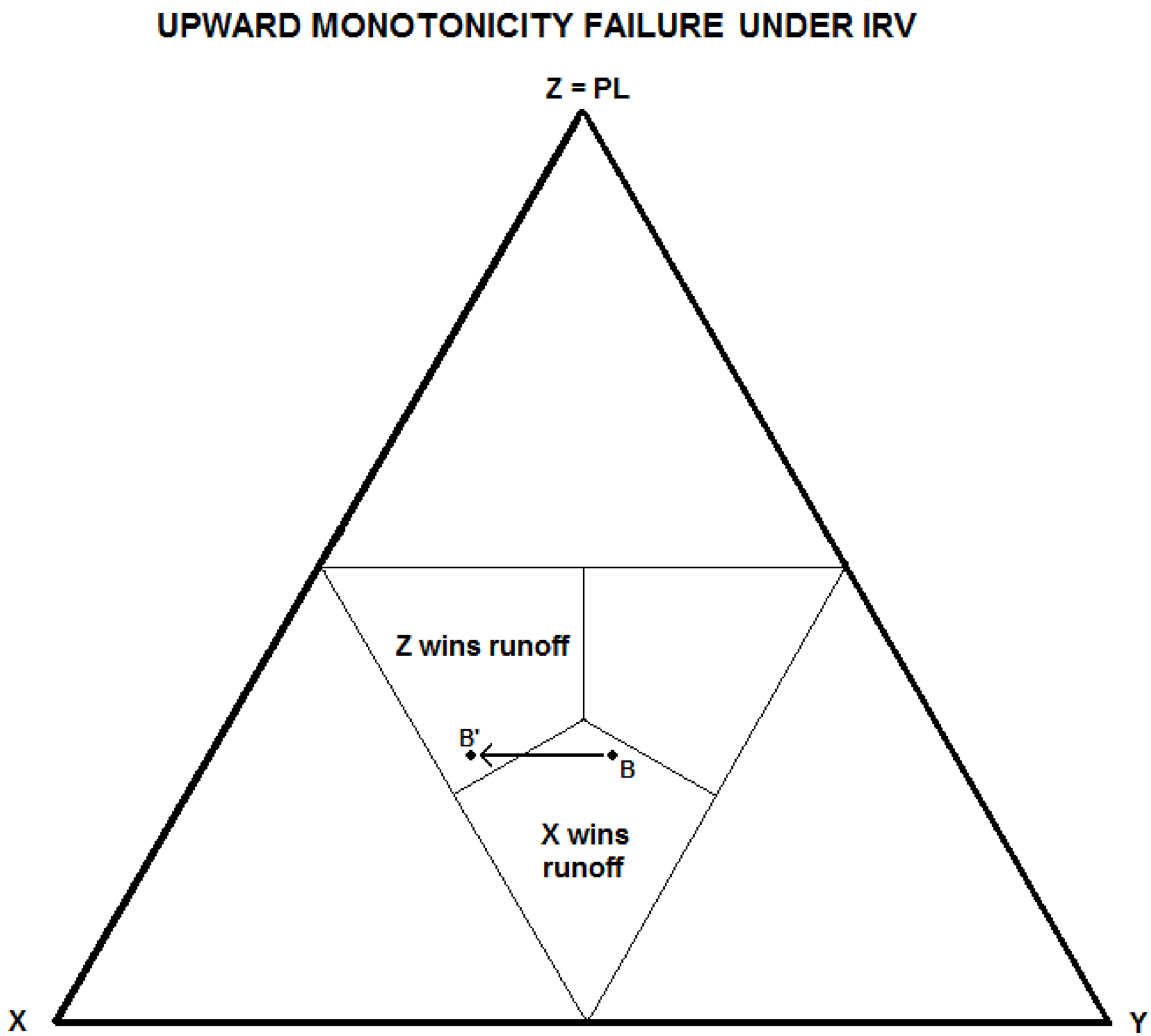
If starting from profile *B*, *X* gains/loses first-preference support from/to *Y*, the resulting profiles lie on the straight line through *B* that is horizontal to the X-Y edge of the triangle.



A Geometrical Example of UMF

Given profile *B*, suppose *X* is the IRV winner and *Z* is the plurality loser and *Z* beats *X*, then the result of *X* gaining 1st-preference support from *Y* may be that *X* becomes the IRV loser.

However, the question arises of whether *Z* still beats *X*, after *X* gains enough 1st-preference support from *Y* to make *Y* the PL; it can be shown algebraically that it must do, provided z > n/4.



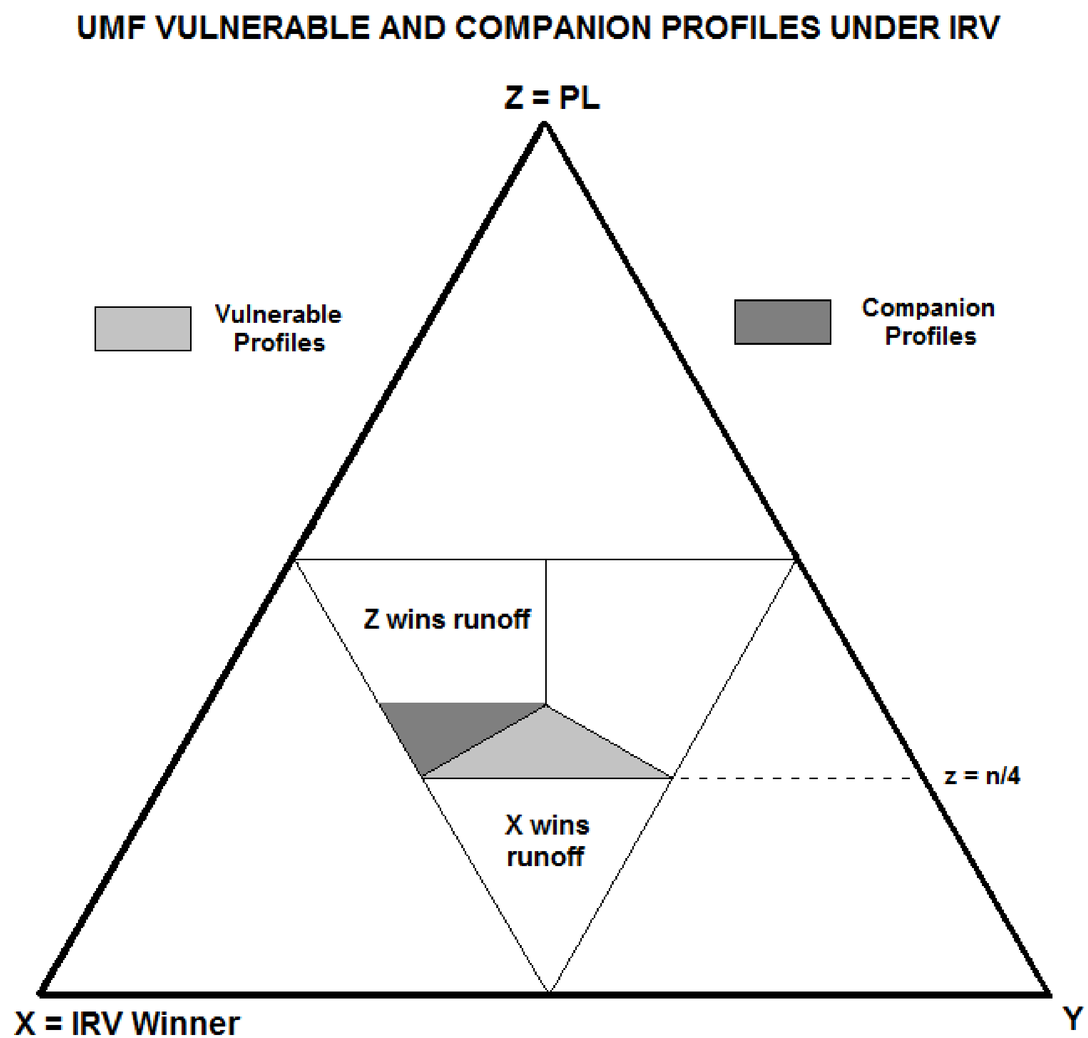
Conditions for UMF

Proposition 1. A three-candidate ballot profile *B* in which *X* is the IRV winner and *Z* is the plurality loser is vulnerable to UMF if and only if:

Condition 1U: *B* is competitive, i.e., $z > n/4$; and

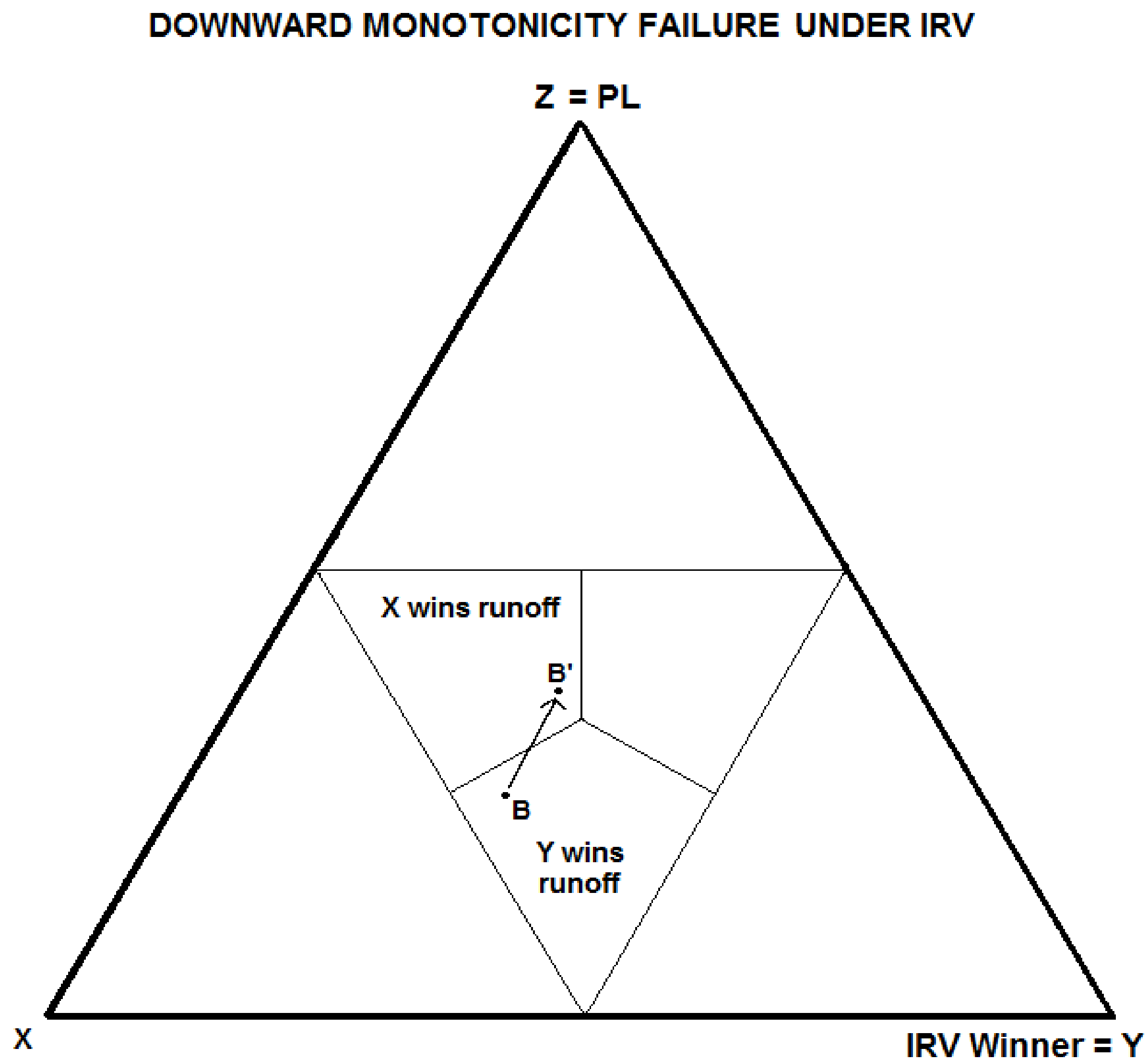
Condition 2U: the plurality loser beats the IRV winner.

Obervation: Both critical and non-critical profiles may be vulnerable to UMF.



A Geometrical Example of DMF

Given profile *B*, suppose *Y* is the IRV winner and *Z* is the plurality loser and *X* beats *Z*, then the result of *X* losing first-preference support to *Z* may be that *X* becomes the IRV loser.



Conditions for DMF

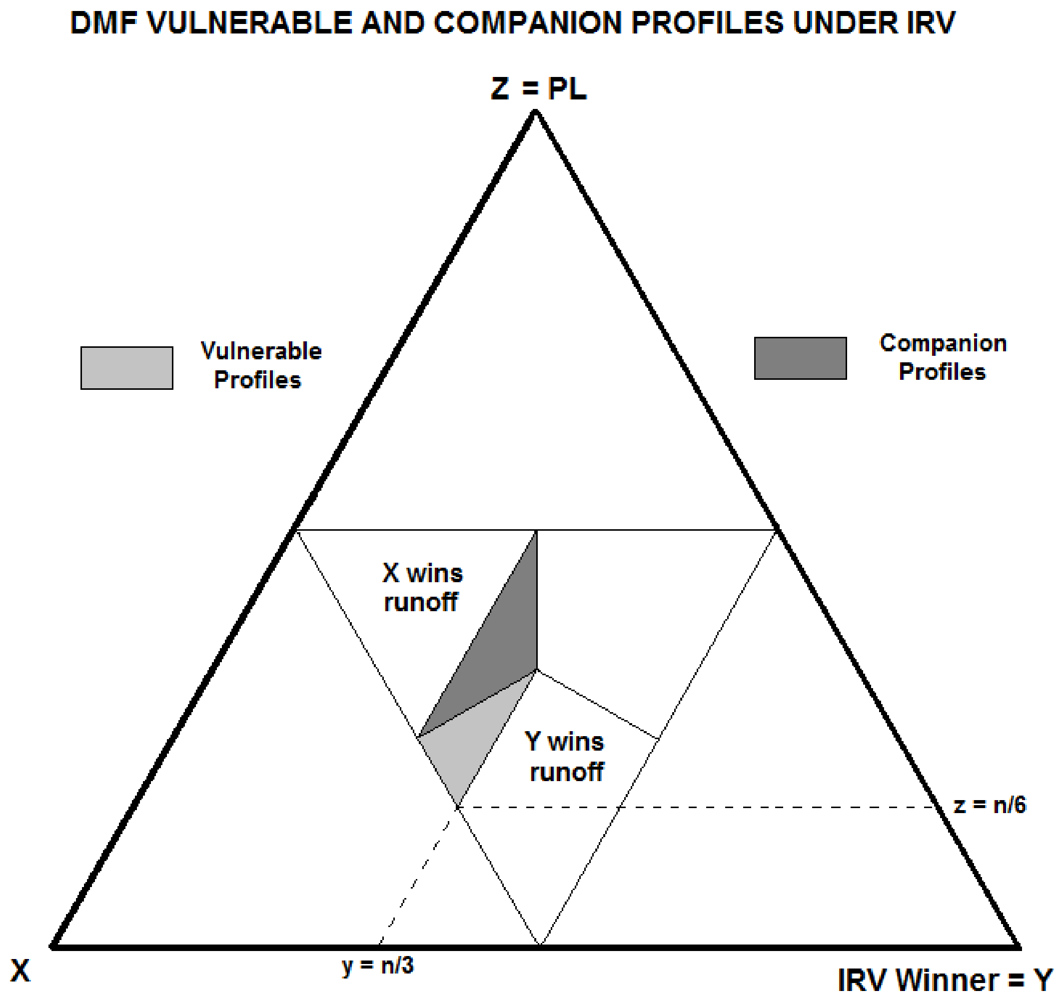
Corollary 2.1. A three-candidate ballot profile *B* is vulnerable to DMF under IRV only if

(1) the IRV winner is supported by the first preferences of fewer than one-third of the voters; and

(2) the plurality winner beats the plurality loser.

Corollary 2.1.1. A three-candidate ballot profile *B* is vulnerable to DMF under IRV only if the plurality is supported is supported by the first preferences of more than one-sixth of the ballots.

Observation: only critical profiles may be vulnerable to DMF



Conditions for DMF (cont.)

However, these necessary conditions are not sufficient for DMF. Additional second-preference conditions must hold.

Proposition 2. A three-candidate ballot profile B in which Y is the IRV winner and Z is the plurality loser is vulnerable to DMF if and only if:

Condition 1D:

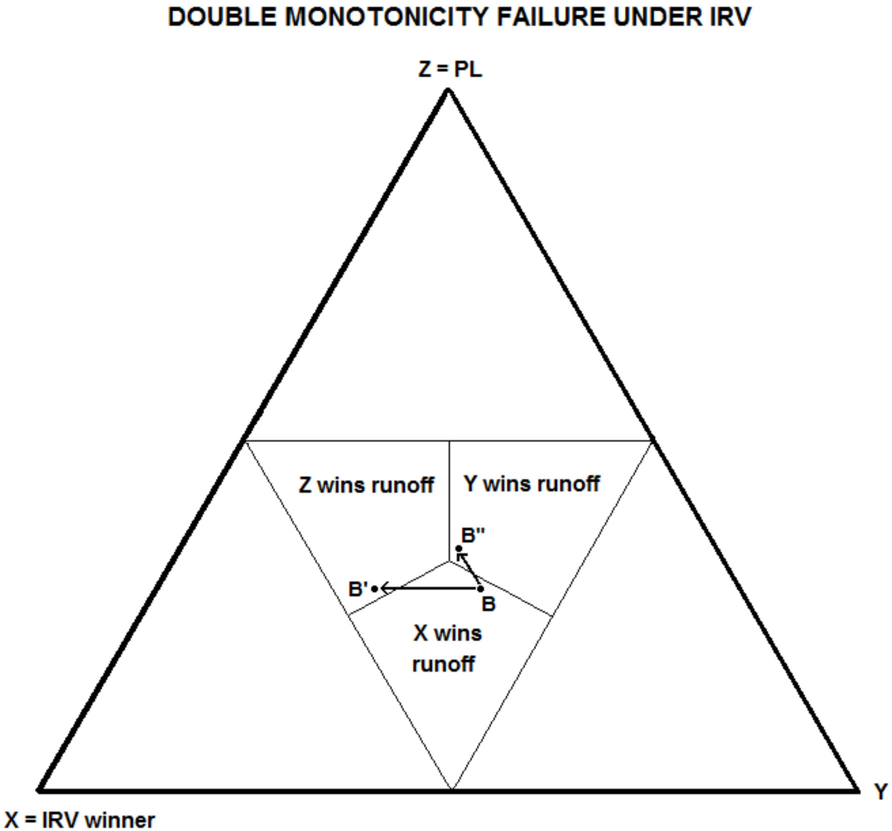
(a) the IRV winner has the first-preference support of less than one-third of the voters, i.e., $y < n/3$, and

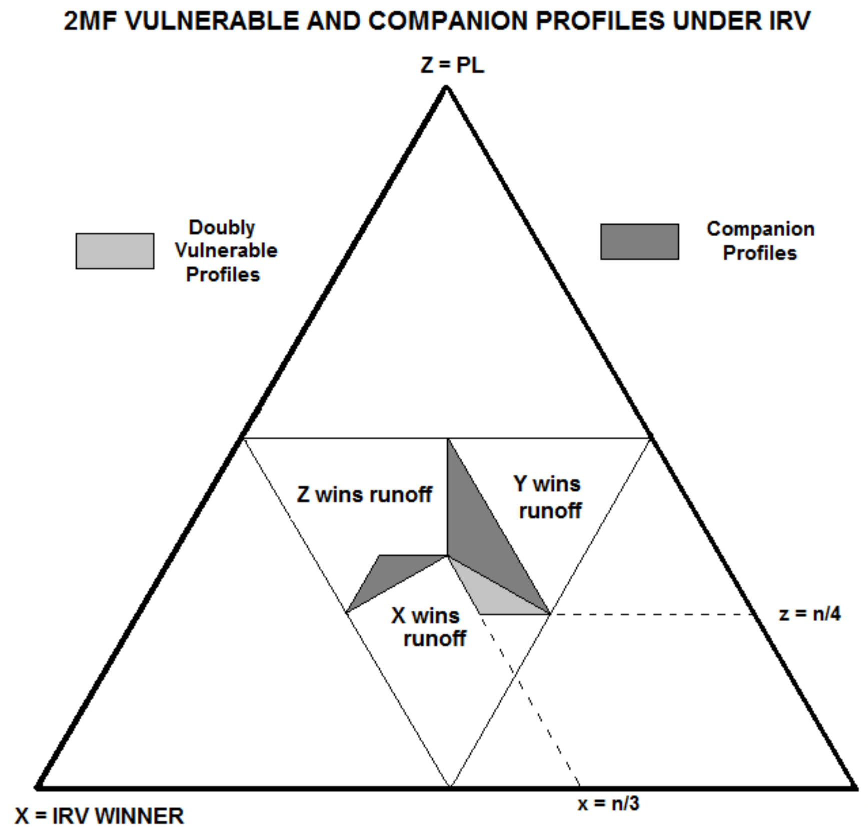
(b) $x_z > y - z$; and

Condition 2D: $y_z < n/2 - y$.

Thus not every ballot profile in the lightly shaded region is vulnerable to DMF.

Double Monotonicity Failure





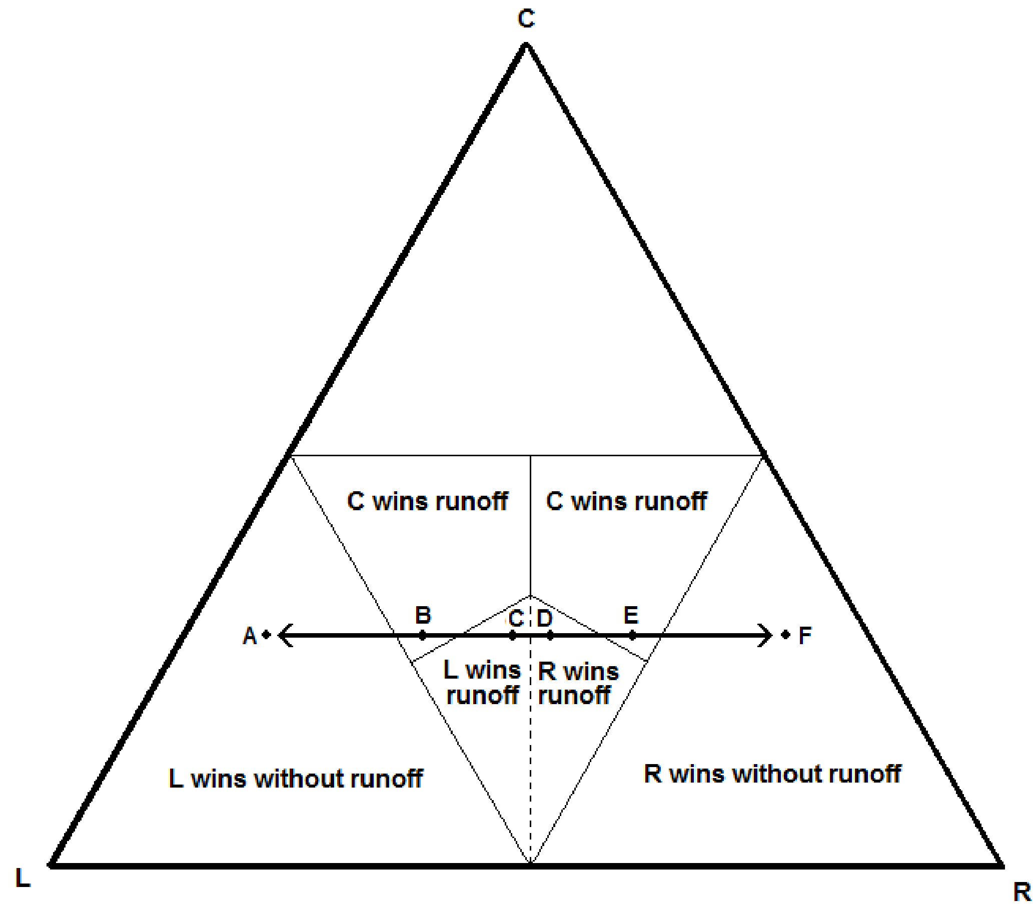
- Profile *B* is vulnerable to 2MF only if it is competitive, critical, and cyclical.

“Left-Right” Monotonicity Failure

Ordering	*L C R*	*C L R*	*C R L*	*R C L*	*Runoff*	*IRV Winner*
Profile A	13	6	1	5	—	L
Profile B	12	5	2	6	L vs. C	C
Profile C	10	4	3	8	L vs. R	L
Profile D	8	3	4	10	L vs. R	R
Profile E	6	2	5	12	C vs. R	C
Profile F	5	1	6	13	—	R

For more elaborate examples, see Ka-Ping Yee’s
Voting Simulation Visualizations
http://zesty.ca/voting/sim/

LEFT-RIGHT MONOTONICITY FAILURE UNDER IRV

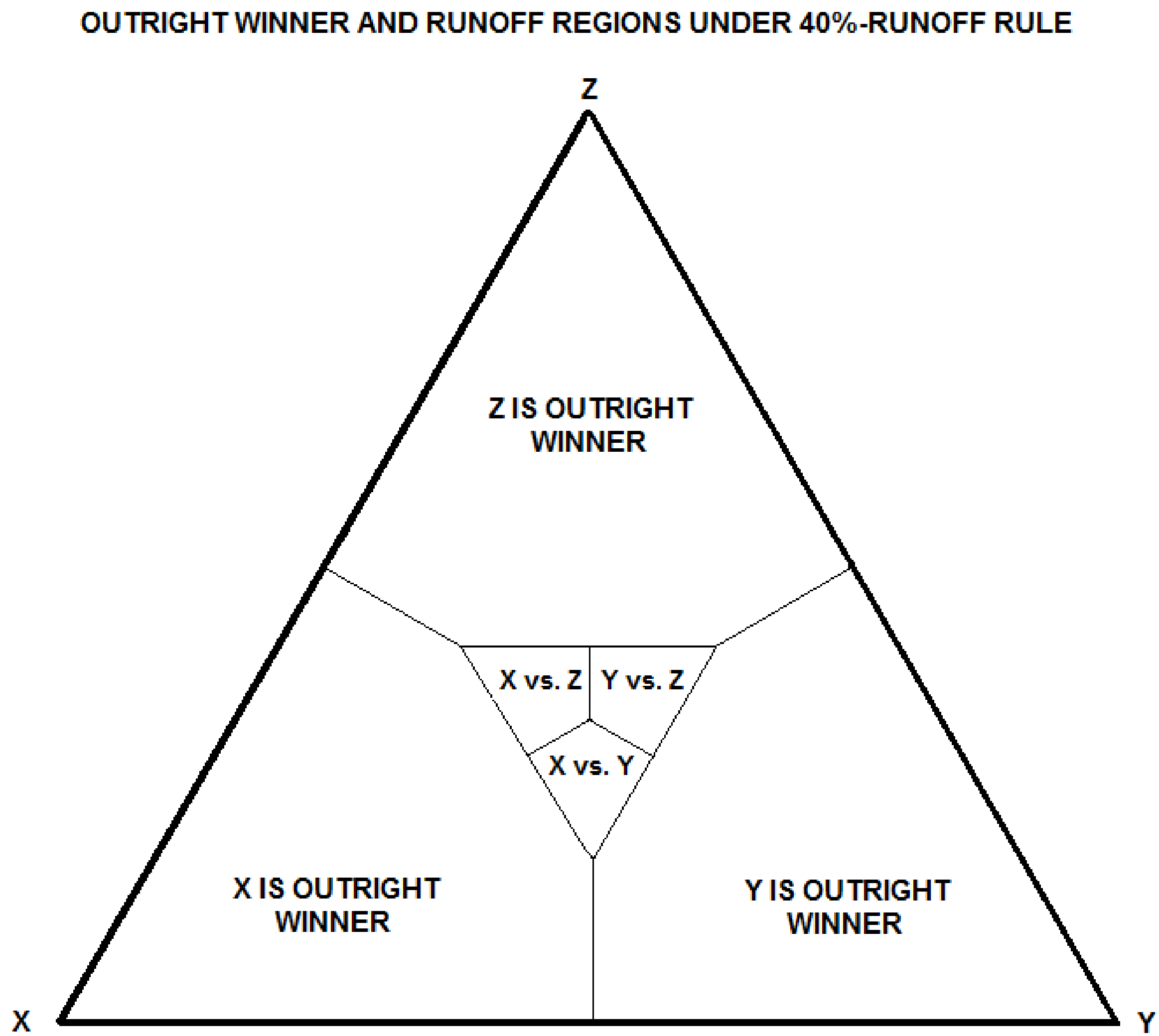


Other Runoff Rules

- Some variants of plurality runoff voting specify less demanding thresholds for outright election than a majority of first preferences.
- For example, one runoff rule elects the plurality winner outright provided the candidate gets at least 40% of the (first-preference) vote, e.g.,
  - the constitutional amendment for direct popular election of the U.S. president passed by the House of Representatives in 1969; and
  - actual rules for direct presidential elections in some countries.
- Does reducing the threshold for outright election in this way mitigate or preclude monotonicity problems?

40% RUNOFF RULE

- The geometry makes it clear that reducing the threshold for outright victory to (e.g.) 40% mitigates but does not preclude the possibility of monotonicity failure.



The "Double Complement" Rule

- Let X = PW, Y = P2, and Z = PL.
- One problem (unrelated to monotonicity) with the 40% (or any fixed threshold less than 50%) rule is that, given a plurality profile such as (x=41%, y=39%, z=20%), a runoff seems to be in order, since if the Z supporters in any substantial degree have more second preferences for Y than for X, a runoff would change the outcome.
- Shelton (1972) proposed this rule: X is elected outright
  - if X is the majority winner, or
  - if X's margin over Y exceeds the margin by which X fails to be a majority winner, i.e.,

$$x > n/2 \quad or \quad x - y > n/2 - x$$

- Subsequently, Shugart and Taagepera reinvented this rule and dubbed it the "Double Complement" Rule, characterizing it as the arithmetic average of plurality rule (X wins if $x > y$) and majority runoff (X wins outright if $x > n/2$), i.e., X wins outright if

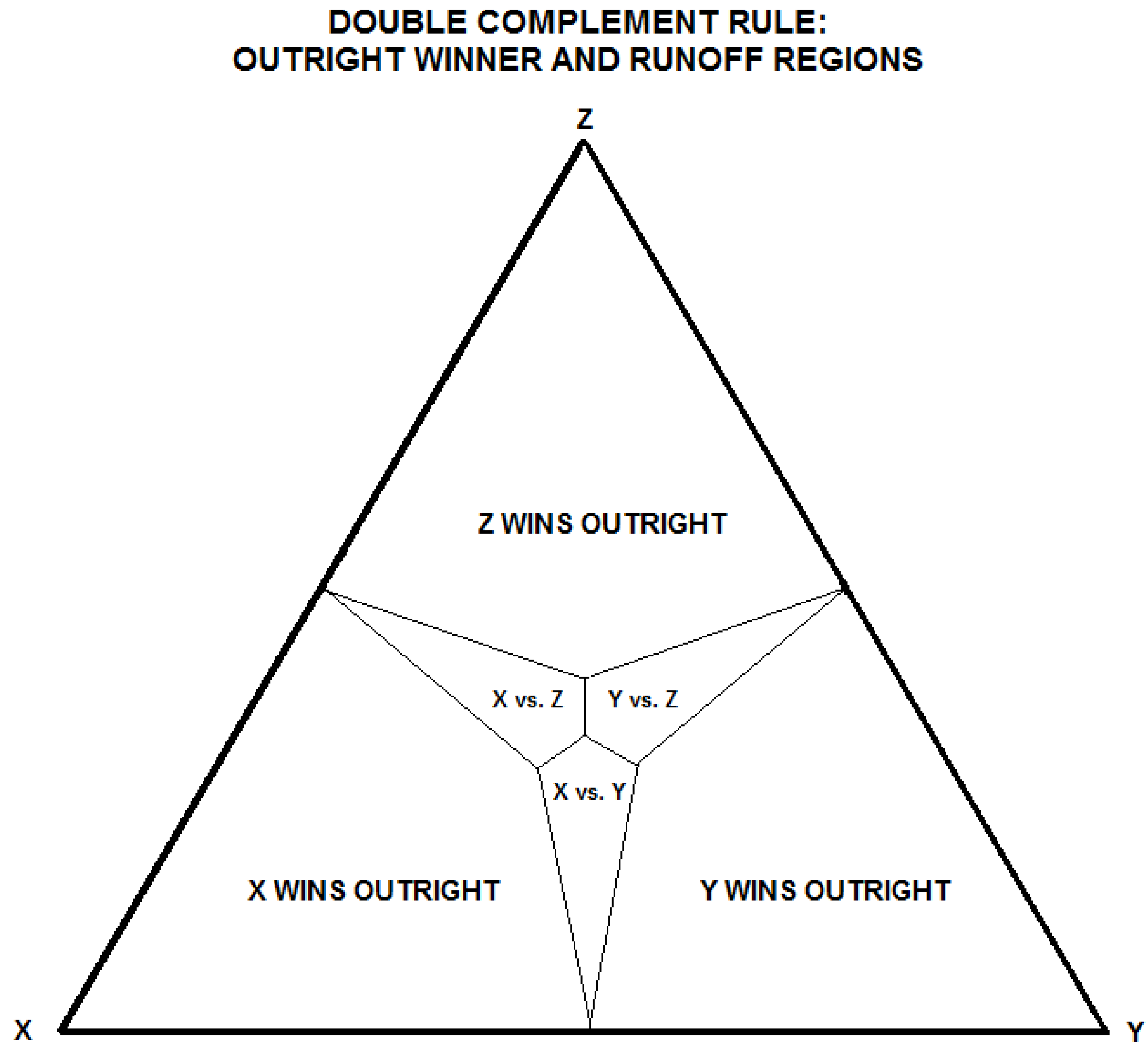
$$x > y/2 + n/4 \quad \text{or} \quad n/2 - y > 2(n/2 - x)$$

M. S. Shugart and R. Taagepera, "Plurality vs. Majority Election of Presidents", Comparative Political Studies, 1994

The "Double Complement" Rule (cont.)

The geometry again makes clear that the Double Complement Rule mitigates but does not preclude the possibility of monotonicity failure.

The DC rule avoids a runoff whenever a runoff would not change the outcome unless *Z* supporters favor *Y* over *X* by more than a 2-1 margin. Other rules can adjust this ratio from ∞-1 to 1-1.

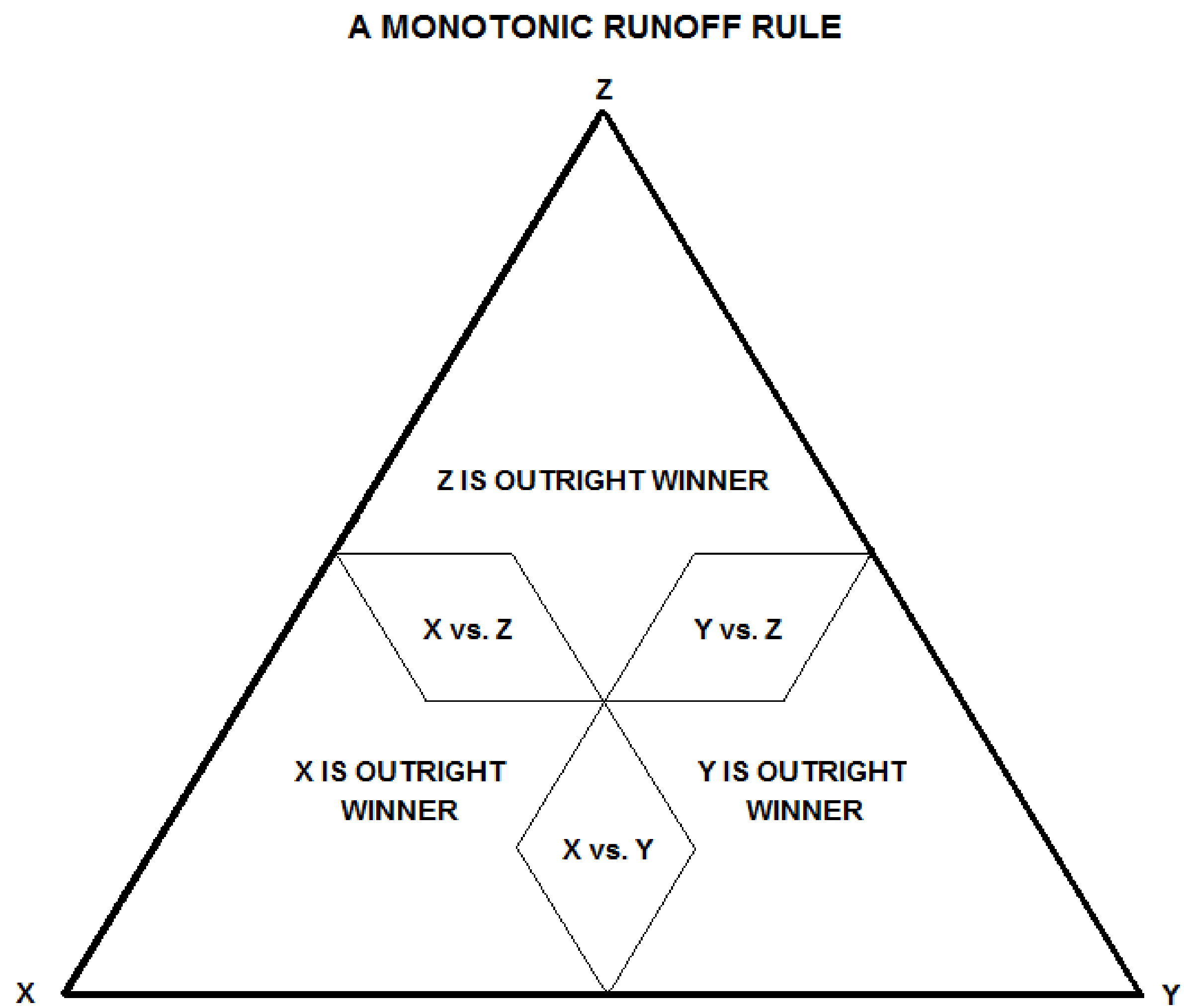


Duddy's "Compromise Rule"

- Duddy proposes a "compromise runoff rule" in which (in the three candidate-case) the relevant threshold pertains to the plurality runner-up, not to the plurality winner (or to both of them).
  - *X* wins outright if the runner-up gets less than one-third of the (first-preference) votes;
  - otherwise there is a runoff.

Duddy's "Compromise Rule" (cont.)

Now the geometry makes clear that the compromise rule precludes the possibility of monoton-icity failure.



Duddy's "Compromise Rule" (cont.)

Note that the compromise method does not avoid monotonicity failure by electing the plurality winner outright in any profile that would be vulnerable to monotonicity failure if a runoff were held.

In the original profile below the plurality runner-up has more than one-third of the vote, so the compromise rule (like IRV) produces a runoff making *Y* the winner. Nevertheless, the profile is vulnerable to UMF under IRV, as shown by the revised profile below.

However, given the revised profile under the compromise rule, the plurality winner *Y* is elected outright since the new plurality runner-up *Z* has less one-third of the vote. Thus the original profile is not vulnerable to UMF under the compromise rule.

Original Profile

36%	34%	30%
X	*Y*	*Z*
Z	*X*	*Y*
Y	*Z*	*X*

Revised Profile

7%	29%	34%	30%
Y	*X*	*Y*	*Z*
X	*Z*	*X*	*Y*
Z	*Y*	*Z*	*X*

Duddy's "Compromise Rule" (cont.)

- An alternative characterization of the compromise rule:
  - *X* wins outright if and only if $x - y > y - z$;
  - otherwise there is a runoff between *X* and *Y*.
- Stated in this manner, the compromise rule can be generalized to any number of candidates.
- And Duddy demonstrates (non-geometrically) that monotonicity of the rule so generalized is preserved.
- Note however, that the compromise rule may not entail a runoff in cases in which a runoff might seem to be in order (and where the Double Complement rule would call for one) e.g.,

$$(x = 34\%,\ y = 33\%,\ z = 33\%)$$